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Rational Approximations of the Number $\sqrt[3]{3}$

Yu. A. Pupyrev*

Moscow State University

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Abstract—We obtain estimates of the form $\|\xi\alpha^n\| > C^k$ for all $k \geq K$ for the values $\xi = 3^{1/3}$ and $\alpha = 2$, where K is an effective constant and $\|\cdot\|$ denotes the distance to the nearest integer.

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1. INTRODUCTION

Sequences of fractional parts of powers of fixed numbers $\alpha > 1$ and more general sequences $\{\xi\alpha^n\}$, $n = 1, 2, \dots$, where $\xi \neq 0$ and $\alpha > 1$ are real numbers have been studied for a long time now and are related to different problems of number theory. For example, from the inequality

$$\{(3/2)^k\} \leq 1 - (3/4)^k, \qquad k \in \mathbb{N},$$

one could obtain a sharp estimate of the maximal number of summands in the Waring problem.

Studying this problem [1], Mahler showed that the inequality

$$\left\|\theta\left(\frac{u}{v}\right)^k\right\| \leq C^k,$$

where u and v are integers, θ is a real number, and $\|\cdot\|$ denotes the distance to the nearest integer, $\|x\| = \min(\{x\}, 1 - \{x\})$, has only a finite number of solutions in integers k for any $C < 1$. In particular, in the case $\theta = 1$, $u/v = 3/2$, $C = 3/4$, the inequality given above was obtained for all $k \geq K$, where K is an absolute, but not an *effective* constant. Similar estimates of the form

$$\left\|\left(\frac{3}{2}\right)^k\right\| > C^k \qquad \text{for all} \quad k \geq K \tag{1}$$

with effective constant K were obtained by Baker and Coates [2], Beukers [3], Dubitskas [4], Habsieger [5] as well as Zudilin [6]. Zudilin's paper contains estimates for the distances $\|(4/3)^n\|$ and $\|(5/4)^n\|$. (See also Habsieger's paper [5] and that of the author [7] dealing with the effectivization of estimates for $\|(3/2)^n\|$, and $\|(4/3)^n\|$.)

In the present paper, the author obtains estimates of the form

$$\|\xi\alpha^k\| > C^k \qquad \text{for all} \quad k \geq K$$

for some particular values of ξ and α. Thus, the following theorem will be proved.

Theorem 1. *The following estimate holds*:

$$\|3^{1/3}2^k\| > 0.3568^k \qquad \textit{for} \quad k > K. \tag{2}$$

*E-mail: pupyrev@mi.ras.ru

Note that the best (at present) effective estimate of the measure of the number $3^{1/3}$ [8]:

$$\mu(3^{1/3}) = 2.692661368\ldots$$

would give the estimate

$$\|3^{1/3}2^k\| > 0.3093^k \qquad \text{for} \quad k > K.$$

2. PADÉ APPROXIMANTS

Using the equality

$$3^2 - 2^3 = 1$$

and choosing a positive integer parameter B, we write

$$\left(\frac{2}{3}\right)^{3B+1} = \left(\frac{27}{8}\right)^{-(B+1/3)} = \left(3\cdot\left(1-\frac{1}{9}\right)^{-1}\right)^{-(B+1/3)} = 3^{-1/3}\cdot 3^{-B}\cdot\left(1-\frac{1}{9}\right)^{B+1/3};$$

hence

$$\begin{aligned} 3^{1/3}2^{3B+1} &= 3^{2B+1}\left(1-\frac{1}{9}\right)^{B+1/3} = 3^{2B+1}\sum_{k=0}^{\infty}\binom{B+1/3}{k}\left(-\frac{1}{9}\right)^k \\ &= 3^{2B+1}\left(\sum_{k=0}^{A-1}\binom{B+1/3}{k}\left(-\frac{1}{9}\right)^k + \sum_{k=A}^{\infty}\binom{B+1/3}{k}\left(-\frac{1}{9}\right)^k\right) \\ &\equiv \left(-\frac{1}{9}\right)^A\sum_{\mu=0}^{\infty}\binom{B+1/3}{A+\mu}\left(-\frac{1}{9}\right)^{\mu} \pmod{\mathbb{Z}} \\ &= (-1)^A 3^{2B-2A+1}\binom{B+1/3}{A}\sum_{\mu=0}^{\infty}\frac{(-B+A-1/3)_\mu}{(A+1)_\mu}\left(\frac{1}{9}\right)^{\mu}. \end{aligned} \tag{3}$$

where $0 \le A \le 4B/7+1$ (for convenience, A is assumed even).

We construct Padé approximants [6] to the function

$$\begin{aligned} F_B(z) &= 3^{2B-2A+1}\sum_{\mu=0}^{\infty}\binom{B+1/3}{A+\mu}(-z)^{\mu} \\ &= 3^{2B-2A+1}\binom{B+1/3}{A}\sum_{\mu=0}^{\infty}\frac{(-B+A-1/3)_\mu}{(A+1)_\mu}\,z^{\mu}. \end{aligned}$$

Choose the polynomial

$$\begin{aligned} Q_n(x) &= \binom{n-B+A-1-1/3}{n}\,{}_2F_1\left(\begin{matrix}-n,\ n+A\\ -B+A-1/3\end{matrix}\,\middle|\,x\right) \\ &= \sum_{\mu=0}^{n}(-x)^{\mu}\binom{n+A+\mu-1}{\mu}\binom{n-B+A-1-1/3}{n-\mu} = \sum_{\mu=0}^{n} q_\mu x^{\mu}. \end{aligned} \tag{4}$$

Then

$$\begin{aligned} Q_n(z^{-1})F_B(z) &= 3^{2B-2A+1}\sum_{\mu=0}^{n} q_{n-\mu}z^{\mu-n}\cdot\sum_{\nu=0}^{\infty}\binom{B+1/3}{A+\nu}(-z)^{\nu} \\ &= 3^{2B-2A+1}\sum_{l=0}^{\infty} z^{l-n}\sum_{\substack{\mu=0\\ \mu\le l}}^{n} q_{n-\mu}\binom{B+1/3}{A+l-\mu}(-1)^{l-\mu} \end{aligned}$$

$$= 3^{2B-2A+1}\sum_{l=0}^{n-1} r_l z^{l-n} + 3^{2B-2A+1}\sum_{l=n}^{\infty} r_l z^{l-n} = P_n(z^{-1}) + R_n(z); \qquad (5)$$

here the polynomial

$$P_n(x) = 3^{2B-2A+1}\sum_{l=0}^{n-1} r_l x^{n-l}, \qquad \text{where} \quad r_l = \sum_{\mu=0}^{l} q_{n-\mu}\binom{B+1/3}{A+l-\mu}(-1)^{l-\mu} \qquad (6)$$

is of degree at most n.

Further, let us prove the following inclusions:

$$3^{[3n/2]}Q_n(x) \in \mathbb{Z}[x], \qquad 3^{[3n/2]}P_n(x) \in \mathbb{Z}[x].$$

For a natural number N and an integer $r = \pm r_1^{\theta_1}\cdots r_t^{\theta_t}$, where $r_1,\dots,r_t$ are primes, we call the number

$${}_N r = \prod_{i:\, r_i \mid N} r_i^{\theta_i}.$$

the Nth part of r. We shall use the following lemma (see [8, Lemma 4.1]).

Lemma 1. *Suppose that $N, r \ge 1$ are integers, and suppose that ${}_N r!$ is the Nth part of $r!$. Then the number*

$$\frac{(Nu-s)\cdots(N(u+r-1)-s)}{r!}\cdot {}_N r!,$$

where $u, v \in \mathbb{Z}$, is an integer.

Proof. Write $r! = {}_N r! \cdot M$, whence $(N, M) = 1$. Hence there exists an N' such that

$$N'N \equiv 1 \pmod M \qquad \text{and} \qquad (N', M) = 1.$$

Then

$$(N')^r \cdot (Nu-s)\cdots(N(u+r-1)-s) \equiv (u-sN')\cdots(u+r-1-sN') \pmod M.$$

But we have

$$(u-sN')\cdots(u+r-1-sN') = \binom{u+r-1-sN'}{r} r!,$$

and hence,

$$(u-sN')\cdots(u+r-1-sN') \equiv 0 \pmod M.$$

Taking the equality $(N', M) = 1$ into account, we obtain

$$(Nu-s)\cdots(N(u+r-1)-s) \equiv 0 \pmod M,$$

whence

$${}_N r!(Nu-s)\cdots(N(u+r-1)-s) \equiv 0 \pmod{r!},$$

as required. □

The coefficients of the remainder

$$R_n(z) = 3^{2B-2A+1}\sum_{l=n}^{\infty} r_l z^{l-n}$$

have the following representation:

$$r_l = \sum_{\mu=0}^{n} q_{n-\mu}\binom{B+1/3}{A+l-\mu}(-1)^{l-\mu}$$

$$= \sum_{\mu=0}^{n} (-1)^{n-\mu} \binom{2n+A-\mu-1}{n-\mu} \binom{n-B+A-1-1/3}{\mu} \binom{B+1/3}{A+l-\mu} (-1)^{l-\mu}$$

$$= (-1)^n \frac{(2n+A-1)\cdots(2n+A-n)\cdot(B+1/3)\cdots(B-A-l+1+1/3)(-1)^l}{n!(A+l)!}$$

$$\times \sum_{\mu=0}^{n} \frac{n!}{(n-\mu)!\mu!} \frac{(n-B+A-1-1/3)\cdots(n-B+A-\mu-1/3)}{(2n+A-1)\cdots(2n+A-\mu)}$$

$$\times \frac{(A+l)\cdots(A+l-\mu+1)}{(B-A-l+\mu+1/3)\cdots(B-A-l+1+1/3)}$$

$$= (-1)^n \frac{(2n+A-1)\cdots(2n+A-n)\cdot(B+1/3)\cdots(B-A-l+1+1/3)(-1)^l}{n!(A+l)!}$$

$$\times {}_3F_2\left(\begin{matrix} -n,\ -A-l,\ -n+B-A+1+1/3 \\ -2n-A+1,\ B-A-l+1+1/3 \end{matrix} \,\middle|\, 1\right).$$

Using the Pfaff–Saalschütz summation formula (see, for example, [9, p. 49, formula (2.3.1.3)])

$${}_3F_2\left(\begin{matrix} -n,\ a,\ b \\ c,\ 1+a+b-c-n \end{matrix} \,\middle|\, 1\right) = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n},$$

we obtain

$$r_l = (-1)^n \frac{(2n+A-1)\cdots(2n+A-n)\cdot(B+1/3)\cdots(B-A-l+1+1/3)(-1)^l}{n!\,(A+l)!}$$

$$\times \frac{(B+1+1/3)_n(n-l)_n}{(B-A-l+1+1/3)_n(n+A)_n}.$$

Note that $(n-l)_n = 0$ for an l from the interval $n \le l \le 2n-1$, so that $r_l = 0$ for such an l, while

$$r_l = r_{2n+\nu} = (-1)^n \frac{(2n+A-1)\cdots(2n+A-n)}{n!}$$

$$\times \frac{(B+1/3)\cdots(B-A-2n-\nu+1+1/3)(-1)^\nu}{(A+2n+\nu)!}$$

$$\times \frac{(B+1+1/3)_n(n-2n-\nu)_n}{(B-A-2n-\nu+1+1/3)_n(n+A)_n}$$

$$= \frac{(n-B+A-1-1/3)\cdots(-B-n-1/3)}{(2n+A)!} \cdot \frac{(n-B+A-1/3)_\nu(n+1)_\nu}{\nu!(2n+A+1)_\nu},$$

whence we obtain the relation

$$R_n(z) = 3^{2B-2A+1} \binom{n-B+A-1-1/3}{2n+A} z^n \cdot {}_2F_1\left(\begin{matrix} n-B+A-1/3,\ n+1 \\ 2n+A+1 \end{matrix} \,\middle|\, z\right) \tag{7}$$

and the integral representation

$$R_n(z) = 3^{2B-2A+1} \frac{(-1)^{n+B+1}}{\Gamma(1-1/3)\Gamma(1/3)} \cdot z^n \cdot \int_0^1 t^{n-B+A-1-1/3}(1-t)^{n+B+1/3}(1-zt)^{-n-1}\,dt. \tag{8}$$

In what follows, we shall need the linear independence of the pair of adjacent approximants, which is a consequence of the following lemma (compare [6, Lemma 2]).

Lemma 2. *The following relation holds*:

$$Q_{n+1}(x)P_n(x) - Q_n(x)P_{n+1}(x) = (-1)^n 3^{2B-2A+1} \binom{2n+A+1}{n+1} \binom{n-B+A-1-1/3}{2n+A} x. \tag{9}$$

Proof. Obviously, the left-hand side of (9) is a polynomial whose constant term is zero in view of the equalities $P_n(0) = P_{n+1}(0) = 0$ (see (6)) while

$$\begin{aligned}&Q_{n+1}(z^{-1})P_n(z^{-1}) - Q_n(z^{-1})P_{n+1}(z^{-1})\\&\quad= Q_{n+1}(z^{-1})(Q_n(z^{-1})F_B(z) - R_n(z)) - Q_n(z^{-1})(Q_{n+1}(z^{-1})F_B(z) - R_{n+1}(z))\\&\quad= Q_n(z^{-1})R_{n+1}(z) - Q_{n+1}(z^{-1})R_n(z);\end{aligned}$$

using (4), (7) and taking into account the fact that only negative powers of the variable z appear in the last sum, we obtain

$$\begin{aligned}&-Q_{n+1}(z^{-1})R_n(z)\\&\quad= (-1)^n\binom{2n+A+1}{n+1}z^{-(n+1)}(1+O(z))\\&\qquad\times 3^{2B-2A+1}\binom{n-B+A-1-1/3}{2n+A}z^n(1+O(z))\\&\quad= (-1)^n 3^{2B-2A+1}\binom{2n+A+1}{n+1}\binom{n-B+A-1-1/3}{2n+A}\frac{1}{z} + O(1)\end{aligned}$$

as $z \to 0$. □

3. ARITHMETIC COMPONENTS

To begin with, note that, for a prime $p > \sqrt{N}$, we have

$$\operatorname{ord}_p N! = \left\lfloor \frac{N}{p} \right\rfloor.$$

We shall also need to find the method of estimating the order of occurrence of a prime p in a product of the form

$$\begin{aligned}\Pi = N^{v-u}\left(u - \frac{s}{N}\right)_{v-u} &= N^{v-u}\left(u - \frac{s}{N}\right)\left(u - \frac{s}{N} + 1\right)\cdots\left(v - 1 - \frac{s}{N}\right)\\&= (Nu - s)(N(u+1) - s)\cdots(N(v-1) - s),\end{aligned} \tag{10}$$

where $u, v, s \in \mathbb{Z}$, $N \in \mathbb{N}$, and $u < v$, and $0 \le s < N$.

Let us use Lemma 4.5 from [8] in a slightly modified form.

Lemma 3. *Under the conditions*

$$p > \sqrt{N\max\{|u|+1, |v|\}}, \quad (p, N) = 1, \qquad \Pi \ne 0,$$

the order of occurrence of a prime p in a product Π *of the form* (10) *is*

$$\operatorname{ord}_p \Pi = \left\lfloor \frac{v-1}{p} - \frac{r}{N} \right\rfloor - \left\lfloor \frac{u-1}{p} - \frac{r}{N} \right\rfloor,$$

where $r \in \mathbb{N}$ *is the minimal number such that*

$$rp \equiv -s \pmod{N}.$$

Proof. The conditions on r are equivalent to the relation $rp = kN - s$ for some $k \in \mathbb{Z}$, and hence, for a given k, we have

$$kN \equiv s \pmod{p}. \tag{11}$$

The question arises: Which ones of the factors of $Nj - s$ are divisible by p? Or: For which j, does the congruence

$$Nj \equiv s \pmod{p} \tag{12}$$

hold? Comparing (11) and (12), we obtain $kN \equiv Nj \pmod{p}$, which, in view of $(p, N) = 1$, is equivalent to the congruence

$$k \equiv j \pmod{p}.$$

Hence all such j are of the form $j = k + pl$ and

$$u \le k + pl \le v - 1. \tag{13}$$

We have $rp = kN - s$, or

$$\frac{k}{p} = \frac{r}{N} + \frac{s}{pN}. \tag{14}$$

First, let us rewrite (13) in the form

$$u - 1 + \frac{s}{N} < kp + l \le v - 1 + \frac{s}{N},$$

and then as

$$\frac{u-1}{p} + \frac{s}{pN} \le \frac{k}{p} + l \le \frac{v-1}{p} + \frac{s}{pN};$$

let us substitute (14) into it, obtaining the following relation from the left-hand inequality:

$$\frac{r}{N} + l > \frac{u-1}{p};$$

from the right-hand inequality, we obtain

$$\frac{r}{N} + l \le \frac{v-1}{p};$$

Therefore, we have

$$\frac{u-1}{p} - \frac{r}{N} < l \le \frac{v-1}{p} - \frac{r}{N}.$$

Hence the number of suitable j is exactly equal to

$$\left\lfloor \frac{v-1}{p} - \frac{r}{N} \right\rfloor - \left\lfloor \frac{u-1}{p} - \frac{r}{N} \right\rfloor,$$

and the conditions

$$p > \sqrt{N(|u|+1)} \qquad \text{and} \qquad p > \sqrt{N|v|}$$

guarantee that p appears in in each of the multipliers raised to the power ≤ 1. □

Further, for each $p > \sqrt{3(A+B+n)}$, let us define the exponent

$$e_p = \min_{\mu\in\mathbb{R}}\biggl(-\left\{\frac{n+A+\mu}{p}\right\} + \left\{\frac{n+A}{p}\right\} + \left\{\frac{\mu}{p}\right\} - \left\{\frac{n-B+A-1}{p} - \frac{r_p}{3}\right\} + \left\{\frac{-B+A+\mu-1}{p} - \frac{r_p}{3}\right\} + \left\{\frac{n-\mu}{p}\right\}\biggr), \tag{15}$$

where r_p is determined from the congruence

$$r_p p \equiv -1 \pmod{3}.$$

Then, obviously,

$$e_p \le \operatorname{ord}_p E_{A,B,\mu}, \qquad \mu = 0, \dots, n,$$

where

$$E_{A,B,\mu} = \binom{n+A+\mu}{\mu}\binom{n-B+A-1-1/3}{n-\mu}.$$

We set

$$\Phi = \Phi(A, B, n) = \prod_{p>\sqrt{3(A+B+n)}} p^{e_p}. \tag{16}$$

We shall need to seek cancellations in coefficients of the form

$$D_{A,B,\mu} = \binom{n+A+\mu-1}{\mu}\binom{n-B+A-1-1/3}{n-\mu} = \frac{n+A}{n+A+\mu}\cdot E_{A,B,\mu},$$
$$\widetilde{D}_{A,B,\mu} = \binom{n+A+\mu}{\mu}\binom{n-B+A-1/3}{n-\mu+1} = \frac{n-B+A-1/3}{n-\mu+1}\cdot E_{A,B,\mu},$$

where $D_{A,B,\mu}$ corresponds to the coefficients at the nth approximation step and $\widetilde{D}_{A,B,\mu}$ corresponds to the $(n+1)$th step.

Lemma 4. *The following inclusions hold*:

$$(3(n+B)+4)\Phi^{-1}3^{[3n/2]}D_{A,B,\mu}\in\mathbb{Z}, \qquad \mu=0,\dots,n,$$
$$(n+1)\Phi^{-1}3^{[3n/2]}\widetilde{D}_{A,B,\mu}\in\mathbb{Z}, \qquad \mu=0,\dots,n+1,$$

whence

$$(3(n+B)+4)\Phi^{-1}3^{[3n/2]}Q_n(x),\ (3(n+B)+4)\Phi^{-1}3^{[3n/2]}P_n(x)\in\mathbb{Z}[x],$$
$$(n+1)\Phi^{-1}3^{[3(n+1)/2]}Q_{n+1}(x),\ (n+1)\Phi^{-1}3^{[3(n+1)/2]}P_{n+1}(x)\in\mathbb{Z}[x].$$

Proof. First, consider the case of the nth step. We have

$$\operatorname{ord}_p D_{A,B,\mu} = \operatorname{ord}_p E_{A,B,\mu} + \operatorname{ord}_p(n+A) - \operatorname{ord}_p(n+A+\mu).$$

If $p\nmid n+A+\mu$, or $p \mid n+A$, then

$$\operatorname{ord}_p D_{A,B,\mu} \ge \operatorname{ord}_p E_{A,B,\mu}.$$

Otherwise, we have $\mu\equiv -(n+A) \pmod p$, so that $\mu/p+(n+A)/p\in\mathbb{Z}$ and

$$\begin{aligned}\operatorname{ord}_p\binom{n+A+\mu-1}{\mu} &= -\left\{\frac{n+A+\mu-1}{p}\right\}+\left\{\frac{n+A-1}{p}\right\}+\left\{\frac{\mu}{p}\right\}\\ &= -\left\{-\frac{1}{p}\right\}+\left\{\frac{n+A-1}{p}\right\}+\left\{-\frac{n+A}{p}\right\}\\ &= -\frac{p-1}{p}+\left\{\frac{n+A-1}{p}\right\}-\left\{\frac{n+A}{p}\right\}+1\\ &= \left\{\frac{1}{p}\right\}+\left\{\frac{n+A-1}{p}\right\}-\left\{\frac{n+A}{p}\right\}=0,\end{aligned} \tag{17}$$

whence

$$\begin{aligned}\operatorname{ord}_p D_{A,B,\mu} = \operatorname{ord}_p\binom{n-B+A-1-1/3}{n-\mu} &= \operatorname{ord}_p\binom{n-B+A-1-1/3}{2n+A}\\ &= \operatorname{ord}_p\binom{n-B+A-1-1/3}{2n+A+1}+\operatorname{ord}_p\frac{2n+A+1}{-n-B-1-1/3}\\ &\ge \operatorname{ord}_p E_{A,B,\mu}|_{\mu/p\equiv(-(n+A)-1)/p \,(\mathrm{mod}\,\mathbb{Z})}-\operatorname{ord}_p(3(n+B)+4).\end{aligned}$$

The last inequality follows from the fact that, for $\mu\equiv-(n+A)-1 \pmod p$, we have the relation

$$\operatorname{ord}_p\binom{n+A+\mu}{\mu}=0;$$

to verify it, we only need to expand it just as (17).

For the case of the $(n+1)$th step, similarly, for $\mu \neq n+1$, we have

$$\operatorname{ord}_p \widetilde{D}_{A,B,\mu} = \operatorname{ord}_p E_{A,B,\mu} + \operatorname{ord}_p(n-B+A-1/3) - \operatorname{ord}_p(n-\mu+1).$$

If $p \nmid n-\mu+1$ and $\mu \neq n+1$, then

$$\operatorname{ord}_p \widetilde{D}_{A,B,\mu} \geq \operatorname{ord}_p E_{A,B,\mu}.$$

Otherwise, $\mu \equiv n+1 \pmod p$, or $\mu/p - (n+1)/p \in \mathbb{Z}$, whence

$$\begin{aligned}
&\operatorname{ord}_p \binom{n-B+A-1/3}{n-\mu+1} \\
&\qquad = -\left\{\frac{n-B+A}{p} - \frac{r_p}{3}\right\} + \left\{\frac{-B+A+\mu-1}{p} - \frac{r_p}{3}\right\} + \left\{\frac{n-\mu+1}{p}\right\} \\
&\qquad = -\left\{\frac{n-B+A}{p} - \frac{r_p}{3}\right\} + \left\{\frac{-B+A+n}{p} - \frac{r_p}{3}\right\} = 0,
\end{aligned}$$

so that

$$\begin{aligned}
\operatorname{ord}_p \widetilde{D}_{A,B,\mu} = \operatorname{ord}_p \binom{n+A+\mu}{\mu} &= \operatorname{ord}_p \binom{2n+A+1}{n+1} \\
&= \operatorname{ord}_p \binom{2n+A}{n} + \operatorname{ord}_p \frac{2n+A+1}{n+1} \\
&\geq \operatorname{ord}_p E_{A,B,\mu}\big|_{\mu/p \equiv n/p \,(\operatorname{mod} \mathbb{Z})} - \operatorname{ord}_p(n+1).
\end{aligned}$$

$\square$

In what follows, we let $\widetilde{\Phi}$ denote one of the numbers $\Phi/(3(n+B)+4)$ or Φ/n, depending on the choice of the approximation step.

4. PROOF OF THEOREM 1

Now suppose that

$$A = \alpha m, \quad B = \beta m - 1, \quad n = \gamma m \text{ or } \gamma m + 1, \qquad m \in \mathbb{N}.$$

It is necessary to estimate the asymptotics of the growth of the polynomial $Q_n(x)$. Note that

$$(1+y)^{n-B+A-1-1/3} = \sum_{k=0}^{\infty} \binom{n-B+A-1-1/3}{k} y^k,$$

$$\frac{1}{(1-y)^{n+A}} = \sum_{k=0}^{\infty} \binom{n+A-1+k}{k} y^k$$

and, also,

$$\begin{aligned}
\sum_{k=0}^{n} \binom{n-B+A-1-1/3}{n-k} y^{-k} &= \frac{1}{y^n} \sum_{k=0}^{n} \binom{n-B+A-1-1/3}{n-k} y^{n-k} \\
&= \frac{1}{y^n} \sum_{k=0}^{n} \binom{n-B+A-1-1/3}{k} y^k,
\end{aligned}$$

whence we find that the constant term of the function

$$G_{n,A,B}(y,x) = (1+y)^{n-B+A-1-1/3} \cdot \frac{1}{y^n} \cdot \frac{1}{(1-(-x)y)^{n+A}} \tag{18}$$

is exactly equal to $Q_n(x)$. Hence, for $c|x| < 1$, $c < 1$, by Cauchy's theorem, we have

$$Q_n(x) = \frac{1}{2\pi i} \int_{|\xi|=c} G_{n,A,B}(\xi, x) \, \frac{d\xi}{\xi}.$$

Then (we consider the case $n = \gamma m$)

$$Q_n(x) = \frac{1}{2\pi i}\int_{|\xi|=c} e^{mg_{\gamma,\alpha,\beta}(\xi,x)}\frac{1}{(1+\xi)^{1/3}}\frac{d\xi}{\xi}, \tag{19}$$

where

$$g_{\gamma,\alpha,\beta}(\xi,x) = (\gamma-\beta+\alpha)\log(1+\xi) - \gamma\log(\xi) - (\gamma+\alpha)\log(1+x\xi);$$

thus, we obtain the asymptotics

$$C_1(z) = \overline{\lim_{m\to\infty}}\frac{\log|Q_n(z^{-1})|}{m} \le \max_{|\xi|=c}\operatorname{Re}(g_{\gamma,\alpha,\beta}(\xi,z^{-1})). \tag{20}$$

(For the case $n = \gamma m + 1$, we obtain the same function $g_{\gamma,\alpha,\beta}(\xi,x)$ and, respectively, the same asymptotics (20) with the additional term $(1+\xi)^{2/3}/\xi(1+x\xi)$ instead of $1/(1+\xi)^{1/3}$ in (19).)

Similarly, using of the Laplace method, from the representation (8) for $R_n(z)$ we obtain

$$\begin{aligned} C_0(z) &= \lim_{m\to\infty}\frac{\log|R_n(z)|}{m} \\ &= (2\beta-2\alpha)\log 3 + \gamma\log|z| \\ &\quad + \max_{0\le t\le 1}((\gamma-\beta+\alpha)\log(t) + (\gamma+\beta)\log(1-t) - \gamma\log(1-zt)). \end{aligned} \tag{21}$$

Further, let us express the product Φ as $\Phi = \Phi_1\cdot\Phi_2$, where

$$\Phi_1 = \prod_{p\,:\,r_p=1} p^{e_p}, \qquad \Phi_2 = \prod_{p\,:\,r_p=2} p^{e_p}.$$

Then, using (15), (16) and the asymptotic distribution law for the primes in an arithmetic progression, we obtain

$$C_2 = \lim_{m\to\infty}\frac{\log\Phi(\alpha m,\beta m-1,\gamma m)}{m} = \frac{1}{2}\int_0^1\varphi_1(x)\,d\psi(x) + \frac{1}{2}\int_0^1\varphi_2(x)\,d\psi(x), \tag{22}$$

where $\psi(x)$ is the logarithmic derivative of the gamma function and the 1-periodic functions $\varphi_1(x)$, $\varphi_2(x)$ are defined as follows:

$$\varphi_i(x) = \min_{0\le y<1}\widehat{\varphi}_i(x,y),$$

$$\begin{aligned} \widehat{\varphi}_i(x,y) &= -\{(\gamma+\alpha)x+y\} + \{(\gamma+\alpha)x\} + \{y\} \\ &\quad - \left\{(\gamma-\beta+\alpha)x - \frac{i}{3}\right\} + \left\{(-\beta+\alpha)x + y - \frac{i}{3}\right\} + \{\gamma x - y\}, \end{aligned}$$

where $i = 1, 2$.

Our aim is to obtain lower bounds for the modulus of ε_k, where

$$3^{1/3}\cdot 2^k = M_k + \varepsilon_k, \qquad M_k\in\mathbb{Z}, \quad 0 < |\varepsilon_k| < \frac{1}{2}.$$

Let us express k as $k = 3(\beta m - 1) + 1 + j$ with nonnegative integers m and $j < 3\beta$. Multiplying (5) by $\widetilde{\Phi}^{-1}2^j 3^{[3n/2]}$ and substituting $z = 1/9$, we find that

$$3^{[3n/2]}Q_n(9)\widetilde{\Phi}^{-1}2^jF_B\left(\frac{1}{9}\right) = 3^{[3n/2]}P_n(9)\widetilde{\Phi}^{-1}2^j + R_n\left(\frac{1}{9}\right)3^{[3n/2]}\widetilde{\Phi}^{-1}2^j. \tag{23}$$

From (3), we obtain

$$2^jF_B\left(\frac{1}{9}\right) \equiv 3^{1/3}2^{3B+1+j} \pmod{\mathbb{Z}} = 3^{1/3}2^k,$$

so that the left-hand side of the last equality can be expressed as $M'_k + \varepsilon_k$ for some $M'_k \in \mathbb{Z}$, and equality (23) can be written in the form

$$3^{[3n/2]} Q_n(9) \widetilde{\Phi}^{-1} \cdot \varepsilon_k = M''_k + R_n\left(\frac{1}{9}\right) 3^{[3n/2]} \widetilde{\Phi}^{-1} 2^j, \tag{24}$$

where

$$M''_k = 3^{[3n/2]} P_n(9) \widetilde{\Phi}^{-1} 2^j - 3^{[3n/2]} Q_n(9) \widetilde{\Phi}^{-1} M'_k \in \mathbb{Z},$$

by Lemma 4. In turn, Lemma 2 guarantees the inequality $M''_k \neq 0$ for at least one of the numbers $n = \gamma m$ or $\gamma m + 1$; let us fix the appropriate choice of n. Further, assuming that

$$C_0\left(\frac{1}{9}\right) - C_2 + \left(\frac{3\gamma}{2}\right) \log 3 < 0 \tag{25}$$

and using (21) and (22), we see that

$$\left| R_n\left(\frac{1}{9}\right) \widetilde{\Phi}^{-1} 3^{[3n/2]} 2^j \right| < \frac{1}{2} \qquad \text{for all} \quad m \geq N_1,$$

where N_1 is an effective constant. Therefore, using (24) and taking the inequality $|M''_k| > 0$ into account, we obtain

$$|3^{[3n/2]} Q_n(9) \widetilde{\Phi}^{-1}| \cdot |\varepsilon_k| \geq |M''_k| - \left| R_n\left(\frac{1}{9}\right) \widetilde{\Phi}^{-1} 3^{[3n/2]} 2^j \right| > \frac{1}{2},$$

so that, using (20), (22) and writing $\widetilde{C}_1 = C_1 + (3\gamma/2)\log 3$, we find that

$$|\varepsilon_k| > \frac{\widetilde{\Phi}}{2|3^{[3n/2]} Q_n(9)|} > e^{-m(\widetilde{C}_1(1/9) - C_2 + \delta)},$$

for any $\delta > 0$ and $m > N_2(\delta)$ under the condition

$$\widetilde{C}_1\left(\frac{1}{9}\right) - C_2 + \delta > 0;$$

here the number $N_2(\delta)$ effectively depends on δ. Finally, in view of $k > 3\beta m - 3$, we obtain the estimate

$$|\varepsilon_k| > e^{-k(\widetilde{C}_1(1/9) - C_2 + \delta)/(3\beta)}, \tag{26}$$

which holds for all $k > K_0(\delta)$, where $K_0(\delta)$ can be explicitly expressed in terms of $\max(N_1, N_2(\delta))$.

Choosing the values $\alpha = \gamma = 4$, $\beta = 7$, and $c = 0.0372\ldots$ in (20), we obtain the constants

$$C_0\left(\frac{1}{9}\right) = -5.601392478500\ldots, \qquad C_1\left(\frac{1}{9}\right) = 16.38920443337\ldots,$$

and also the following collection of intervals: for $r_p = 1$,

$$\left[\frac{8}{33}, \frac{1}{4}\right) \cup \left[\frac{1}{3}, \frac{3}{8}\right) \cup \left[\frac{14}{33}, \frac{1}{2}\right) \cup \left[\frac{20}{33}, \frac{5}{8}\right) \cup \left[\frac{23}{33}, \frac{3}{4}\right) \cup \left[\frac{32}{33}, 1\right);$$

for $r_p = 2$,

$$\left[\frac{4}{33}, \frac{1}{8}\right) \cup \left[\frac{7}{33}, \frac{1}{4}\right) \cup \left[\frac{16}{33}, \frac{1}{2}\right) \cup \left[\frac{2}{3}, \frac{3}{4}\right) \cup \left[\frac{28}{33}, \frac{7}{8}\right) \cup \left[\frac{31}{33}, 1\right).$$

Hence

$$C_2 = 1.34311237783\ldots.$$

Now we verify condition (25):

$$C_0\left(\frac{1}{9}\right) + \frac{3\gamma}{2}\log 3 - C_2 = -0.352831123491\ldots$$

and, choosing $\delta = 0.000210078869\ldots$, we obtain

$$e^{-(\widetilde{C}_1(1/9)-C_2+\delta)/(3\beta)} = 0.3568.$$

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