

Runge's Method and Modular Curves

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Abstract

We bound the j -invariant of S -integral points on arbitrary modular curves over arbitrary fields, in terms of the congruence group defining the curve, assuming a certain *Runge condition* is satisfied by our objects. We then apply our bounds to prove that for sufficiently large prime p , the points of $X_0^+(p^r)(\mathbb{Q})$ with $r > 1$ are either cusps or CM points. This can be interpreted as the non-existence of quadratic elliptic \mathbb{Q} -curves with higher prime-power degree.

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1 Introduction

Let $N \geq 2$ be an integer and $X(N)$ the principal modular curve of level N . Further, let G a subgroup of $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ containing -1 and let X_G be the corresponding modular curve. This curve is defined over $\mathbb{Q}(\zeta_N)^{\det(G)}$, so in particular it is defined over \mathbb{Q} if $\det(G) = (\mathbb{Z}/N\mathbb{Z})^\times$. (Through all this paper, we say that an algebraic curve is *defined* over a field if it has a geometrically integral model over this basis.) As usual, we denote by Y_G the finite part of X_G (that is, X_G deprived of the cusps). If X_G is defined over a number field K , the curve X_G has a natural (modular) model over $\mathcal{O} = \mathcal{O}_K$ that we still denote by X_G . The cusps define a closed subscheme of X_G over \mathcal{O} , and we also call Y_G the relative curve over \mathcal{O} which is X_G deprived of the cusps. If S is a finite set of places of K containing the infinite places, the set of S -integral points $Y_G(\mathcal{O}_S)$ consists of those $P \in Y_G(K)$ for which $j(P) \in \mathcal{O}_S$, where j is, as usual, the modular invariant and $\mathcal{O}_S = \mathcal{O}_{K,S}$ is the ring of S -integers.

In its simplest form, the first principal result of this article gives an explicit upper bound for $j(P) \in \mathbb{Z}$ under certain Galois condition on the cusps.

Theorem 1.1 *Assume that X_G is defined over \mathbb{Q} , and assume that the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts non-transitively on the cusps of X_G . Then for any $P \in Y_G(\mathbb{Z})$ we have*

$$\log |j(P)| \leq 12|G|N^2 \log 3N. \quad (1)$$

This result was announced in [1]. Because of an inaccuracy in the proof given in [1], the log-factor is missing therein (this, however, does not affect the arithmetical applications of that theorem, which by the way has since been drastically improved in [2], see below).

Actually, we obtain a more general Theorem 1.2 below, which applies to any number field and any ring of S -integers in it. To state Theorem 1.2 we need to introduce some notations. We denote by $h(\cdot)$ the usual absolute logarithmic height (see Subsection 1.1). For $P \in X_G(\bar{\mathbb{Q}})$ we shall write $h(P) = h(j(P))$. For a number field K we denote by $\mathcal{C} = \mathcal{C}(G)$ the set of cusps of X_G , and by $\mathcal{C}(G, K)$ the set of $\text{Gal}(\bar{K}/K)$ -orbits of \mathcal{C} .

Theorem 1.2 *Let K be a number field and S a finite set of places of K (containing all the infinite places). Let G be a subgroup of $\text{GL}_2(\mathbb{Z}/N\mathbb{Z})$ such that X_G is defined over K . Assume that $|\mathcal{C}(G, K)| > |S|$ (the “Runge condition”). Then for any $P \in Y_G(\mathcal{O}_S)$ we have*

$$h(P) \leq 36s^{s/2+1} (N^2|G|/2)^s \log 2N, \quad (2)$$

where $s = |S|$. If $S = M_K^\infty$, we even have

$$h(P) \leq 24s^{s/2+1} (N^2|G|/2)^s \log 3N. \quad (3)$$

Theorem 1.1 is a special case of this theorem.

This theorem is proved in Section 5 by a variation of the method of Runge, after some preparation in Sections 2, 3 and 4. For a general discussion of Runge’s method see [3, 12].

Theorems 1.1 and 1.2 apply for a general group G . For a specific G , one may obtain much stronger results. One such example can be found in [2]. In Section 6 of the present article we apply our general strategy in the case when $N = p$ is a prime number, the field K is quadratic and G is a split Cartan subgroup; in this case we obtain a much sharper estimate for the integral points, than what follows by direct application of Theorem 1.2. In Section 7 we apply this result to the arithmetic of modular curves, using the integrality property established in Section 8.

Theorem 1.3 *There is an absolute constant p_0 such that, if $r > 1$ and $p > p_0$, the set $X_0^+(p^r)(\mathbb{Q})$ is made of cusps and CM points. Equivalently, there is no quadratic \mathbb{Q} -curve of prime power degree p^r with exponent $r > 1$ when this prime is large enough.*

Recall that a \mathbb{Q} -curve is an elliptic curve without complex multiplication over $\bar{\mathbb{Q}}$ which is isogenous to each of its conjugates over \mathbb{Q} . When such a curve is quadratic (that is, defined over a quadratic field), we say it *has degree N* if the minimal degree of an isogeny from the curve to its only non-trivial conjugate is N . For concrete examples of quadratic \mathbb{Q} -curves see [7] and references therein.

Theorem 1.3 extends the principal result of [2] and essentially settles the problem tackled in [18], [19] or [21].

1.1 Notations, Conventions

Everywhere in this article \log and \arg stand for the principal branches of the complex logarithm and argument functions; that is, for any $z \in \mathbb{C}^\times$ we have $-\pi < \text{Im} \log z = \arg z \leq \pi$. In Section 2 of this article we shall systematically use, often without special reference, the estimates of the kind

$$\begin{aligned} |\log(1+z)| &\leq \frac{|\log(1-r)|}{r} |z|, & |e^z - 1| &\leq \frac{e^r - 1}{r} |z|, \\ |(1+z)^A - 1 - Az| &\leq \frac{|(1+\varepsilon r)^A - 1 - \varepsilon Ar|}{r^2} |z|^2 \quad (A \in \mathbb{R}, \varepsilon = \text{sign } A), \end{aligned} \quad (4)$$

etc., for $|z| \leq r < 1$. They can be easily deduced from the maximum principle.

Let \mathcal{H} denote Poincaré upper half-plane: $\mathcal{H} = \{\tau \in \mathbb{C} : \text{Im } \tau > 0\}$. For $\tau \in \mathcal{H}$ we put $q_\tau = e^{2\pi i \tau}$ (or simply q if there is no ambiguity). We put $\bar{\mathcal{H}} = \mathcal{H} \cup \mathbb{Q} \cup \{i\infty\}$. If Γ is the pull-back of

$G \cap \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ to $\mathrm{SL}_2(\mathbb{Z})$, then the set $X_G(\mathbb{C})$ of complex points is analytically isomorphic to the quotient $X_\Gamma = \tilde{\mathcal{H}}/\Gamma$, supplied with the properly defined topology and analytic structure [11, 22].

We denote by D the standard fundamental domain of $\mathrm{SL}_2(\mathbb{Z})$ (the hyperbolic triangle with vertices $e^{\pi i/3}$, $e^{2\pi i/3}$ and $i\infty$, together with the geodesic segments $[i, e^{2\pi i/3}]$ and $[e^{2\pi i/3}, i\infty]$). Notice that for $\tau \in D$ we have $|q_\tau| \leq e^{-\pi\sqrt{3}} < 0.005$, which will be systematically used without special reference.

For $\mathbf{a} = (a_1, a_2) \in \mathbb{Q}^2$ we put $\ell_{\mathbf{a}} = B_2(a_1 - \lfloor a_1 \rfloor)/2$ where $B_2(T) = T^2 - T + 1/6$ is the second Bernoulli polynomial. The quantity $\ell_{\mathbf{a}}$ is \mathbb{Z}^2 -periodic in \mathbf{a} and is thereby well-defined for $\mathbf{a} \in (\mathbb{Q}/\mathbb{Z})^2$ as well: for such \mathbf{a} we have $\ell_{\mathbf{a}} = B_2(\tilde{a}_1)$, where \tilde{a}_1 is the lifting of the first coordinate of \mathbf{a} to the interval $[0, 1)$. Obviously, $|\ell_{\mathbf{a}}| \leq 1/12$; this will also be often used without special reference.

We fix, once and for all, an algebraic closure $\bar{\mathbb{Q}}$ of \mathbb{Q} , which is assumed to be a subfield of \mathbb{C} . In particular, for every $a \in \mathbb{Q}$ we have the well defined root of unity $e^{2\pi ia} \in \bar{\mathbb{Q}}$. Every number field used in this article is presumed to be contained in the fixed $\bar{\mathbb{Q}}$. If K is such a number field and v is a valuation on K , then we tacitly assume that v is somehow extended to $\bar{\mathbb{Q}} = \bar{K}$; equivalently, we fix an algebraic closure \bar{K}_v and an embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{K}_v$. In particular, the roots of unity $e^{2\pi ia}$ are well-defined elements of \bar{K}_v .

For a number field K we denote by M_K the set of all valuations (or places) of K normalized to extend the usual infinite and p -adic valuations of \mathbb{Q} : $|2|_v = 2$ if $v \in M_K$ is infinite, and $|p|_v = p^{-1}$ if v extends the p -adic valuation of \mathbb{Q} . In the finite case we sometimes use the additive notation $v(\cdot)$, normalized to have $v(p) = 1$. We denote by M_K^∞ and M_K^0 the subsets of M_K consisting of the infinite (archimedean) and the finite (non-archimedean) valuations, respectively.

Recall the definition of the absolute logarithmic height $h(\cdot)$. For $\alpha \in \bar{\mathbb{Q}}$ we pick a number field K containing α and put $h(\alpha) = [K : \mathbb{Q}]^{-1} \sum_{v \in M_K} [K_v : \mathbb{Q}_v] \log^+ |\alpha|_v$, where the valuations on K are normalized to extend standard infinite and p -adic valuations on \mathbb{Q} and $\log^+ x = \log \max\{x, 1\}$. The value of $h(\alpha)$ is known to be independent on the particular choice of K . As usual, we extend the definition of the height to $\mathbb{P}^1(\bar{\mathbb{Q}}) = \bar{\mathbb{Q}} \cup \{\infty\}$ by setting $h(\infty) = 0$. If α is a rational integer or an imaginary quadratic integer then $h(\alpha) = \log |\alpha|$.

If E is an elliptic over $\bar{\mathbb{Q}}$, define $h(E) := h(j_E)$ for j_E its j -invariant. For $P \in X(\bar{\mathbb{Q}})$ a point with values in $\bar{\mathbb{Q}}$ of a modular curve X , the height we will use (unless explicitly mentioned otherwise) will be this naive Weil height of the elliptic curve associated to P , that is $h(P) = h(j(P))$.

2 Estimates for Modular Functions at Infinity

The results of this section must be known, but we did not find them in the available literature, so we state and prove them here. For the sake of further applications, we have tried to give rather sharp constants, even if we do not actually need this precision in the sequel of the present paper.

2.1 Estimating the j -Function

Recall that the modular j -invariant $j : \mathcal{H} \rightarrow \mathbb{C}$ is defined by $j(\tau) = (12c_2(\tau))^3/\Delta(\tau)$, where

$$c_2(\tau) = \frac{(2\pi)^4}{12} \left(1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - q^n} \right)$$

(see, for instance, [10, Section 4.2]) and $\Delta(\tau) = (2\pi)^{12} q \prod_{n=1}^{\infty} (1 - q^n)^{24}$. Also, j has the familiar Fourier expansion $j(\tau) = q^{-1} + 744 + 196884q + \dots$

Proposition 2.1 *For $\tau \in \mathcal{H}$ such that $|q| (= |q_\tau|) \leq 0.005$ (and, in particular, for every $\tau \in D$) we have*

$$|j(\tau) - q^{-1} - 744| \leq 330000|q|. \quad (5)$$

(Recall that D is the standard fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$.)

Proof Using the estimate $n^3 \leq 3^n$ for $n \geq 3$, we find that for $|q| < 1/3$

$$\begin{aligned} \left| \frac{12}{(2\pi)^4} c_2(\tau) - 1 - 240q \right| &\leq 240 \left(\frac{|q|^2}{1-|q|} + \sum_{n=2}^{\infty} \frac{n^3 |q|^n}{1-|q|^n} \right) \\ &\leq \frac{240}{1-|q|} \left(|q|^2 + 8|q|^2 + \sum_{n=3}^{\infty} |3q|^n \right) \\ &= \frac{2160}{(1-|q|)(1-3|q|)} |q|^2, \end{aligned}$$

and for $|q| \leq 0.005$ we obtain

$$\left| \frac{12}{(2\pi)^4} c_2(\tau) - 1 - 240q \right| \leq 2204|q|^2. \quad (6)$$

Further, using (4), we obtain, for $|q| \leq 0.005$,

$$\left| \log \frac{(2\pi)^{12} q (1-q)^{24}}{\Delta(\tau)} \right| = 24 \left| \sum_{n=2}^{\infty} \log(1-q^n) \right| \leq 24.1 \sum_{n=2}^{\infty} |q|^n \leq 24.3|q|^2.$$

Hence

$$\begin{aligned} \left| \frac{(2\pi)^{12} q}{\Delta(\tau)} - 1 - 24q \right| &\leq |(1-q)^{-24}| \left| \frac{(2\pi)^{12} q (1-q)^{24}}{\Delta(\tau)} - 1 \right| + |(1-q)^{-24} - 1 - 24q| \\ &\leq 1.13 \left| \log \frac{(2\pi)^{12} q (1-q)^{24}}{\Delta(\tau)} \right| + 314|q|^2 \leq 342|q|^2. \end{aligned}$$

Combining this with (6), we obtain (5) after a tiresome, but straightforward calculation. \square

The following consequences are obtained by direct numerical computations.

Corollary 2.2 *For $\tau \in D$ the following statements are true.*

- (i) We have $|\log |q_\tau|| \leq \log(|j(\tau)| + 2400)$.
- (ii) We have either $|j(\tau)| \leq 3500$ or $|q_\tau| < 0.001$.
- (iii) If $|j(\tau)| > 3500$ then $|j(\tau) - q_\tau^{-1}| \leq 1100$ and $\frac{3}{2}|j(\tau)| \geq |q_\tau^{-1}| \geq \frac{1}{2}|j(\tau)|$. \square

2.2 Estimating Siegel's Functions

For a rational number a we define $q^a = e^{2\pi i a \tau}$. Let $\mathbf{a} = (a_1, a_2) \in \mathbb{Q}^2$ be such that $\mathbf{a} \notin \mathbb{Z}^2$, and let $g_{\mathbf{a}} : \mathcal{H} \rightarrow \mathbb{C}$ be the corresponding *Siegel function* [9, Section 2.1]. Then we have the following infinite product presentation for $g_{\mathbf{a}}$ [9, page 29] (where $B_2(T)$ is the second Bernoulli polynomial):

$$g_{\mathbf{a}}(\tau) = -q^{B_2(a_1)/2} e^{\pi i a_2 (a_1 - 1)} \prod_{n=0}^{\infty} (1 - q^{n+a_1} e^{2\pi i a_2}) (1 - q^{n+1-a_1} e^{-2\pi i a_2}). \quad (7)$$

We also have [9, pages 29–30] the relations

$$g_{\mathbf{a}} \circ \gamma = g_{\mathbf{a}\gamma} \cdot (\text{a root of unity}) \quad \text{for } \gamma \in \Gamma(1), \quad (8)$$

$$g_{\mathbf{a}} = g_{\mathbf{a}'} \cdot (\text{a root of unity}) \quad \text{when } \mathbf{a} \equiv \mathbf{a}' \pmod{\mathbb{Z}^2}. \quad (9)$$

Remark that the root of unity in (8) is of order dividing 12, and in (9) of order dividing $2N$, where N is the denominator of \mathbf{a} (the common denominator of a_1 and a_2 ; see [2] for more references).

The order of vanishing of $g_{\mathbf{a}}$ at $i\infty$ (that is, the only rational number ℓ such that the limit $\lim_{\tau \rightarrow i\infty} q_\tau^{-\ell} g_{\mathbf{a}}(\tau)$ exists and is non-zero) is equal to the number $\ell_{\mathbf{a}}$, defined in Subsection 1.1 (see [9, page 31]).

Proposition 2.3 *Let \mathbf{a} be an element of $\mathbb{Q}^2 \setminus \mathbb{Z}^2$ and $N > 1$ an integer such that $N\mathbf{a} \in \mathbb{Z}^2$. Then for $\tau \in D$ we have*

$$\left| \log |g_{\mathbf{a}}(\tau)| - \ell_{\mathbf{a}} \log |q_{\tau}| \right| \leq \log N. \quad (10)$$

Proof Due to (9), we may assume that $0 \leq a_1 < 1$ and distinguish between the cases $0 < a_1 < 1$ and $a_1 = 0$. According to (7), the left-hand side of (10) is equal to

$$\left| \log |1 - q^{a_1} \zeta| + \log |1 - q^{1-a_1} \bar{\zeta}| + \sum_{n=1}^{\infty} (\log |1 - q^{n+a_1} \zeta| + \log |1 - q^{n+1-a_1} \bar{\zeta}|) \right|,$$

where $\zeta = e^{2\pi i a_2}$. Recall that, since $\tau \in D$, we have $|q| \leq e^{-\pi\sqrt{3}}$. Hence, for $n \geq 1$ each of $|q^{n+a_1}|$ and $|q^{n+1-a_1}|$ does not exceed 0.005, whence

$$\left| \sum_{n=1}^{\infty} (\log |1 - q^{n+a_1} \zeta| + \log |1 - q^{n+1-a_1} \bar{\zeta}|) \right| \leq 1.005 \frac{|q^{1+a_1}| + |q^{2-a_1}|}{1 - |q|} \leq 3|q| \leq 0.02. \quad (11)$$

To estimate

$$\left| \log |1 - q^{a_1} \zeta| + \log |1 - q^{1-a_1} \bar{\zeta}| \right|, \quad (12)$$

assume first that $0 < a_1 < 1$. Then among the numbers $|q^{a_1}|$ and $|q^{1-a_1}|$ one is bounded by $|q|^{1/N}$ and the other is bounded by $|q|^{1/2}$, which bounds (12) by

$$\left| \log |1 - e^{-\pi\sqrt{3}/N}| \right| + \left| \log |1 - e^{-\pi\sqrt{3}/2}| \right| \leq \log N - 0.1.$$

This proves (10) in the case $0 < a_1 < 1$. When $a_1 = 0$ then $a_2 \notin \mathbb{Z}$ and $\zeta \neq 1$, which bounds (12) by

$$\left| \log |1 - e^{2\pi i/N}| \right| + \left| \log |1 - e^{-\pi\sqrt{3}}| \right| \leq \log N - 0.1,$$

proving (10) in this case as well. \square

Since Siegel's functions have no pole nor zero on the upper half plane \mathcal{H} , they should be bounded from above and from below on any compact subset of \mathcal{H} . In particular, they should be bounded where j is bounded. Here is a quantitative version of this.

Corollary 2.4 *Let $\mathbf{a} \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$ and $N > 1$ satisfy $N\mathbf{a} \in \mathbb{Z}^2$. Then for any $\tau \in \mathcal{H}$ we have*

$$\left| \log |g_{\mathbf{a}}(\tau)| \right| \leq \frac{1}{12} \log (|j(\tau)| + 2400) + \log N. \quad (13)$$

Proof Replacing τ by $\gamma\tau$ and $g_{\mathbf{a}}$ by $g_{\mathbf{a}\gamma^{-1}}$ with a suitable $\gamma \in \Gamma(1)$, we may assume that $\tau \in D$, in particular (10) holds. Combining (10) with item (i) of Corollary 2.2, and using the inequality $|\ell_{\mathbf{a}}| \leq 1/12$, we obtain (13). \square

2.3 Non-Archimedean Versions

We also need non-archimedean versions of some of the above inequalities. In this subsection K_v is a field complete with respect to a non-archimedean valuation v and \bar{K}_v its algebraic closure. Let $q \in K_v$ satisfy $|q|_v < 1$. For $\mathbf{a} = (a_1, a_2) \in \mathbb{Q}^2$ such that $\mathbf{a} \notin \mathbb{Z}^2$ define $g_{\mathbf{a}}(\tau)$ as in (7). Recall (see Subsection 1.1) that the roots of unity $e^{2\pi i a}$ with $a \in \mathbb{Q}$ are defined as elements of \bar{K}_v .

The right-hand side of (7) is, formally, not well-defined because we use rational powers of q . However, if we fix $q^{1/12N^2} \in \bar{K}_v$, where N is the order of \mathbf{a} in $(\mathbb{Q}/\mathbb{Z})^2$, then everything becomes well-defined, and we again have (8) and (9). The statement of the following proposition is independent on the particular choice of $q^{1/12N^2}$.

Proposition 2.5 *Let \mathbf{a} be an element of $\mathbb{Q}^2 \setminus \mathbb{Z}^2$ and $N > 1$ an integer such that $N\mathbf{a} \in \mathbb{Z}^2$. Then*

$$\left| \log |g_{\mathbf{a}}(q)|_v - \ell_{\mathbf{a}} \log |q|_v \right| \begin{cases} = 0 & \text{if } v(N) = 0 \\ \leq \frac{\log p}{p-1} & \text{if } v \mid p \mid N, \end{cases}$$

where here and below $v \mid p \mid N$ means that $v(N) > 0$ and p is the prime below v .

Proof We again may assume that $0 \leq a_1 < 1$. When $0 < a_1 < 1$ we have $\log |g_{\mathbf{a}}(q)|_v = \ell_{\mathbf{a}} \log |q|_v$. When $a_1 = 0$ we have

$$\log |g_{\mathbf{a}}(q)|_v = \ell_{\mathbf{a}} \log |q|_v + \log |1 - e^{2\pi i a_2}|_v,$$

and $|1 - e^{2\pi i a_2}|_v = 1$ if $v(N) = 0$, while if $v \mid p \mid N$ we have $1 \geq |1 - e^{2\pi i a_2}|_v \geq p^{-1/(p-1)}$. \square

3 Locating a “Nearby Cusp”

Let N be a positive integer, G a subgroup of $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ and X_G the corresponding modular curve, defined over a number field K . We fix a valuation v of K . Recall we denote by \mathcal{O} the ring of integers of K and by K_v the v -completion of K . When v is non-archimedean, we denote by \mathcal{O}_v the ring of integers of K_v , and by k_v its residue field at v . As usual ζ_N will denote a primitive N -th root of unity. Recall that when we say that a curve “is defined over” a field, it means that this curve has a geometrically integral model over that field.

Let P be a point on $X_G(K_v)$ such that $|j(P)|$ is “large”. Then it is intuitively clear that, in the v -adic metric, P is situated “near” a cusp of X_G . The purpose of this section is to make this intuitive observation precise and explicit. We shall locate this “nearby” cusp and specify what the word “near” means.

We first recall the following description of the cuspidal locus of $X(N)$ (for more details see e.g. [4, Chapitres V and VII]). The cusps of $X(N)$ define a closed subscheme of the smooth locus of the modular model of $X(N)$ over $\mathbb{Z}[\zeta_N]$. Fix a uniformization $X(N)(\mathbb{C}) \simeq \bar{\mathcal{H}}/\Gamma(N)$, let c_{∞} be the cusp corresponding to $i\infty \in \bar{\mathcal{H}}$, and write $q^{1/N} = e^{2i\pi\tau/N}$ the usual parameter. If $c = \gamma(c_{\infty})$, for some $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, is another cusp, denote by $q_c := q \circ \gamma^{-1}$ the parameter on $X(N)(\mathbb{C})$ at c . It follows from [4, Chapitre VII, Corollaire 2.5] that the completion of the curve $X(N)$ over $\mathbb{Z}[\zeta_N]$ along the section c is isomorphic to $\mathrm{Spec}(\mathbb{Z}[\zeta_N][[q_c^{1/N}]])$. In other words, the parameter $q_c^{1/N}$ at c on $X(N)(\mathbb{C})$ is actually defined over $\mathbb{Z}[\zeta_N]$, that is $q_c^{1/N}$ comes from an element of the completed local ring $\hat{\mathcal{O}}_{X(N),c}$ of the modular model of $X(N)$ over $\mathbb{Z}[\zeta_N]$, along the section c .

Next we describe the local parameters at the cusps on an arbitrary X_G . As before, let Γ be the pull-back of $G \cap \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ to $\Gamma(1)$. For each cusp c of X_G we obtain a parameter at c on X_G by picking a lift \tilde{c} of c on $X(N)$ and taking the norm $\prod q_{\tilde{c}}^{1/N} \circ \gamma$, where γ runs through a set of representatives of $\Gamma/\Gamma(N)$. We denote by t_c this parameter in the sequel. Note that it is defined over a (possibly strict) subring of $\mathbb{Z}[\zeta_N]$.

It is clear from the definition that the above parameter t_c defines a v -analytic function on a v -adic neighborhood of $c \in X_G(\bar{K}_v)$ which satisfies the initial condition $t_c(c) = 0$. Further, if e_c is the ramification index of the covering $X_G \rightarrow X(1)$ at c (note that e_c divides N) then, setting $q_c := t_c^{e_c}$, the familiar expansion $j = q_c^{-1} + 744 + 196884q_c + \dots$ holds in a v -adic neighborhood of c , the right-hand side converging v -adically.

Let us define explicitly a set $\Omega_c = \Omega_{c,v}$ on which t_c and q_c are defined and analytic. Assume first that v is archimedean. Let \tilde{D} be the fundamental domain D modified as follows:

$$\tilde{D} = D \cup \{i\infty\} \setminus (\text{the arc connecting } i \text{ and } e^{2\pi i/3}). \quad (14)$$

Then the set $\Delta = \tilde{D} + \mathbb{Z}$ of translated of \tilde{D} by the rational integers has the following properties.

- (i) If for some $\tau \in \Delta$ and $\gamma \in \Gamma(1)$ we have $\gamma(\tau) \in \Delta$ then $\tau \equiv \gamma(\tau) \pmod{1}$.
- (ii) For any $\gamma \in \Gamma(1)$ either $\gamma(\Delta) = \Delta$ or $\gamma(\Delta) \cap \Delta = \emptyset$.

(iii) The quotient Δ/Γ is an open subset of $\tilde{\mathcal{H}}/\Gamma = X_G(\mathbb{C})$. (Recall that Γ is the pull-back of $G \cap \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ to $\mathrm{SL}_2(\mathbb{Z})$.)

The same properties hold for the set $\gamma(\Delta)$ for any $\gamma \in \Gamma(1)$. Now we define $\Omega_c = \gamma(\Delta)/\Gamma$, where $\gamma \in \Gamma(1)$ is chosen so that $\gamma(i\infty)$ represents the cusp c . It follows from the properties above that the sets Ω_c are pairwise disjoint, and that q_c and t_c are defined and analytic on Ω_c .

If v is non-archimedean, then we define $\Omega_c = \Omega_{c,v}$ as the set of the points from $X_G(\bar{K}_v)$ having reduction c at v .

Proposition 3.1 *Put*

$$X_G(\bar{K}_v)^+ = \begin{cases} \{P \in X_G(\bar{K}_v) : |j(P)| > 3500\} & \text{if } v \in M_K^\infty, \\ \{P \in X_G(\bar{K}_v) : |j(P)| > 1\} & \text{if } v \in M_K^0. \end{cases}$$

Then

$$X_G(\bar{K}_v)^+ \subseteq \bigcup_{c \in \mathcal{C}} \Omega_{c,v} \quad (15)$$

with equality for the non-archimedean v . Also, for $P \in \Omega_{c,v}$ we have

$$|j(P) - q_c(P)^{-1}|_v \leq 1100, \quad (16)$$

$$\frac{3}{2}|j(P)|_v \geq |q_c(P)^{-1}|_v \geq \frac{1}{2}|j(P)|_v \quad (17)$$

if v is archimedean, and $|j(P)|_v = |q_c(P)^{-1}|_v$ if v is non-archimedean.

Proof For the non-archimedean v both statements are immediate. For archimedean v they follow from Corollary 2.2. Indeed, fix a uniformization $X(\bar{K}_v) = \tilde{\mathcal{H}}/\Gamma$ and let $\tau_0 \in \tilde{\mathcal{H}}$ be a lift of $P \in X_G(\bar{K}_v)^+$. Pick $\gamma \in \Gamma(1)$ such that $\tau = \gamma(\tau_0) \in D$. Then $|j(\tau)| = |j(\tilde{P})| > 3500$, and item (ii) of Corollary 2.2 implies that $|q_\tau| < 0.001 < e^{-2\pi}$. It follows that $\tau \in \tilde{D}$, which is equivalent to saying that $P \in \Omega_c$ with $c = \gamma^{-1}(c_\infty)$. This proves (15), and item (iii) of Corollary 2.2 implies (16) and (17). \square

The proposition implies that for every $P \in X_G(\bar{K}_v)^+$ there exists a cusp c such that $P \in \Omega_{c,v}$. We call it a v -nearby cusp, or simply a nearby cusp to P .

Remark 3.2 As we have already seen, the sets Ω_c are pairwise disjoint when v is archimedean. The same is true if v is non-archimedean and $v(N) = 0$, as in this case the cusps define a finite étale scheme over \mathcal{O}_v . Thus, in these case the nearby cusp is well-defined.

However, if $v(N) > 0$ then the sets Ω_c are not disjoint, see the addendum below; in particular, in this case a point in $X_G(\bar{K}_v)$ may have several nearby cusps. This is absolutely harmless for our arguments, but for further applications it would be nice to refine the sets Ω_c to make them pairwise disjoint; in particular, this would allow us to define “the v -nearest cusp” rather than just “a v -nearby cusp” for any v . In the addendum below we examine more carefully geometry of cusps and their neighborhoods in the non-archimedean case. In particular, we explicitly exhibit a pairwise disjoint system of v -adic cusp neighborhoods.

Addendum: more on the cusps and their v -adic neighborhoods¹

Recall first of all that the modular interpretation of $X(N)$ associates with each cusp a Néron polygon C with N sides on $\mathbb{Z}[\zeta_N]$, endowed with its structure of generalized elliptic curve, and enhanced with a basis of $C[N] \simeq \mathbb{Z}/N\mathbb{Z} \times \mu_N = \langle q^{1/N}, \zeta_N \rangle$ such that the determinant of this basis is 1, and two bases are identified if they are conjugate by the subgroup $\pm U = \pm \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ of $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$, the action being $\begin{pmatrix} \epsilon & a \\ 0 & \epsilon \end{pmatrix} : (q^{1/N}, \zeta_N) \mapsto (q^{\epsilon/N} \zeta_N^a, \zeta_N^\epsilon)$, for $\epsilon = \pm 1$ and $a \in \mathbb{Z}/N\mathbb{Z}$. We

¹The material of this addendum will not be used in the present article, but we include it for future references.

may, for instance, interpret c_∞ as the orbit $\{(C, (q^{\epsilon/N} \zeta_N^a, \zeta_N^\epsilon)), \epsilon \in \{\pm 1\}, a \in \mathbb{Z}/N\mathbb{Z}\}$ of enhanced Néron polygons over $\mathbb{Z}[\zeta_N]$.

It follows that the modular interpretation of X_G associates to each cusp an orbit of our enhanced Néron polygon $(C, (q^{1/N}, \zeta_N))$ under the action of the group generated by G and $\pm U$. We see from the above that the cusps of X_G have values in a subring of $\mathbb{Z}[\zeta_N]$. Assume moreover that X_G is defined over K , of which v is a place of characteristic p , with $N = p^n N'$ and $p \nmid N'$. Extending v to a place of $\mathcal{O}_v[\zeta_{N'}]$ if necessary, and setting $\mathcal{O}'_v := (\mathcal{O}[\zeta_{N'}])_v$, one sees that the closed subscheme of cusps over \mathcal{O}'_v may be written as a sum of connected components of shape $\text{Spec}(R)$ where R is a subring of $\mathcal{O}'_v[\zeta_{p^n}]$. Therefore if $v(N) = 0$, the subscheme of cusps is étale over \mathcal{O}_v , but this may not be the case if $v(N) > 0$. In the latter case, however, the ramification is well controlled. Indeed, with the preceding notations, set $\pi := (1 - \zeta_{p^n})$. Any two different p^n -th roots of unity $\zeta_{p^n}^a$ and $\zeta_{p^n}^b$ satisfy $(\zeta_{p^n}^a - \zeta_{p^n}^b) = \pi^{p^k} \alpha$ with α a v -invertible element and $0 \leq k \leq n - 1$. As $v(\pi) = 1/p^{n-1}(p-1)$ (normalizing v so that $v(p) = 1$), it follows that the Néron polygons enhanced with a level- N structure are distinct over $\mathbb{Z}[\zeta_N]/(\pi^{p^{n-1}+1})$. The modular interpretation shows more precisely that if two different cusps c_1 and c_2 have same reduction at v , then $t_{c_1}(c_2)$ has v -adic valuation less or equal to $1/(p-1)$ (where t_c is the parameter defined at the beginning of this Section). This remark will be used later on.

To illustrate all this with a familiar example, letting $G := \begin{pmatrix} 1 & 0 \\ * & * \end{pmatrix} \subset \text{GL}_2(\mathbb{Z}/N\mathbb{Z})$, which gives rise to the modular curve $X_1(N)$, one finds that there are $|(\mathbb{Z}/N\mathbb{Z})^\times|$ cusps, with modular interpretation corresponding to $\{(C, \zeta_N^{\epsilon a}) : \epsilon \in \{\pm 1\}\}$ where a runs through $(\mathbb{Z}/N\mathbb{Z})^\times / \pm 1$, and $\{(C, q^{\epsilon a/N} \zeta_N^\alpha) : \epsilon \in \{\pm 1\}, \alpha \in (\mathbb{Z}/N\mathbb{Z})\}$, where a runs through the same set. The curve $X_1(N)$ is defined over \mathbb{Q} and has a modular model over \mathbb{Z} . The cusps in the former subset above have values in $\mathbb{Z}[\zeta_N + \zeta_N^{-1}]$, and the cusps in the latter subset have values in \mathbb{Z} . In other words, the closed subscheme of cusps over \mathbb{Z} is isomorphic to the disjoint union of $\text{Spec}(\mathbb{Z}[\zeta_N + \zeta_N^{-1}])$ and $|(\mathbb{Z}/N\mathbb{Z})^\times|/2$ copies of $\text{Spec}(\mathbb{Z})$.

Let us examine the v -adic neighborhoods of the cusps. If $v(N) = 0$ then the cusps of X_G define a finite étale closed subscheme of X_G over \mathcal{O}_v , so the sets Ω_c are obviously pairwise disjoint.

Now assume that $v(N) > 0$. Let p be the residue characteristic of v and $p^n \parallel N$ be the largest power of p dividing $N = p^n N'$. The scheme of cusps on X_G may be no longer étale over \mathcal{O}_v . We can however still partition it into connected components, which totally ramify in the fiber at v . More precisely, setting as above $\mathcal{O}'_v := (\mathcal{O}[\zeta_{N'}])_v$, each connected component over \mathcal{O}'_v is schematically a $\text{Spec}(R)$ for R a subring of $\mathcal{O}'_v[\zeta_{p^n}]$. Each set Ω_c as in Proposition 3.1 contains exactly one such connected component of cusps, so when R does ramify nontrivially at v , then Ω_c is clearly “too large” (one has $\Omega_{c_1} = \Omega_{c_2}$ exactly when c_1 and c_2 have same reduction at v). We want to show that, nevertheless, the refined sets Ω_c^+ , defined by

$$\Omega_c^+ = \Omega_{c,v}^+ = \{P \in \Omega_c : |q_c(P)|_v < p^{-N/(p-1)}\}$$

are pairwise disjoint.

If the cusps c_1 and c_2 belong to distinct connected components, then already Ω_{c_1} and Ω_{c_2} are disjoint, so $\Omega_{c_1}^+$ and $\Omega_{c_2}^+$ are disjoint *a fortiori*. Now assume that c_1 and c_2 belong to the same component, *i.e.*, have same reduction at v . In this case, as explained above, one may write $t_{c_1}(c_2) = \pi^{p^k} a \in \mathcal{O}'_v[\zeta_{p^n}]$, for π a certain uniformizer (e.g. $\pi := (\zeta_{p^n} - 1)$), where the element a is v -invertible and $0 \leq k \leq n - 1$. As $\Omega_{c_1}^+$ is contained in $\{P \in X_G(\bar{K}_v) : |t_{c_1}(P)|_v < p^{-1/(p-1)}\}$ (recall $q_c = t_c^{e_c}$, with $e_c \mid N$), we see that c_2 does not belong to $\Omega_{c_1}^+$, which implies that the sets $\Omega_{c_1}^+$ and $\Omega_{c_2}^+$ are disjoint.

4 Modular Units

In this section we recall the construction of modular units on the modular curve X_G . By a *modular unit* we mean a rational function on X_G having poles and zeros only at the cusps.

4.1 Integrality of Siegel's Function

For $\mathbf{a} \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$ Siegel's function $g_{\mathbf{a}}$ (see Subsection 2.2) is algebraic over the field $\mathbb{C}(j)$: this follows from the fact that $g_{\mathbf{a}}^{12}$ is automorphic of certain level [9, page 29]. Since $g_{\mathbf{a}}$ is holomorphic and does not vanish on the upper half-plane \mathcal{H} , both $g_{\mathbf{a}}$ and $g_{\mathbf{a}}^{-1}$ must be integral over the ring $\mathbb{C}[j]$. Actually, a stronger assertion holds.

Proposition 4.1 *Both $g_{\mathbf{a}}$ and $(1 - \zeta_N) g_{\mathbf{a}}^{-1}$ are integral over $\mathbb{Z}[j]$. Here N is the exact order of \mathbf{a} in $(\mathbb{Q}/\mathbb{Z})^2$ and ζ_N is a primitive N -th root of unity.*

This is, essentially, established in [9], but is not stated explicitly therein. For a complete proof, see [2, Proposition 2.2].

4.2 Modular Units on $X(N)$

From now on, we fix an integer $N > 1$. Recall that the curve $X(N)$ is defined over the field $\mathbb{Q}(\zeta_N)$. Moreover, the field $\mathbb{Q}(X(N)) = \mathbb{Q}(\zeta_N)(X(N))$ is a Galois extension of $\mathbb{Q}(j)$, the Galois group being isomorphic to $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\pm 1$. The isomorphism

$$\mathrm{Gal}(\mathbb{Q}(X(N))/\mathbb{Q}(j)) \cong \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\pm 1 \quad (18)$$

is defined up to an inner automorphism; once it is fixed, we have the well-defined isomorphisms

$$\mathrm{Gal}(\mathbb{Q}(X(N))/\mathbb{Q}(\zeta_N, j)) \cong \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\pm 1, \quad \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q}) \cong (\mathbb{Z}/N\mathbb{Z})^\times \quad (19)$$

(which give the geometric and arithmetic parts of the Galois group respectively). We may identify the groups on the left and on the right in (18 and 19). Our choice of the isomorphism (18) will be specified in Proposition 4.2.

According to Theorem 1.2 from [9, Section 2.1], given $\mathbf{a} = (a_1, a_2) \in (N^{-1}\mathbb{Z})^2 \setminus \mathbb{Z}^2$, the function $g_{\mathbf{a}}^{12N}$ is $\Gamma(N)$ -automorphic of weight 0. Hence $g_{\mathbf{a}}^{12N}$ defines a rational function on the modular curve $X(N)$, to be denoted by $u_{\mathbf{a}}$ (one checks that $u_{\mathbf{a}} = u_{-\mathbf{a}}$). Since the root of unity in (9) is of order dividing $12N$, we have $u_{\mathbf{a}} = u_{\mathbf{a}'}$ when $\mathbf{a} \equiv \mathbf{a}' \pmod{\mathbb{Z}^2}$. Hence $u_{\mathbf{a}}$ is well-defined when \mathbf{a} is a non-zero element of the abelian group $(N^{-1}\mathbb{Z}/\mathbb{Z})^2$, which will be assumed in the sequel. We put $\mathbf{A} = (N^{-1}\mathbb{Z}/\mathbb{Z})^2 \setminus \{0\}$.

The functions $u_{\mathbf{a}}$ have the following properties.

Proposition 4.2 (i) *The functions $u_{\mathbf{a}}$ and $(1 - \zeta_{N_{\mathbf{a}}})^{12N} u_{\mathbf{a}}^{-1}$ are integral over $\mathbb{Z}[j]$, where $N_{\mathbf{a}}$ is the exact order of \mathbf{a} in $(N^{-1}\mathbb{Z}/\mathbb{Z})^2$. In particular, $u_{\mathbf{a}}$ has zeros and poles only at the cusps of $X(N)$.*

(ii) *The functions $u_{\mathbf{a}}$ belong to the field $\mathbb{Q}(X(N))$, and the Galois action on the set $\{u_{\mathbf{a}}\}$ is compatible with the (right) linear action of $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ on \mathbf{A} in the following sense: the isomorphism (18) can be chosen so that for any $\sigma \in \mathrm{Gal}(\mathbb{Q}(X(N))/\mathbb{Q}(j)) = \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\pm 1$ and any $\mathbf{a} \in \mathbf{A}$ we have $u_{\mathbf{a}}^\sigma = u_{\mathbf{a}\sigma}$.*

(iii) *For the cusp c_∞ at infinity we have $\mathrm{ord}_{c_\infty} u_{\mathbf{a}} = 12N^2 \ell_{\mathbf{a}}$, where $\ell_{\mathbf{a}}$ is defined in Subsection 1.1. For an arbitrary cusp c we have $|\mathrm{ord}_c u_{\mathbf{a}}| \leq N^2$.*

Proof Item (i) follows from Proposition 4.1. Item (ii) is Proposition 1.3 from [9, Chapter 2]. We are left with item (iii). The q -order of vanishing of $u_{\mathbf{a}}$ at i_∞ is $12N\ell_{\mathbf{a}}$. Since the ramification index of the morphism $X(N) \rightarrow X(1)$ at every cusp is N , we obtain $\mathrm{ord}_{c_\infty} u_{\mathbf{a}} = 12N^2 \ell_{\mathbf{a}}$. Since $|\ell_{\mathbf{a}}| \leq 1/12$, we have $|\mathrm{ord}_{c_\infty} u_{\mathbf{a}}| \leq N^2$. The case of arbitrary c reduces to the case $c = c_\infty$ upon replacing \mathbf{a} by $\mathbf{a}\sigma$ where $\sigma \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ is such that $\sigma(c) = c_\infty$. \square

The group generated by the principal divisors $(u_{\mathbf{a}})$, where $\mathbf{a} \in \mathbf{A}$, is contained in the group of cuspidal divisors on $X(N)$ (that is, the divisors supported at the set $\mathcal{C}(N) = \mathcal{C}(\Gamma(N))$ of cusps).

Since principal divisors are of degree 0, the rank of the former group is at most $|\mathcal{C}(N)| - 1$. It is fundamental for us that this rank is indeed maximal. The following proposition is Theorem 3.1 in [9, Chapter 2].

Proposition 4.3 *The group generated by the set $\{(u_{\mathbf{a}}) : \mathbf{a} \in \mathbf{A}\}$ is of rank $|\mathcal{C}(N)| - 1$. \square*

We also need to know the behavior of the functions $u_{\mathbf{a}}$ near the cusps, and estimate them in terms of the modular invariant j . In the following proposition K is a number field containing ζ_N and v is a valuation of K , extended somehow to \bar{K} . For $v \in M_K$ we define

$$\rho_v = \begin{cases} 12N \log N & \text{if } v \in M_K^\infty, \\ 0 & \text{if } v \in M_K^0 \text{ and } v(N) = 0, \\ 12N \frac{\log p}{p-1} & \text{if } v \in M_K^0 \text{ and } v \mid p \mid N. \end{cases} \quad (20)$$

We use the notation of Section 3.

Proposition 4.4 *Let c be a cusp of $X(N)$ and v a place of K . For $P \in \Omega_{c,v}$ we have*

$$|\log |u_{\mathbf{a}}(P)|_v - \text{ord}_c u_{\mathbf{a}} \log |t_c(P)|_v| \leq \rho_v.$$

For $v \in M_K^\infty$ and $P \in X(N)(K_v)$ we have

$$|\log |u_{\mathbf{a}}(P)|_v| \leq N \log(|j(P)|_v + 2400) + \rho_v.$$

Proof The first statement for $c = c_\infty$ is an immediate consequence of Propositions 2.3 and 2.5 (notice that $\log |q_c|_v = N \log |t_c|_v$ for every cusp c). The general case reduces to the case $c = c_\infty$ by applying a suitable Galois automorphism. The second statement follows from Corollary 2.4. \square

4.3 K -Rational Modular Units on X_G

Now let K be a number field, and let G be a subgroup of $\text{GL}_2(\mathbb{Z}/N\mathbb{Z})$ containing -1 . Let $\det G$ be the image of G under the determinant map $\det: \text{GL}_2(\mathbb{Z}/N\mathbb{Z}) \rightarrow (\mathbb{Z}/N\mathbb{Z})^\times = \text{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$ (recall that we have a well-defined isomorphism (19)). In the sequel we shall assume that $K \supseteq \mathbb{Q}(\zeta_N)^{\det G}$, where $\mathbb{Q}(\zeta_N)^{\det G}$ is the subfield of $\mathbb{Q}(\zeta_N)$ stable under $\det G$. This assumption implies that the curve X_G is defined over K . Then $G' := \text{Gal}(K(X(N))/K(X_G))$ is a subgroup of $G/\pm 1$, which contains the geometric part $(G \cap \text{SL}_2(\mathbb{Z}/N\mathbb{Z}))/\pm 1$. For every $\mathbf{a} \in \mathbf{A}$ we put $w_{\mathbf{a}} = \prod_{\sigma \in G'} u_{\mathbf{a}\sigma}$. Since $u_{\mathbf{a}\sigma} = u_{\mathbf{a}}^\sigma$, the functions $w_{\mathbf{a}}$ are contained in $K(X_G)$. They have the following properties.

Proposition 4.5 (i) *The functions $w_{\mathbf{a}}$ have zeros and poles only at the cusps of X_G . If c is such a cusp, then $|\text{ord}_c w_{\mathbf{a}}| \leq |G'|N^2$.*

(ii) *For every $\mathbf{a} \in \mathbf{A}$ there exists an algebraic integer $\lambda_{\mathbf{a}} \in \mathbb{Z}[\zeta_N]$, which is a product of $|G'|$ factors of the form $(1 - \zeta_{N'})^{12N}$, where $N' \mid N$, such that the functions $w_{\mathbf{a}}$ and $\lambda_{\mathbf{a}} w_{\mathbf{a}}^{-1}$ are integral over $\mathbb{Z}[j]$.*

(iii) *Let c be a cusp of X_G . Then for $v \in M_K$ and $P \in \Omega_{c,v}$ we have*

$$|\log |w_{\mathbf{a}}(P)|_v - \text{ord}_c w_{\mathbf{a}} \log |t_c(P)|_v| \leq |G'| \rho_v.$$

(iv) *For $v \in M_K^\infty$ and $P \in X_G(K_v)$ we have*

$$|\log |w_{\mathbf{a}}(P)|_v| \leq |G'|N \log(|j(P)|_v + 2400) + |G'| \rho_v.$$

(v) *The group generated by the principal divisors $(w_{\mathbf{a}})$ is of rank $|\mathcal{C}(G, K)| - 1$.*

Proof Items (i) and (ii) follow from Proposition 4.2, items (iv) and (iii) follow from Proposition 4.4. Finally, item (v) follows from Proposition 4.3. \square

4.4 A Unit Vanishing at the Given Cusps

Item (v) of Proposition 4.5 implies that for any proper subset Σ of $\mathcal{C}(G, K)$ there is a K -rational unit on X_G vanishing at this subset; moreover, such a unit can be expressed as a multiplicative combination of the units $w_{\mathbf{a}}$. We call it a *Runge unit* for Σ . In this subsection we give a quantitative version of this fact. We shall use the following simple lemma, where we denote by $\|\cdot\|_1$ the ℓ_1 -norm.

Lemma 4.6 *Let M be an $s \times t$ matrix of rank s with entries in \mathbb{Z} . Assume that the entries of M do not exceed A in absolute value. Then there exists a vector $\mathbf{b} \in \mathbb{Z}^t$ such that $\|\mathbf{b}\|_1 \leq s^{s/2+1} A^{s-1}$, and such that all the s coordinates of the vector $M\mathbf{b}$ (in the standard basis) are strictly positive.*

Proof Assume first that $s = t$. Let d be the determinant of M . Then the column vector $(|d|, \dots, |d|)$ can be written as $M\mathbf{b}$, where $\mathbf{b} = (b_1, \dots, b_s)$ with b_k being (up to the sign) the determinant of the matrix obtained from M upon replacing the k -th column by $(1, \dots, 1)$. Using Hadamard's inequality, we bound $|b_k|$ by $s^{s/2} A^{s-1}$. This proves the lemma in the case $s = t$. The general case reduces to the case $s = t$ by selecting a non-singular $s \times s$ sub-matrix, which gives s entries of the vector \mathbf{b} ; the remaining $t - s$ entries are set to be 0. \square

Now let G, K and G' be as in Subsection 4.3.

Proposition 4.7 *Let Σ be a proper subset of $\mathcal{C}(G, K)$ and s a positive integer satisfying $s \geq |\Sigma|$. Then one can associate to every $\mathbf{a} \in \mathbf{A}$ an integer $b_{\mathbf{a}}$ such that*

$$B := \sum_{\mathbf{a} \in \mathbf{A}} |b_{\mathbf{a}}| \leq s^{s/2+1} (|G'|N^2)^{s-1} \quad (21)$$

and the unit $w := \prod_{\mathbf{a} \in \mathbf{A}} w_{\mathbf{a}}^{b_{\mathbf{a}}}$ has the following properties.

- (i) *If c is a cusp such that the orbit of c is in Σ then $\text{ord}_c w > 0$ (that is, w is a Runge unit for Σ).*
- (ii) *There exists an algebraic integer λ , which is a product of at most $|G'|B$ factors of the form $(1 - \zeta_{N'})^{12N}$, where $N' \mid N$, such that λw is integral over $\mathbb{Z}[j]$.*
- (iii) *Let c be a cusp of X_G . Then for $v \in M_K$ and $P \in \Omega_{c,v}$ we have*

$$|\log |w(P)|_v - \text{ord}_c w \log |t_c(P)|_v| \leq B|G'| \rho_v.$$

- (iv) *For $v \in M_K^\infty$ and $P \in X_G(K_v)$ we have*

$$|\log |w(P)|_v| \leq B|G'|N \log(|j(P)|_v + 2400) + B|G'| \rho_v.$$

Proof The K -rational Galois orbit of a cusp c has $[K(c) : K]$ elements. Fix a representative in every such orbit and consider the $|\mathcal{C}(G, K)| \times |\mathbf{A}|$ matrix $(\text{ord}_c w_{\mathbf{a}})$, where c runs over the set of selected representatives. According to item (v) of Proposition 4.5, this matrix is of rank $|\mathcal{C}(G, K)| - 1$, and the only (up to proportionality) linear relation between the rows is $\sum_c [K(c) : K] \text{ord}_c w_{\mathbf{a}} = 0$ for every $\mathbf{a} \in \mathbf{A}$. It follows that any proper subset of the rows of our matrix is linearly independent. In particular, if we select the rows corresponding to the set Σ , we get a sub-matrix of rank $|\Sigma|$. Applying to it Lemma 4.6, where we may take $A = |G'|N^2$ due to item (i) of Proposition 4.5, we find integers $b_{\mathbf{a}}$ such that (21) holds and the function $w = \prod_{\mathbf{a} \in \mathbf{A}} w_{\mathbf{a}}^{b_{\mathbf{a}}}$ is as desired, by Proposition 4.5, (ii)–(iv). \square

5 Proof of Theorem 1.2

We use the notations of Section 3. We also write $d_v = [K_v : \mathbb{Q}_v]$ and $d = [K : \mathbb{Q}]$. We fix an extension of every $v \in M_K$ to \bar{K} and denote this extension by v as well. We shall use the following obvious estimates for the quantities ρ_v defined in (20):

$$\sum_{v \in M_K^\infty} d_v \rho_v = 12dN \log N, \quad \sum_{v \in M_K^0} d_v \rho_v = 12dN \sum_{p|N} \frac{\log p}{p-1} \leq 12dN \log N. \quad (22)$$

5.1 A Runge Unit

Fix $P \in Y_G(\mathcal{O}_S)$. Let S_1 consist of the places $v \in M_K$ such that $P \in X_G(K_v)^+$. Plainly, $S_1 \subset S$. For $v \in S_1$ let c_v be a v -nearby cusp to P (if there are several, choose any of them) and let Σ be the set of all $\text{Gal}(\bar{K}/K)$ -orbits of cusps containing some of the c_v . Then $|\Sigma| \leq |S_1| \leq |S|$, and since $|S| < |\mathcal{C}(G, K)|$ by assumption, Σ is a proper subset of $\mathcal{C}(G, K)$. It follows from Proposition 4.7 that there exists a K -rational modular unit $w = \prod_{\mathbf{a} \in \mathbf{A}} w_{\mathbf{a}}^{b_{\mathbf{a}}}$ such that $\text{ord}_{c_v} w > 0$ for every $v \in S_1$ (a *Runge unit*) for which $B := \sum_{\mathbf{a} \in \mathbf{A}} |b_{\mathbf{a}}|$ satisfies (21), where we may put $s = |S|$.

Since w is a modular unit and P is not a cusp, we have $w(P) \neq 0, \infty$, and the product formula gives $\sum_{v \in M_K} d_v \log |w(P)|_v = 0$. We want to show that this is impossible when $h(P)$ is too large.

5.2 Partitioning the Places of K

We partition the set of places M_K into three pairwise disjoint subsets: $M_K = S_1 \cup S_2 \cup S_3$, where $S_i \cap S_j = \emptyset$ for $i \neq j$. The set S_1 is already defined. The set S_2 consists of the *archimedean* places not belonging to S_1 . Obviously, $S_1 \cup S_2 \subset S$. Finally, the set S_3 consists of the places not belonging to $S_1 \cup S_2$; in other words, $v \in S_3$ if and only if v is non-archimedean and $|j(P)|_v \leq 1$.

We will estimate from above the quantities

$$\Xi_i = d^{-1} \sum_{v \in S_i} d_v \log |w(P)|_v \quad (i = 1, 2, 3).$$

We will show that $\Xi_1 \leq -N^{-1} dh(P) + O(1)$, where the $O(1)$ -term is independent of P (it will be made explicit). Further, we will bound Ξ_2 and Ξ_3 independently of P . Since

$$\Xi_1 + \Xi_2 + \Xi_3 = 0, \quad (23)$$

this would bound $h(P)$.

5.3 Estimating Ξ_1

For $v \in S_1$ we have $P \in \Omega_{c_v, v}$, so we may apply item (iii) of Proposition 4.7. Since $\text{ord}_{c_v} w \geq 1$ and $\log |q_{c_v}(P)|_v = e_{c_v} \log |t_{c_v}(P)|_v$ with $e_{c_v} | N$, we have

$$\Xi_1 \leq d^{-1} \sum_{v \in S_1} d_v \frac{\text{ord}_{c_v} w}{e_{c_v}} \log |q_{c_v}(P)|_v + B|G'|d^{-1} \sum_{v \in M_K} d_v \rho_v \quad (24)$$

$$\leq N^{-1} d^{-1} \sum_{v \in S_1} d_v \log |q_{c_v}(P)|_v + 24B|G'|N \log N \quad (25)$$

$$\leq -N^{-1} d^{-1} \sum_{v \in S_1} d_v \log |j(P)|_v + N^{-1} \log 2 + 24B|G'|N \log N, \quad (26)$$

where we use (22) and (17). Further, for $v \in M_K \setminus S_1$ we have $|j(P)|_v \leq 3500$ if v is archimedean, and $|j(P)|_v \leq 1$ if v is non-archimedean. It follows that

$$d^{-1} \sum_{v \in M_K \setminus S_1} d_v \log^+ |j(P)|_v \leq \log 3500.$$

Since $|j(P)|_v > 1$ for $v \in S_1$, one may also replace $\log |j(P)|_v$ by $\log^+ |j(P)|_v$ in (26). Hence

$$\begin{aligned}\Xi_1 &\leq -N^{-1}d^{-1} \sum_{v \in M_K} d_v \log^+ |j(P)|_v + N^{-1} \log 7000 + 24B|G'|N \log N \\ &= -N^{-1}h(P) + N^{-1} \log 7000 + 24B|G'|N \log N.\end{aligned}\tag{27}$$

5.4 Estimating Ξ_2 , Ξ_3 and Completing the Proof

Item (iv) of Proposition 4.7 together with (22) implies that

$$\Xi_2 \leq B|G'|N \log 5900 + 12B|G'|N \log N\tag{28}$$

Further, let λ be from item (ii) of Proposition 4.7. Then $h(\lambda) \leq 12B|G'|N \log 2$, because $h(1 - \zeta) \leq \log 2$ for a root of unity ζ . For $v \in S_3$ the number $j(P)$ is a v -adic integer. Hence so is the number $\lambda w(P)$. It follows that $|w(P)|_v \leq |\lambda^{-1}|_v$ for $v \in S_3$, and

$$\Xi_3 \leq d^{-1} \sum_{v \in S_3} d_v \log |\lambda^{-1}|_v \leq h(\lambda^{-1}) = h(\lambda) \leq 12B|G'|N \log 2.$$

Combining this with (23), (27) and (28), we obtain $h(P) \leq 36B|G'|N^2 \log 2N$, which, together with (21) implies (2) with $|G|/2$ replaced by $|G'|$.

If $S = M_K^\infty$ then in the second sum in (24) one can replace $v \in M_K$ by $v \in M_K^\infty$. Hence in (25), (26) and (27) one may replace 24 by 12, which allows $h(P) \leq 24B|G'|N^2 \log 3N$. This completes the proof of the theorem. \square

6 Special Case: the Split Cartan Group

When G is a particular group, one can usually obtain a stronger result than in general. For instance in [2] we examined the case when $N = p$ is a prime number, G is the normalizer of a split Cartan subgroup of $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$ and $K = \mathbb{Q}$. In this case we obtained the estimate $h(P) \leq C\sqrt{p}$ for $P \in Y_G(\mathbb{Z})$, with an absolute constant C . In this section we consider the case when $N = p$ is a prime number, G is the Cartan subgroup itself, and K is a quadratic field. (Note that the fact we are not working over \mathbb{Q} any longer makes a significant difference with the normalizer-of-Cartan case studied in [2] (appearance of the Ξ_2 -term), and requires much of the generality of the first part of the present work.) Without loss of generality we may assume that G is the diagonal subgroup of $\mathrm{GL}_2(\mathbb{F}_p)$. The modular curve X_G , corresponding to this subgroup, is denoted by $X_{\mathrm{sp.C}}(p)$. It parametrizes geometric isomorphism classes $(E, (A, B))$ of elliptic curves endowed with an ordered pair of p -isogenies. There is an isomorphism $\phi: X_0(p^2) \rightarrow X_{\mathrm{sp.C}}(p)$ over \mathbb{Q} defined functorially as

$$(E, A_{p^2}) \mapsto (E/pA_{p^2}, (A_{p^2}/pA_{p^2}, E[p]/pA_{p^2})).$$

On the Poincaré upper half-plane \mathcal{H} , the map ϕ is induced by the map $\tau \mapsto p\tau$.

The homographic action of the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on \mathcal{H} defines an involution of $X_{\mathrm{sp.C}}$, which modularly is $(E, (A, B)) \mapsto (E, (B, A))$.

Theorem 6.1 *Let $p \geq 3$ be a prime number and K a number field of degree at most 2. Then for any $P \in Y_{\mathrm{sp.C}}(p)(\mathcal{O}_K)$ we have $h(P) \leq 24p \log 3p$.*

Remark 6.2 With some little additional effort, one can obtain a stronger estimate $h(P) \leq Cp$ (and probably even $h(P) \leq C\sqrt{p}$) with an absolute constant C . However, the bound of Theorem 6.1 is easier to obtain and sufficient for our purposes.

The curve $X_{\mathrm{sp.C}}(p)$ has $p + 1$ cusps $c_\infty, c_0, c_1, \dots, c_{p-1}$ corresponding, respectively, to the points $i\infty, 0, 1/p, \dots, (p-1)/p$ of $\bar{\mathcal{H}}$. The morphism $X(p) \rightarrow X_{\mathrm{sp.C}}(p)$ is unramified at the cusps,

and the morphism $X_{\text{sp.C}} \rightarrow X(1)$ is ramified with index p at all the cusps (as can be immediately seen by computing ramification indices), so the local parameter at every cusp c of $X_{\text{sp.C}}$ is $t_c = q_c^{1/p}$.

The cusps c_∞ and c_0 are defined over \mathbb{Q} . The cusps c_1, \dots, c_{p-1} are defined over the cyclotomic field $\mathbb{Q}(\zeta_p)$ and are conjugate over \mathbb{Q} . Thus the set $\mathcal{C}(G, \mathbb{Q})$ consists of 3 elements, and the group generated by the principal divisors $(w_{\mathbf{a}})$ is of rank 2 by Proposition 4.5, item (v). Moreover, it is clearly generated by the divisors $(w_{\mathbf{a}})$ with $\mathbf{a} \in \{(1/p, 0), (0, 1/p)\}$. More precisely, we have the following.

Proposition 6.3 *The principal divisors $(w_{\mathbf{a}})$ with $\mathbf{a} \in \{(1/p, 0), (0, 1/p)\}$ are*

$$(w_{(1/p, 0)}) = -\frac{1}{2}p(p-1)^2(c_\infty - pc_0 + c_1 + \dots + c_{p-1}), \quad (29)$$

$$(w_{(0, 1/p)}) = -\frac{1}{2}p(p-1)^2(-pc_\infty + c_0 + c_1 + \dots + c_{p-1}). \quad (30)$$

Proof Denote by $p^{-1}\mathbb{F}_p^\times$ the set of non-zero elements of $p^{-1}\mathbb{Z}/\mathbb{Z}$. The G' -orbits of $(1/p, 0)$ and $(0, 1/p)$ are $\{(a, 0) : a \in p^{-1}\mathbb{F}_p^\times / \pm 1\}$ and $\{(0, a) : a \in p^{-1}\mathbb{F}_p^\times / \pm 1\}$ respectively, each element of each orbit occurring exactly $p-1$ times. Since the morphism $X(p) \rightarrow X_{\text{sp.C}}$ is unramified at the cusps, we have

$$\text{ord}_{c_\infty} w_{(1/p, 0)} = (p-1) \sum_{a \in p^{-1}\mathbb{F}_p^\times / \pm 1} \text{ord}_{c_\infty} u_{(a, 0)} = 12p^2(p-1) \sum_{a \in p^{-1}\mathbb{F}_p^\times / \pm 1} \ell_{(a, 0)}$$

by item (iii) of Proposition 4.2. It follows that

$$\text{ord}_{c_\infty} w_{(1/p, 0)} = 3p^2(p-1) \sum_{k=1}^{p-1} B_2\left(\frac{k}{p}\right) = -\frac{1}{2}p(p-1)^2,$$

where we use the identity

$$\sum_{k=1}^N B_2\left(\frac{k}{N}\right) = -\frac{N-1}{6N}$$

for any positive integer N . In a similar fashion,

$$\text{ord}_{c_\infty} w_{(0, 1/p)} = 12p^2(p-1) \sum_{a \in p^{-1}\mathbb{F}_p^\times / \pm 1} \ell_{(0, a)} = 3p^2(p-1)^2 B_2(0) = \frac{1}{2}p^2(p-1)^2.$$

Further, the involution induced by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ defined before Theorem 6.1 exchanges the cusps c_0 and c_∞ and the units $w_{(1/p, 0)}$ and $w_{(0, 1/p)}$. It follows that

$$\text{ord}_{c_0} w_{(1/p, 0)} = \text{ord}_{c_\infty} w_{(0, 1/p)} = p^2(p-1)^2/2, \quad \text{ord}_{c_0} w_{(0, 1/p)} = \text{ord}_{c_\infty} w_{(1/p, 0)} = -p(p-1)^2/2.$$

Finally, the Galois conjugation over \mathbb{Q} shows that

$$\text{ord}_{c_1} w_{\mathbf{a}} = \dots = \text{ord}_{c_{p-1}} w_{\mathbf{a}},$$

which implies that $\text{ord}_{c_k} w_{\mathbf{a}} = -p(p-1)^2/2$ for $k \in \{1, \dots, p-1\}$. This proves the proposition. \square

Proof of Theorem 6.1 We have $S = M_K^\infty$, the set of the archimedean places of the quadratic field K . In particular, $s = |S|$ is 1 or 2. Fix $P \in Y_{\text{sp.C}}(\mathcal{O}_K)$. We use the notation S_i and Ξ_i of Subsection 5.2; in the present context this means that $S_2 = S \setminus S_1$ and $S_3 = M_K^0$. We again

pick for every $v \in S_1$ a v -nearby² cusp c_v and set $\Sigma = \{c_v : v \in S_1\}$. The set Σ has at most two elements, and we have one of the following three possibilities:

$$\Sigma \subset \{c_\infty, c_1, \dots, c_{p-1}\}, \quad (31)$$

$$\Sigma \subset \{c_0, c_1, \dots, c_{p-1}\}, \quad (32)$$

$$\Sigma \subset \{c_\infty, c_0\}. \quad (33)$$

We define $w = w_{(1/p, 0)}^{-1}$ in the case (31), $w = w_{(0, 1/p)}^{-1}$ in the case (32) and $w = w_{(1/p, 0)} w_{(0, 1/p)}$ in the case (33). With the notation of Subsection 4.3 and Proposition 4.7, we have

$$N = p, \quad |G'| \leq |G|/2 = (p-1)^2/2, \quad B = \begin{cases} 1 & \text{in the cases (31) and (32),} \\ 2 & \text{in the case (33).} \end{cases}$$

Now we argue as Subsection 5.3, with one very significant distinction: instead of the estimate $(\text{ord}_{c_v} w)/e_{c_v} \geq N^{-1}$ we use the *identity*

$$\frac{\text{ord}_{c_v} w}{e_{c_v}} = \begin{cases} (p-1)^2/2 & \text{in the cases (31) and (32),} \\ (p-1)^3/2 & \text{in the case (33).} \end{cases}$$

In the cases (31) and (32) we obtain

$$\begin{aligned} \Xi_1 &\leq d^{-1} \sum_{v \in S_1} d_v \frac{\text{ord}_{c_v} w}{e_{c_v}} \log |q_{c_v}(P)|_v + \frac{1}{2} (p-1)^2 d^{-1} \sum_{v \in M_K^\infty} d_v \rho_v \\ &= \frac{1}{2} (p-1)^2 d^{-1} \sum_{v \in S_1} d_v \log |q_{c_v}(P)|_v + 6(p-1)^2 p \log p \\ &\leq -\frac{1}{2} (p-1)^2 d^{-1} \sum_{v \in S_1} d_v \log |j(P)|_v + \frac{1}{2} (p-1)^2 \log 2 + 6(p-1)^2 p \log p \\ &\leq -\frac{1}{2} (p-1)^2 d^{-1} \sum_{v \in M_K} d_v \log^+ |j(P)|_v + \frac{1}{2} (p-1)^2 \log 7000 + 6(p-1)^2 p \log p \\ &= -\frac{1}{2} (p-1)^2 h(P) + \frac{1}{2} (p-1)^2 \log 7000 + 6(p-1)^2 p \log p. \end{aligned} \quad (34)$$

In the case (33) a similar calculation gives

$$\Xi_1 \leq -\frac{1}{2} (p-1)^3 h(P) + \frac{1}{2} (p-1)^3 \log 7000 + 12(p-1)^2 p \log p. \quad (35)$$

We estimate Ξ_2 exactly as in Subsection 5.4:

$$\Xi_2 \leq \frac{B}{2} (p-1)^2 p \log 5900 + 6B(p-1)^2 p \log p. \quad (36)$$

Further, in the cases (31) and (32) we may take $\lambda = (1 - \zeta_p)^{6p(p-1)^2}$, which is equal to $p^{6p(p-1)^2}$ times a unit. And in the case (33) we may take $\lambda = 1$. We obtain

$$\Xi_3 \begin{cases} \leq 6(p-1)p \log p & \text{in the cases (31) and (32),} \\ = 0 & \text{in the case (33).} \end{cases}$$

Combining all the previous estimates, we obtain $h(P) \leq 24p \log 3p$ in the cases (31) and (32), and in the case (33) we obtain a much sharper estimate $h(P) \leq 72 \log 3p$. This proves the theorem. \square

²Since v is archimedean, we can write here “the v -nearby cusp”.

7 An Application: \mathbb{Q} -curves of Prime Power Degree

In this section we apply Theorem 6.1 to the study of \mathbb{Q} -curves, proving Theorem 1.3. Let us recall some definitions. For a positive integer N with prime decomposition $N = p_1^{a_1} \cdots p_k^{a_k}$, we set $X_0^*(N) := X_0(N)/\langle w_{p_i} \rangle$, where $\langle w_{p_i} \rangle$ is the group of automorphisms of $X_0(N)$ spanned by the Atkin-Lehner involutions w_{p_i} , and $X_0^+(N) := X_0(N)/w_N$. If K is a number field, it is a theorem of Elkies [5] that any K -curve of degree N gives rise to a point in $X_0^*(N)(K)$. The curve $X_0^+(N)$ ($= X_0^*(N)$ if N is a prime power) parameterizes quadratic K -curves of degree N (that is, elliptic curves defined over a quadratic extension of K and admitting a cyclic isogeny of degree N to the K -conjugate curve).

Theorem 1.3 is a consequence of the following three statements.

Theorem 7.1 *For a prime $p \geq 37$ and $r > 1$, the rational noncuspidal points of $X_0^+(p^r)$ are integral; that is, the j -invariant $j(P)$ of any lift $P \in X_0(p^r)(\mathbb{C})$ of any non-cuspidal point of $X_0^+(p^r)(\mathbb{Q})$ belongs to $\overline{\mathbb{Z}}$.*

Theorem 7.2 *For every d there is a positive number $\kappa(d)$ such that the following holds. Let E be a non-CM elliptic curve defined over a number field K of degree d , and admitting a cyclic isogeny over K of degree δ . Then $\delta \leq \kappa(d)(1 + h(j_E))^2$.*

Theorem 7.3 *Let $p \geq 3$ be a prime number, K be a quadratic number field, $r > 1$ an integer, and P a point of $Y_0(p^r)(\mathcal{O}_K)$. Then $h(P) \leq 110p \log p$.*

Proof of Theorem 1.3 (assuming Theorems 7.1, 7.2 and 7.3) Existence of the degeneracy morphisms $X_0^+(p^{r+2}) \rightarrow X_0^+(p^r)$ over \mathbb{Q} (see, for instance, [18]) shows it is enough to prove the result for $r = 2$ and $r = 3$. The case $r = 2$, where $X_0^+(p^2)$ is isomorphic to $X_{\text{split}}(p)$, was settled in [2], so we are left with the case $r = 3$.

Thus, let $P \in X_0^+(p^3)(\mathbb{Q})$ be a non-cuspidal and non-CM point, and $Q \in X_0(p^3)(K)$ a lift of it, with values in a quadratic number field K . From Theorem 7.1 we know that Q is an integral point if p is large enough. Theorem 7.3 implies that $h(Q) \leq 110p \log p$.

Call E the elliptic curve associated to Q . It is a non-CM elliptic curve admitting a cyclic isogeny of degree p^3 over K . Theorem 7.2 implies that $p^3 \leq C(1 + h(Q))^2$ with an absolute constant C . Therefore $p^3 \leq C'(p \log p)^2$ with another absolute constant C' and p is bounded. \square

Theorem 7.1 will be proved in Section 8. It is a generalization of the one used in [2] when $r = 2$, which in that case was originally due to Mazur, Momose and Merel. Theorem 7.2 is a straightforward consequence of the isogeny bounds due to Masser and Wüstholz [13] and Pellarin [20]. See [2, Corollary 5.4] for the details.

Theorem 7.3 is deduced below from Theorem 6.1 and the following lemma.

Lemma 7.4 *Let E, E' be elliptic curves defined over some number field and linked by an isogeny of degree δ . Then*

$$|h(j_E) - h(j_{E'})| \leq 13 \log(1 + h(j_{E'})) + 7 \log \delta + 100.$$

Proof Denote by $h_{\mathcal{F}}(E)$ the Faltings semistable height of the elliptic curve E . Recall that $h_{\mathcal{F}}(E)$ is defined as $[K : \mathbb{Q}]^{-1} \deg \omega$, where K is a number field such that E has semi-stable reduction at every place of K , and ω is a Néron differential on $E|_K$; it is independent of the choice of K and ω . A result of Faltings [6, Lemma 5] implies that

$$|h_{\mathcal{F}}(E) - h_{\mathcal{F}}(E')| \leq \frac{1}{2} \log \delta.$$

Further, for any elliptic curve E over a number field we have

$$|h(j_E) - 12h_{\mathcal{F}}(E)| \leq 6 \log(1 + h(j_E)) + C,$$

with an absolute constant C , see [23, Proposition 2.1]. Pellarin shows that one take $C = 47.15$, see [20], equation (51) on page 240. Combining all this, we find

$$|\mathrm{h}(j_E) - \mathrm{h}(j_{E'})| \leq 6 \log(1 + \mathrm{h}(j_E)) + 6 \log(1 + \mathrm{h}(j_{E'})) + 6 \log \delta + 95,$$

which implies the result after a routine calculation. \square

Proof of Theorem 7.3 We may assume $r = 2$. Let $\phi : X_0(p^2) \rightarrow X_{\mathrm{sp.C}}(p)$ be the isomorphism defined in the beginning of Section 6. Then the elliptic curve implied by a point P on $X_0(p^2)$ is p -isogenous to the curve implied by the point $P' = \phi(P)$ on $X_{\mathrm{sp.C}}(p)$. Lemma 7.4 implies that

$$|\mathrm{h}(P) - \mathrm{h}(P')| \leq 13 \log(1 + \mathrm{h}(P')) + 7 \log p + 100. \quad (37)$$

Since $P \in Y_0(p^2)(\mathcal{O}_K)$ and good reduction is preserved under isogeny, P' belongs to $Y_{\mathrm{sp.C}}(p)(\mathcal{O}_K)$ as well. Applying Theorem 6.1 to P' , we find $\mathrm{h}(P') \leq 24p \log 3p$, which, combined with (37), implies the result. \square

8 Integrality of $Y_0^+(p^3)(\mathbb{Q})$

We show that rational points on $X_0^+(p^3)$ are, in fact, integral.

Theorem 8.1 *For a prime $p \geq 37$, and $P \in X_0^+(p^3)(\mathbb{Q})$ a non-cuspidal point, the j -invariant of any lift of P to $X_0(p^3)(\mathbb{C})$ belongs to \mathbb{Z} .*

The proof of this theorem is an adaptation of the one we proposed in [2], relying on results and observations of Mazur, Momose and Merel. Actually, Theorem 8.1 was already proven, except integrality at 2 when $p \not\equiv 1 \pmod{8}$, by Momose in [18, Theorem 3.8]; in the present paper we however do need the whole statement of Theorem 8.1. The theorem is probably true for $p \geq 11$, $p \neq 13$ (some cases are indeed given by Momose in loc. cit.); but our assumption that $p \geq 37$ simplifies our arguments.

If M, N are natural integers and M is a divisor of N , we write $\pi_{N,M} : X_0(N) \rightarrow X_0(M)$ for the degeneracy morphism which is defined functorially as $(E, A_N) \mapsto (E, A_M)$, where $A_M := E[M] \cap A_N$. If M and N/M are relatively prime, let w_M for the corresponding Atkin-Lehner involution; recall that $X_0^+(N) := X_0(N)/w_N$. As usual, we write $J_0(N)$ for the jacobian over \mathbb{Q} of $X_0(N)$, and $J_0^-(N) := J_0(N)/(1 + w_N)J_0(N)$. Models over rings of integers for abelian varieties will be Néron models. Recall that, in this paper, the model for $X_0(N)$ over \mathbb{Z} that we consider is the modular one. Models for those modular curves over arbitrary schemes will be deduced by base change. We denote by $X_0(N)_{\mathbb{Z}}^{\mathrm{sm}}$ the smooth locus of $X_0(N)_{\mathbb{Z}}$ (obtained, when $N = p$ is prime, by removing the (super)singular points in the fiber at p).

As already mentioned in Section 6, the curve $X_{\mathrm{sp.C}}(N)$ parametrizes elliptic curves endowed with an ordered pair of independent N -isogenies. With “ordered” replaced by “unordered”, the same is true (at least when N is a prime power) for the curve $X_{\mathrm{split}}(N)$. For each prime p dividing N there is an involution on $X_{\mathrm{sp.C}}(N)$, here also denoted by w_p , defined functorially by

$$\left(E, \left(A = \prod_q A_q, B = \prod_q B_q \right) \right) \mapsto \left(E, \left(\prod_{q \neq p} A_q \times B_p, \prod_{q \neq p} B_q \times A_p \right) \right),$$

so that $X_{\mathrm{split}}(N) = X_{\mathrm{sp.C}}(N)/\langle w_p : p|N \rangle$. The map $z \mapsto Nz$ on the upper half-plane defines the \mathbb{Q} -isomorphism $\phi : X_0(N^2) \simeq X_{\mathrm{sp.C}}(N)$ of Subsection 6, inducing an isomorphism

$$X_{\mathrm{split}}(N) \simeq X_0(N^2)/\langle w_p : p|N \rangle$$

on the quotients.

We recall certain properties of the modular Jacobian $J_0(p)$ and its *Eisenstein quotient* $\tilde{J}(p)$ (see [14]).

Proposition 8.2 *Let p be a prime number. Then we have the following.*

- (i) [14, Theorem 1] *The group $J_0(p)(\mathbb{Q})_{\text{tors}}$ is cyclic and generated by $\text{cl}(0-\infty)$, where 0 and ∞ are the cusps of $X_0(p)$. Its order is equal to the numerator of the quotient $(p-1)/12$.*
- (ii) [14, Theorem 4] *The group $\tilde{J}(p)(\mathbb{Q})$ is finite. Moreover, the natural projection $J_0(p) \rightarrow \tilde{J}(p)$ defines an isomorphism $J_0(p)(\mathbb{Q})_{\text{tors}} \rightarrow \tilde{J}(p)(\mathbb{Q})$.*

Remark 8.3 As Mazur notices, Raynaud's theorem on group schemes of type (p, \dots, p) insures that $J_0(p)(\mathbb{Q})_{\text{tors}}$ defines a \mathbb{Z} -group scheme which, being constant in the generic fiber, is constant outside 2, and which at 2 has étale quotient of rank at least half that of $J_0(p)(\mathbb{Q})_{\text{tors}}$.

For a point $Q \in X_0^+(p^3)(\mathbb{Q})$ and $z \in X_0(p^3)(K)$ a lifting of Q with K a quadratic number field, the point z corresponds to a couple (E, C_{p^3}) over K , by [4], Proposition VI.3.2. Set $\pi := \pi_{p^3, p}$, $x := w_p \pi(z)$ and $x_0 := \pi w_{p^3}(z) \in X_0(p)(K)$. Writing $D_p := p^2 C_{p^3}$, the modular interpretation of x and x_0 is therefore $(E/D_p, E[p] \bmod D_p)$ and $(E/C_{p^3}, E[p] + C_{p^3} \bmod C_{p^3})$ respectively. For t an element in the \mathbb{Z} -Hecke algebra for $\Gamma_0(p)$, define the morphism g_t from $X_0(p^3)_{/\mathbb{Z}}^{\text{sm}}$ to $J_0(p)_{/\mathbb{Z}}$, which extends by the universal property of Néron models the morphism on generic fibers:

$$g_t: \begin{cases} X_0(p^3) & \rightarrow J_0(p) \\ Q & \mapsto t \cdot \text{cl}(w_p \pi(Q) - \pi w_{p^3}(Q)). \end{cases}$$

Let $J_0(p) \xrightarrow{\Pi} \tilde{J}(p)$ be the projection to the Eisenstein quotient, and $\tilde{g}_t := \Pi \circ g_t$.

Lemma 8.4 *The morphism \tilde{g}_t factorizes through a \mathbb{Q} -morphism \tilde{g}_t^+ from $X_0^+(p^3)$ to $\tilde{J}(p)$. If $t \cdot (1 + w_p) = 0$, the same is true for g_t and we similarly denote by $g_t^+ : X_0^+(p^3) \rightarrow J_0(p)$ the factor morphism.*

Proof We compute that, in $J_0(p)$:

$$\text{cl}((w_p \pi(z)) - (\pi w_{p^3}(z)) - (w_p \pi w_{p^3}(z)) + (\pi(z))) = (1 + w_p) \text{cl}((\pi(z)) - (\pi w_{p^3}(z))),$$

from which we derive the second assertion when $t(1 + w_p) = 0$. As for the first statement, we know that $(1 + w_p)$ acts trivially on $\tilde{J}(p)$ from [14, Proposition 17.10]. \square

By the universal property of Néron models, we may extend g_t^+ and \tilde{g}_t^+ to maps from $X_0^+(p^3)_{/\mathbb{Z}}^{\text{sm}}$ to $J_0(p)_{/\mathbb{Z}}$ and $\tilde{J}(p)_{/\mathbb{Z}}$, respectively. We still denote those extended morphisms in the same way and we put $\tilde{g}^+ = \tilde{g}_1^+$.

Proposition 8.5 *Let $P \in X_0^+(p^3)(\mathbb{Q})$ for some $p \geq 37$, let x, x_0 and K be defined as above, and let \mathcal{O}_K be the ring of integers of K . Then:*

- (i) *The isogeny class of elliptic curves associated to P is not potentially supersingular at p .*
- (ii) *The points x and x_0 coincide in the fibers of characteristic p of $X_0(p)_{/\mathcal{O}_K}$.*

Proof Point (i) is Lemma 2.2 (ii) together with Theorem 3.2 of [18]. Point (ii) is proved in Proposition 3.1 of [19]. \square

Proof of Theorem 8.1 Let P be a non-cuspidal point on $X_0^+(p^3)$ with values in \mathbb{Q} , and Q a lift of P to $X_0(p^3)(\mathbb{Q})$. If \mathcal{L} is a finite place of \mathcal{O}_K dividing the denominator of $j(Q)$, then Q specializes to a cusp at \mathcal{L} . Recall that $X_0(p^3)$ has two cusps defined over \mathbb{Q} , and two other Galois orbits of cusps, with fields of definition $\mathbb{Q}(\zeta_p)^+$ and $\mathbb{Q}(\zeta_{p^2})^+$ respectively. We first claim that Q specializes to one of the rational cusps (which, by changing our lift of P , may be assumed to be the ∞ -cusp, as w_p switches the rational cusps). Indeed, it follows from Propositions 8.5 (ii) that $\tilde{g}^+(P)(\mathbb{F}_p) = 0(\mathbb{F}_p)$, and by the remark after Proposition 8.2, $\tilde{g}^+(P)(\mathbb{Q}) = 0(\mathbb{Q})$ (recall $p > 2$).

The non-rational cusps of $X_0^+(p^3)(\mathbb{C})$ map to $\text{cl}(0 - \infty)$ in $J_0(p)(\mathbb{C})$ (indeed, w_{p^3} preserves each non-rational Galois orbit of cusps. For more details see for instance the proof of [17, Proposition 2.5]). Therefore, as $p \geq 37$, Proposition 8.2 implies that if Q specializes to a non-rational cusp at \mathcal{L} then $\tilde{g}^+(P)$ would not be 0 at the characteristics ℓ of \mathcal{L} , a contradiction.

Now let \mathcal{I} be the ideal of the Hecke algebra such that $\tilde{J}(p) = J_0(p)/\mathcal{I}J_0(p)$. Choose an ℓ -adically maximal element $t \neq 0$ in the Hecke algebra such that $t \cdot \mathcal{I} = 0$. Again, as $t(1 + w_p) = 0$, the morphism g_t factorizes through a morphism $g_t^+ : X_0^+(p^3)_{/\mathbb{Z}}^{\text{sm}} \rightarrow t \cdot J_0(p)_{/\mathbb{Z}}$. Moreover $g_t^+(P)$ belongs to $t \cdot J_0(p)(\mathbb{Q})$, hence is a torsion point, as $t \cdot J_0(p)$ is isogenous to a quotient of $\tilde{J}(p)$. We see as above by looking at the fiber at p that $g_t^+(P) = 0$ at p , hence generically, because of Proposition 8.2 (or more generally by Raynaud’s well-known result on group schemes of type (p, \dots, p) on a not-too-ramified discrete valuation ring). We then easily check by using the q -expansion principle, as in [16, Theorem 5], that g_t^+ is a formal immersion at the specialization $\infty(\mathbb{F}_\ell)$ of the rational cusp ∞ on $X_0^+(p^3)$. This allows us to apply the classical argument of Mazur (see e.g. [15, proof of Corollary 4.3]), yielding a contradiction with the fact that P is not generically cuspidal. Therefore P is not cuspidal at \mathcal{L} . \square

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