Representation Theorems for R.E. Sets and a Conjecture Related to Poonen's Large Subring of $\mathcal{Q}$.

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Abstract

The proof of the Novikoff-Boone Theorem provides an example of a group with an unsolvable word problem. Higman proved much more: he provided a characterization of recursive enumerability in terms of group theory. Such representation theorems are often available in connection with unsolvability theorems. Thus the unsolvability proof for Hilbert's 10th Problem was by means of such a representation theorem, the Matiyasevich/MRDP/DPRM Theorem to the effect that a set $S \subseteq N$ (where N is the set of natural numbers) is recursively enumerable if and only if it has a Diophantine definition.

Poonen has shown how to construct a subring $\mathcal{U}$ of the ring $\mathcal{Q}$ of rational numbers such that Hilbert's 10th Problem is unsolvable over $\mathcal{U}$, where $\mathcal{U}$ consists of the fractions whose denominators are products of primes from a particular set of primes of density 1 in the set of all primes. Poonen's proof makes use of a computable map $a \to y_a$ that maps a given $a \in N$ into an element $y_a \in \mathcal{U}$. The proof can be used to show more, namely that given any recursively enumerable set $S \subseteq N$, there is a polynomial p with coefficients in $\mathcal{U}$ such that

$$S = \{a \in N \mid (\exists x \in \mathcal{U}^\ell)[p(y_a, x) = 0]\}.$$

Writing

$$\hat{S}_p = \{a \in N) \mid (\exists x \in \mathcal{Q}^\ell)[p(y_a, x) = 0]\},$$

we have $S \subseteq \hat{S}_p \subseteq N$. I conjecture:

> *There is a* **simple** *recursively enumerable set $S \subseteq N$ and corresponding polynomial p such that $\hat{S}_p$ is also simple.*

This conjecture clearly implies that Hilbert's 10th Problem is unsolvable over $\mathcal{Q}$.