

# A survey on Büchi's problem : new presentations and open problems

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## Abstract

In any commutative ring  $A$  with unit, *Büchi sequences* are those sequences whose second difference of squares is the constant sequence  $(2)$ . Sequences of elements of the form  $(x + n)^2$  for some fixed  $x$  are Büchi sequences that we call *trivial*. Since we want to study sequences whose elements do not belong to certain subrings (e.g. for fields of rational functions  $F(z)$  over a field  $F$  we are interested in sequences that are not over  $F$ ) the concept of *trivial sequences* may vary. Büchi's Problem for a ring  $A$  asks whether there exists a positive integer  $M$  such that any Büchi sequence of length  $M$  or more is trivial.

We survey the current status of knowledge for Büchi's problem and its analogues for higher-order differences and higher powers. We propose several new and old open problems. We present a few new results and various sketches of proofs of old results (in particular : Vojta's conditional proof for the case of integers and a quite detailed proof for the case of polynomial rings in characteristic zero), and present a new and short proof of the positive answer to Büchi's problem over finite fields with  $p$  elements (originally proved by Hensley). We discuss applications to Logic (which were the initial aim for solving these problems).