INHOMOGENEOUS EXTREME FORMS

MATHIEU DUTOUR SIKIRIĆ, ACHILL SCHÜRMANN, AND FRANK VALLENTIN

Abstract. G.F. Voronoi (1868–1908) wrote two memoirs in which he describes two reduction theories for lattices, well-suited for sphere packing and covering problems. In his first memoir a characterization of locally most economic packings is given, but a corresponding result for coverings has been missing. In this paper we bridge the two classical memoirs.

By looking at the covering problem from a different perspective, we discover the missing analogue. Instead of trying to find lattices giving economical coverings we consider lattices giving, at least locally, very uneconomical ones. We classify the covering maxima up to dimension 6 and prove their existence in all dimensions beyond.

New phenomena arise: Many highly symmetric lattices turn out to give uneconomical coverings; the covering density function is not a topological Morse function. Both phenomena are in sharp contrast to the packing problem.

1. Introduction

A basis of the n-dimensional Euclidean space $\mathbb{R}^n$ defines a lattice consisting of all integer linear combinations. A lattice defines a sphere packing in the following way: One centers congruent balls at the lattice points with maximum radius such that interiors do not intersect. Similarly, it defines a sphere covering: One places congruent balls with minimum radius such that each point in $\mathbb{R}^n$ is covered by a ball.

The *(lattice sphere) packing problem* asks for a lattice which gives the most economical packing, i.e. the one which maximizes the fraction of space covered by the balls. The *(lattice sphere) covering problem* asks for a lattice which gives the most economical covering, i.e. the one which minimizes the average number of balls covering a point in $\mathbb{R}^n$.

Many researchers were attracted by the packing problem. One important reason for this is that low-dimensional lattices which give good packings are often related to objects of exceptional beauty in combinatorics, geometry, and number theory. A vivid account of this is the monograph [8] by Conway and Sloane with over 100 pages of references which since the appearance of its first edition in 1988 spurred a tremendous amount of activity.

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Our computational studies in [33], [30], [31], [17] show that the covering problem behaves very differently. Many of the best known coverings could only be found with computer assistance. They were found by a numerical convex continuous optimization procedure; some of them do not have a rational representations, and their beauty is not immediately apparent.

Furthermore, in [30] it came as a surprise that the root lattice E_8 does not even give a locally optimal covering whereas the Leech lattice Λ_{24} does. Both lattices are the unique optimum, up to scaling and isometries, for the lattice packing problem which was proved by Blichfeldt [3] (optimality of E_8), Vetchinkin [35] (uniqueness of E_8) and Cohn, Kumar [6] (optimality and uniqueness of Λ_{24}). In many respects both lattices behave similarly. The shortest vectors of both lattices give spherical point configurations which are optimal for many other extremal questions in geometry, like the kissing number problem and more generally for potential energy minimization which is proved in Cohn and Kumar's work on universally optimal point configurations on spheres [5].

From further experimental studies it became apparent that E_8 is almost a local covering maximum, that is, the covering density decreases for almost all perturbations of E_8. We say that E_8 is a covering pessimum. This raised the question: Do local covering maxima exist (although local packing minima do not exist)? The first local covering maximum E_6 is found in [29].

In this paper we develop the theory of (local) covering maxima. It turns out that our theory gives the missing bridge between Voronoi's two classical memoirs [36], [37].

In Section 2 we start by formulating a characterization of covering maxima in the spirit of Voronoi. In [36] Voronoi gives a similar characterization of packing maxima extending earlier work of Korkine and Zolotarev. Then, Section 3 contains a proof of our characterization. It is based on using the Karush-Kuhn-Tucker condition from nonlinear optimization.

In Section 4 we formulate and prove a sufficient condition for being a covering maximum in the spirit of Venkov: It uses the t-design property of spherical point configurations. In [34] Venkov gives a similar condition for packing maxima. It turns out that many interesting lattices satisfy this condition.

In Section 5 we show that there are only finitely many covering maxima in every dimension and we give a classification which is complete up to dimension 6. For dimension 7 and 8 we give a list of all known covering maxima. There is strong numerical evidence that our list is complete.

One important difference between the packing problem and the covering problem is discussed in Section 6: Ash [1] proved that the packing density function is a topological Morse function. We show that the covering density function does not have this property if the dimension is at least four.

In the last section we give a construction showing that there are covering maxima in all dimensions $n \geq 6$.

2. Extremality = Perfectness and Eutaxy

In his first memoir Voronoi gives a characterization of locally optimal packings, building on previous works by Korkine and Zolotarev. For this he uses the notions of extremality, perfectness and eutaxy, which are naturally defined in the language of positive definite quadratic forms (PQFs).

Some preliminaries: There is a one-to-one correspondence between lattice bases up to orthogonal transformations and PQFs by taking the Gram matrix of the lattice basis. We identity the space of quadratic forms in n variables with the space of real symmetric $n \times n$-matrices. It is an $\binom{n+1}{2}$-dimensional Euclidean space with inner product $\langle Q, Q' \rangle = \operatorname{trace}(QQ')$, where Q and Q' are quadratic forms. By this identification we can evaluate a quadratic form Q at a vector $x \in \mathbb{R}^n$ by

$$Q[x] = x^t Q x = \langle Q, xx^t \rangle.$$

Now we review Voronoi's characterization for the homogeneous packing case where we refer to the monographs [23] of Martinet and [29] of Schürmann for proofs and further information. Then we present our characterization for the inhomogeneous covering case.

2.1. **Homogeneous case.** Let Q be a positive definite quadratic form in n variables. The *Hermite invariant* of Q is

$$\gamma(Q) = \frac{\lambda(Q)}{(\det Q)^{1/n}},$$

where

$$\lambda(Q) = \min_{v \in \mathbb{Z}^n \setminus \{0\}} Q[v],$$

is the *homogeneous minimum* of Q. It is scale-invariant. Maximizing the packing density among lattices is equivalent to maximizing the Hermite invariant among PQFs.

Voronoi gave a characterization of the local maxima of the Hermite invariant using the geometry of the *shortest vectors*

$$\operatorname{Min} Q = \{v \in \mathbb{Z}^n : Q[v] = \lambda(Q)\}.$$

Definition 2.1. *Let Q be a PQF.*

(i) *It is called* extreme *if it is a local maximum of the Hermite invariant.*
(ii) *It is called* perfect *if the linear space spanned by*

$$\{vv^t : v \in \operatorname{Min} Q\}$$

has maximal possible rank $\binom{n+1}{2}$.
(iii) *It is called* eutactic *if there are positive constants α_v so that*

$$Q^{-1} = \sum_{v \in \operatorname{Min} Q} \alpha_v vv^t.$$

It is called semieutactic *if the constants are nonnegative, and* weakly eutactic *if the constants are real, i.e. if they exist at all.*

The extended notion of semieutaxy and weak eutaxy is due to Bergé and Martinet [2]

Theorem 2.2 (Voronoi [36]). *A PQF is extreme if and only if it is perfect and eutactic.*

2.2. **Inhomogeneous case.** We define the *inhomogeneous Hermite invariant* of a PQF Q as

$$H(Q) = \frac{\mu(Q)}{(\det Q)^{1/n}},$$

where

$$\mu(Q) = \max_{x \in \mathbb{R}^n} \min_{v \in \mathbb{Z}^n} Q[x - v]$$

is the *inhomogeneous minimum* of Q. Like γ it is scale-invariant. Finding extrema for the covering density among lattices is equivalent to finding extrema for the inhomogeneous Hermite invariant among PQFs.

In the literature, so far only the local minima of the inhomogeneous Hermite invariant have been considered, as they give economical coverings. However, to link the homogeneous with the inhomogeneous case we have to consider the local maxima.

In this paper we characterize local maxima of the inhomogeneous Hermite invariant using the geometry of closest vectors. For each point $c \in \mathbb{R}^n$ attaining $\mu(Q)$ we define the *closest vectors*

$$\operatorname{Min}_c Q = \{v \in \mathbb{Z}^n : Q[v - c] = \mu(Q)\}.$$

Geometrically, the closest vectors give the vertices of the *Delone polytope* defined by the PQF Q which has center c: We have $Q[v - c] = \mu(Q)$ only for $v \in \operatorname{Min}_c Q$ and for all other lattice points $v \in \mathbb{Z}^n$ we have strict inequality $Q[v - c] > \mu(Q)$. The set of all Delone polytopes is called the *Delone subdivision* of Q which is a $\mathbb{Z}^n$-periodic polyhedral subdivision of $\mathbb{R}^n$. The inhomogeneous minimum of Q is at the same time the maximum squared circumradius of its Delone polytopes.

Definition 2.3. *Let Q be a PQF.*

(i) *It is called* inhomogeneous extreme *if it is a local maximum of the inhomogeneous Hermite invariant.*
(ii) *It is called* inhomogeneous perfect*, if for each $c \in \mathbb{R}^n$ attaining $\mu(Q)$, the linear space spanned by*

$$\left\{ \begin{pmatrix} 1 \\ v \end{pmatrix} \begin{pmatrix} 1 \\ v \end{pmatrix}^t : v \in \operatorname{Min}_c Q \right\}$$

has maximal possible rank $\binom{n+2}{2} - 1$.
(iii) *It is called* inhomogeneous eutactic*, if for each $c \in \mathbb{R}^n$ attaining $\mu(Q)$, there are positive constants α_v so that*

$$\begin{pmatrix} 1 & c^t \\ c & cc^t + \frac{\mu(Q)}{n} Q^{-1} \end{pmatrix} = \sum_{v \in \operatorname{Min}_c Q} \alpha_v \begin{pmatrix} 1 \\ v \end{pmatrix} \begin{pmatrix} 1 \\ v \end{pmatrix}^t.$$

It is called inhomogeneous semieutactic *if the constants are nonnegative, and* inhomogeneous weakly eutactic *if the constants are real.*

Now we are ready to state our principal result.

Theorem 2.4. *A PQF is inhomogeneous extreme if and only if it is inhomogeneous perfect and inhomogeneous eutactic.*

We prove this theorem in Section 3 after giving a reformulation in the following subsection.

2.3. **Quadratic functions.** Before we go on, a remark why in the definition of inhomogeneous perfect forms the maximal possible rank is $\binom{n+2}{2} - 1$ instead of $\binom{n+2}{2}$ is in order: It is $\binom{n+2}{2} - 1$ because the vectors v of $\mathrm{Min}_c\, Q$ satisfy the equation $Q[v-c] = \mu(Q)$ which translates into one *linear* equation in the space of quadratic functions. This observation, due to Erdahl and Ryshkov [19], [20], [28], will be the key to the proof of our principal result. Let us elaborate on this.

Instead of using one quadratic form, which (implicitly) defines the inhomogeneous minimum $\mu(Q)$ and the points $c \in \mathbb{R}^n$ attaining $\mu(Q)$, we make things explicit by using several quadratic functions; one for each c.

A *quadratic function* in n variables can be written as

$$f(x) = \alpha_f + 2b_f \cdot x + Q_f[x],$$

where $\alpha_f \in \mathbb{R}$, $b_f \in \mathbb{R}^n$, and Q_f is a quadratic form in n variables. By $b_f \cdot x$ we denote the standard inner product of the two n-dimensional vectors b_f and x. We equip the space of quadratic functions with the inner product

$$(f, g) = \alpha_f \alpha_g + 2b_f \cdot b_g + \langle Q_f, Q_g \rangle.$$

For $x \in \mathbb{R}^n$ we define the quadratic function

$$\mathrm{ev}_x(y) = (1 + x \cdot y)^2,$$

which can be used to evaluate a quadratic function f at x by $(\mathrm{ev}_x, f) = f(x)$. We define the *Erdahl cone* by

$$\mathcal{E}_{\geq 0} = \{f : f(v) \geq 0 \text{ for all } v \in \mathbb{Z}^n\}.$$

If a quadratic function f lies in the Erdahl cone, then Q_f is positive semidefinite (see e.g. [19, Proposition 1.3]). We define the *positive Erdahl cone* by

$$\mathcal{E}_{>0} = \{f \in \mathcal{E}_{\geq 0} : Q_f \text{ is positive definite}\}.$$

Let f be a quadratic function lying in the Erdahl cone. The zero set of f is an ellipsoid whose interior is free of integral points, points lying in $\mathbb{Z}^n$. The convex hull of the integral zeroes of f is called the *Delone polyhedron of* f,

$$\mathrm{Del}\, f = \mathrm{conv}\{v \in \mathbb{Z}^n : f(v) = 0\}.$$

Note that a Delone polyhedron might be empty, bounded or unbounded. We define the function

$$\mu(f) = -\min_{x \in \mathbb{R}^n} f(x) = \max_{x \in \mathbb{R}^n} -f(x).$$

We will make extensive use of the fact that μ is a convex function. This follows because evaluation is linear in f. The function μ is negative exactly for those f having an empty zero set so that the Delone polyhedron of f is empty.

Let f be a quadratic function lying in the positive Erdahl cone. If the zero set of f is a non-degenerate ellipsoid (i.e. it is non-empty and bounded), then its center is $c_f = -Q_f^{-1} b_f$ and its squared circumradius (with respect to Q_f) is $\mu(f)$. In this case one can write

$$f(x) = Q_f[x - c_f] - \mu(f), \quad \text{and} \quad \mu(f) = Q_f[c_f] - \alpha_f.$$

The *inhomogeneous Hermite invariant* of $f \in \mathcal{E}_{>0}$ is

$$H(f) = \frac{\mu(f)}{(\det Q_f)^{1/n}}.$$

Note that it is invariant under multiplication by positive scalars.

Definition 2.5. *Let f be a quadratic function lying in the positive Erdahl cone.*

(i) *It is called* extreme *if it is a local maximum of the inhomogeneous Hermite invariant.*

(ii) *It is called* perfect*, if the linear space spanned by* ev_v*, with* $v \in \operatorname{vert} \operatorname{Del} f$*, has maximal possible rank* $\binom{n+2}{2} - 1$*.*

(iii) *It is called* eutactic *if there are positive real numbers* α_v*, with* $v \in \operatorname{vert} \operatorname{Del} f$*, so that the following conditions hold*

$$\sum_{v \in \operatorname{vert} \operatorname{Del} f} \alpha_v \, \mathrm{ev}_v = \mathrm{ev}_{c_f} + \frac{\mu(f)}{n} Q_f^{-1}.$$

It is called semieutactic *if the constants are nonnegative, and* weakly eutactic *if the constants are real.*

The equation in the definition of eutaxy (iii) has the following geometric interpretation: A negative multiple of the gradient of the function H, which is given on the right hand side (see Lemma 3.2), lies in the interior of the *inhomogeneous Voronoi cone*

$$\mathcal{V}(f) = \operatorname{cone}\{\mathrm{ev}_x : f(x) = 0, x \in \mathbb{Z}^n\}.$$

Going back to PQFs: Let Q be a PQF and $c \in \mathbb{R}^n$ be a point attaining the inhomogeneous minimum $\mu(Q)$. Then the closest vectors $\operatorname{Min}_c Q$ are the vertices of the Delone polytope $\operatorname{Del} f$ of the quadratic function f given by $Q_f = Q$, $b_f = Q^{-1}c$, $\mu(f) = \mu(Q)$. Hence, the inhomogeneous minimum of Q is

$$\mu(Q) = \max\{\mu(f) : f \text{ quadratic function with } Q_f = Q\}.$$

A side remark: The convexity of $f \mapsto \mu(f)$ immediately implies the convexity of $Q \mapsto \mu(Q)$, i.e. the main result of Delone, Dolbilin, Ryshkov, Shtogrin in [11], see also [31, Proposition 7.1] or [29, Proposition 5.1].

We can reformulate the definition of inhomogeneous perfectness and eutaxy: A PQF Q is inhomogeneous perfect if all quadratic functions f with $Q_f = Q$ and $\mu(f) = \mu(Q)$ are perfect. A PQF Q is inhomogeneous eutactic if all quadratic functions f with $Q_f = Q$ and $\mu(f) = \mu(Q)$ are eutactic. With this, Theorem 2.4 follows immediately from the following theorem.

Theorem 2.6. *A quadratic function lying in the positive Erdahl cone is extreme if and only if it is perfect and eutactic.*

3. Proof of Theorem 2.6

The proof of our principal theorem is an analysis of local maxima of a differentiable function satisfying inequality constraints. We first recall some background from nonlinear optimization: sufficient and necessary criteria for a function to have a local maximum. Then we specialize this to our situation of the inhomogeneous Hermite invariant.

3.1. **Nonlinear optimization.** We just state the result and refer to any book on nonlinear optimization for more details, e.g. the book by Boyd and Vandenberghe [4, Chapter 5].

Let E be a Euclidean space with inner product $x \cdot y$ and let $p : E \to \mathbb{R}$ and $q_1, \ldots, q_k : E \to \mathbb{R}$ be differentiable functions. Assume, we want to determine whether or not p has a local maximum x_0 on the boundary of the set

$$G = \{x \in E : q_i(x) \geq 0 \text{ for } i = 1, \ldots, k\}.$$

In a sufficiently small neighborhood of x_0, the functions p and q_i can be linearized and approximated by affine functions:

$$x \mapsto p(x_0) + (\operatorname{grad} p)(x_0) \cdot (x - x_0).$$

We define the *normal cone* of G at x_0 by

$$N(x_0) = \operatorname{cone}\{-(\operatorname{grad} q_i)(x_0) : i = 1, \dots, k\}.$$

Proposition 3.1. *Suppose x_0 satisfies $(\operatorname{grad} p)(x_0) \neq 0$ and $q_i(x_0) = 0$, as well as $(\operatorname{grad} q_i)(x_0) \neq 0$, for $i = 1, \dots, k$.*

(i) *The function p attains an isolated local maximum on G at x_0, if*

$$(\operatorname{grad} p)(x_0) \in \operatorname{int} N(x_0),$$

where $\operatorname{int} N(x_0)$ *is the* interior *of the normal cone.*

(ii) *The function p does not attain a local maximum on G at x_0, if*

$$(\operatorname{grad} p)(x_0) \notin N(x_0).$$

3.2. **Proof of Theorem 2.6.** First we compute the gradient of the inhomogeneous Hermite invariant:

Lemma 3.2. *The Taylor series of the inhomogeneous Hermite invariant H at the quadratic function f_0 lying in the positive Erdahl cone is*

$$\frac{1}{(\det Q_{f_0})^{1/n}} \left(\mu(f_0) - \left(\mathrm{ev}_{c_{f_0}} + \frac{\mu(f_0)}{n} Q_{f_0}^{-1}, f - f_0 \right) + h.o.t. \right),$$

where h.o.t. stands for higher order terms.

Proof. The Taylor series of the functional μ at f_0 is

$$\mu(f_0) - (\mathrm{ev}_{c_{f_0}}, f - f_0) + \text{h.o.t.},$$

and the gradient of the determinant is $(\operatorname{grad} \det)(Q) = (\det Q)Q^{-1}$. □

We need the following convexity result. It implies that local maxima of the inhomogeneous Hermite invariant can only be attained at the extreme rays of the positive Erdahl cone.

Lemma 3.3. *Let f_1 and f_2 be two quadratic functions in the positive Erdahl cone having positive Hermite invariants. Then, the maximum of the inhomogeneous Hermite invariant H on* $\operatorname{cone}\{f_1, f_2\}$ *is only attained at its extreme rays* $\operatorname{cone}\{f_1\}$ *or* $\operatorname{cone}\{f_2\}$.

Proof. We may assume that f_1 and f_2 are not collinear. Since H is scale-invariant for positive scalars we may assume that $\mu(f_1) = \mu(f_2)$. It is sufficient to prove that

$$H(tf_1 + (1 - t)f_2) < tH(f_1) + (1 - t)H(f_2) \tag{1}$$

holds for all $0 < t < 1$. The convexity of the function μ and the convexity of the function $Q \mapsto (\det Q)^{-1/n}$, immediately give the inequality (1), but only with "$\leq$" instead of "$<$".

Since the function $Q \mapsto (\det Q)^{-1/n}$ is strictly convex (originally due to Minkowski [27, §8]) we have equality in (1) if and only if both functions

$$t \mapsto \mu(tf_1 + (1 - t)f_2), \text{ and } \quad t \mapsto Q_{tf_1 + (1-t)f_2}$$

are constant for $0 \leq t \leq 1$. Suppose this is the case, then

$$0 = \mu(tf_1 + (1 - t)f_2) - t\mu(f_1) - (1 - t)\mu(f_2) = -t(1 - t)Q_{f_1}[b_{f_1} - b_{f_2}],$$

and hence $b_{f_1} = b_{f_2}$. From this it follows that $\alpha_{f_1} = \alpha_{f_2}$, and hence $f_1, tf_1+(1-t)f_2$ and f_2 all coincide which contradicts the assumption. □

Note that the lemma and its proof show that the function H is strictly convex on the line segment connecting f_1 and f_2 if $\mu(f_1) = \mu(f_2)$ and if μ is positive on the line segment.

Now we can finish the proof.

Proof of Theorem 2.6. Suppose that f_0 is perfect and eutactic. Since the inhomogeneous Hermite function is invariant with respect to positive scaling, we can work with the Erdahl cone intersected with the affine hyperplane H_{f_0} orthogonal to f_0 and containing f_0. Consider the set

$$G_{f_0} = \{f \in \mathcal{E}_{>0} \cap H_{f_0} : (\mathrm{ev}_v, f) \geq 0, v \in \operatorname{vert} \operatorname{Del} f_0\}.$$

Since f_0 is perfect, the functions ev_v, with $v \in \operatorname{vert}\operatorname{Del} f_0$, span a subspace of codimension 1 in the $\binom{n+2}{2}$-dimensional space of quadratic functions. Hence, for a sufficiently small neighborhood N_{f_0} of the point f_0 we have

$$N_{f_0} \cap G_{f_0} = N_{f_0} \cap (\mathcal{E}_{>0} \cap H_{f_0}).$$

Since f_0 is eutactic and because of the gradient computation in Lemma 3.2 we have that $-(\operatorname{grad} H)(f_0)$ lies in the interior of the inhomogeneous Voronoi cone $\mathcal{V}(f_0)$. Here we take the interior within the affine hyperplane H_{f_0}. Applying Proposition 3.1 (i) shows that f_0 is a local maximum of H.

Conversely, suppose that f_0 is extreme. Then by Lemma 3.3 we know that f_0 has to lie on an extreme ray of the Erdahl cone, hence it is perfect. Suppose that f_0 is not eutactic. Proposition 3.1 (ii) shows that the only situation which can occur is that $-(\operatorname{grad} H)(f_0)$ lies on the boundary of the inhomogeneous Voronoi cone $\mathcal{V}(f_0)$. Then, by Farkas lemma, there exists a quadratic function h in the affine hyperplane H_{f_0} orthogonal to f_0 and containing f_0 so that

$$\begin{cases} (\mathrm{ev}_v, h) \geq 0, \quad \text{for all } v \in \operatorname{vert}\operatorname{Del} f_0, \\ ((\operatorname{grad} H)(f_0), h) = 0. \end{cases}$$

For $\lambda \geq 0$, consider the univariate function

$$\varphi_\alpha(\lambda) = \mu(f_0 + \lambda(h + \alpha f_0)).$$

We can choose α so that

$$0 = \frac{\partial \varphi_\alpha}{\partial \lambda}(0) = ((\operatorname{grad} \mu)(f_0), h + \alpha f_0),$$

because $((\operatorname{grad} \mu)(f_0), f_0) = \mu(f_0) \neq 0$. Since φ_α is convex and because $\frac{\partial \varphi_\alpha}{\partial \lambda}(0) = 0$, we have

$$\varphi_\alpha(\lambda) \geq \varphi_\alpha(0).$$

For $\lambda \geq 0$ consider the univariate function

$$\psi_\alpha(\lambda) = \det(Q_{f_0} + \lambda(Q_h + \alpha Q_{f_0}))^{-1/n}.$$

Since ψ_α is strictly convex, we have for $\lambda > 0$

$$\psi_\alpha(\lambda) > \psi_\alpha(0) + \frac{\partial \psi_\alpha}{\partial \lambda}(0)\lambda.$$

Taking the product shows

$$H(f_0 + \lambda(h + \alpha f_0)) = \varphi_\alpha(\lambda)\psi_\alpha(\lambda) > \varphi_\alpha(0)\psi_\alpha(0) = H(f_0),$$

because $\frac{\partial \psi_\alpha}{\partial \lambda}(0) \geq 0$. Hence, f_0 is not extreme. $\square$

A technical remark: The existence of extreme forms, which we will establish in the next section, shows that the positive Erdahl cone is *not* the interior of the Erdahl cone.

4. Examples — Strongly inhomogeneous perfect forms

Venkov introduced strongly perfect forms in [34]. *Strongly perfect forms* are PQFs in which the shortest vectors carry a spherical 4-design.

Theorem 4.1 (Venkov [34])**.** *Strongly perfect forms are extreme.*

The notion of spherical designs is due to Delsarte, Goethals, Seidel [12]. Generally, finitely many points X in $\mathbb{R}^n$ carry a *spherical t-design* (with respect to a PQF Q) if they lie on a sphere

$$S_Q(c,r) = \{x \in \mathbb{R}^n : Q[x-c] = r^2\}, \quad \text{with } c \in \mathbb{R}^n \text{, and } r \in \mathbb{R},$$

and so that for all polynomials f up to degree t we have

$$\frac{1}{|X|}\sum_{x \in X} f(x) = \int_{S_Q(c,r)} f(x) d\omega(x),$$

where ω is the normalized surface measure on $S_Q(c,r)$. The maximal t for which X carries a spherical t-design is called its *strength* which we denote by $s(X)$. An equivalent, alternative characterization of spherical t-designs is the following: The points X carry a spherical t-design (with respect to a PQF Q) if there exists $c \in \mathbb{R}^n$ and $r \in \mathbb{R}$ so that the following equalities hold for all $k \leq t$ and all $y \in \mathbb{R}^n$:

$$\sum_{x \in X} \left\langle Q, (x-c)(y-c)^t \right\rangle^k = \begin{cases} 0, & \text{for all odd } k, \\ \frac{1 \cdot 3 \cdots (k-1)}{n(n+2)\cdots(n+k-2)} |X| r^{k/2} Q[y-c]^{k/2}, \\ & \text{for all even } k. \end{cases}$$

For the proof of Theorem 4.1 Venkov used Voronoi's characterization of extreme PQFs in Theorem 2.2. He shows that having a spherical 2-design already implies eutaxy, and having a spherical 4-design implies perfectness.

Theorem 4.1 gives a uniform way for showing that many remarkable PQFs are extreme. It applies e.g. to the forms of the root lattices D_4, E_6, E_7, E_8, the Coxeter-Todd lattice K_{12}, the Barnes-Wall lattice BW_{16}, the laminated lattice Λ_{23}, the shorter Leech lattice O_{23}, the Leech lattice Λ_{24}, the Thompson-Smith lattice Λ_{248}. All but the last case are treated in Venkov [34]. For the Thompson-Smith lattice see Lempken, Schröder, Tiep [26]. Here it is interesting to note that one can show the strong perfectness of Λ_{248} *without* having the list of all minimal vectors (in fact at the time of writing not even the inhomogeneous minimum is known) but using properties of the automorphism group of Λ_{248} only.

Now we adapt the concept of strong perfection to the inhomogeneous case.

Definition 4.2. *Let Q be a PQF. It is called* strongly inhomogeneous perfect*, if for each $c \in \mathbb{R}^n$ attaining $\mu(Q)$, the closest vectors* $\operatorname{Min}_c Q$ *carry a spherical 4-design.*

Theorem 4.3. *Inhomogeneous strongly perfect forms are inhomogeneous extreme.*

We also adapt the definitions to the setting of quadratic functions.

Definition 4.4. *Let f be a quadratic function lying in the positive Erdahl cone. It is called* strongly perfect*, if the vertices of its Delone polytope carry a spherical 4-design.*

Theorem 4.5. *Strongly perfect quadratic functions are extreme.*

Like previously, Theorem 4.3 immediately follows from Theorem 4.5. The proof of the second theorem uses our characterization of inhomogeneous extreme forms in Theorem 2.2. It shows, like in the homogeneous case, that having spherical 2-designs already implies eutaxy, and that having spherical 4-designs implies perfectness.

Proof of Theorem 4.5. Let f be a strongly perfect quadratic function. The set $X = \operatorname{vert}\operatorname{Del} f$ carries a spherical 4-design with respect to the quadratic form Q_f.

We shall show that f is eutactic: If we unfold the equation in the definition of eutactic quadratic functions, we get

$$\begin{cases} 1 & = \sum_{x \in X} \alpha_x, \\ 0 & = \sum_{x \in X} \alpha_x (x - c_f), \\ \frac{\mu(f)}{n} Q_f^{-1} & = \sum_{x \in X} \alpha_x (x - c_f)(x - c_f)^t. \end{cases}$$

We set $\alpha_x = \frac{1}{|X|}$ with $x \in X$, so that the first condition in Definition 2.5 (iii) is satisfied. Then, by looking at the alternative definition of spherical 1- and 2-designs, we see that the other two conditions are satisfied, see e.g. [30, Lemma 5.1].

We shall show that f is perfect: Let g be a quadratic function which satisfies the linear equations

$$(\mathrm{ev}_x, g) = g(x) = 0 \quad \text{for all } x \in X.$$

Since X carries a spherical 4-design, we have

$$0 = \frac{1}{|X|} \sum_{x \in X} g(x)^2 = \int_{S_{Q_f}(c_f, \sqrt{\mu(f)})} g(x)^2 d\omega(x).$$

So, g vanishes on $S_{Q_f}(c_f, \sqrt{\mu(f)}) = \{x \in \mathbb{R}^n : f(x) = 0\}$. Hence, it has to be a multiple of f. So the space spanned by the functions ev_x, with $x \in X$, has codimension 1 in the $\binom{n+2}{2}$-dimensional space of quadratic functions. In other words, f is perfect. □

Using Theorem 4.5 one can show that the PQFs belonging to the lattices $\mathsf{E}_6, \mathsf{E}_7$, BW_{16}, Λ_{23}, O_{23} are inhomogeneous strongly perfect and hence inhomogeneous extreme. Geometrically this says that these lattices yield local covering maxima. These are all inhomogeneous strongly PQFs we know of. In Table 4.1 we give some details about these PQFs and the Delone polytopes: The second column gives the number of orbits of Delone polytopes. In all these cases there is only one orbit corresponding to points c where $\mu(Q)$ is attained. In the last column we give a reference where a description of the orbits can be found.

The PQFs belonging to the lattices $\mathbb{Z}^n$, D_n, E_6^*, E_7^*, E_8, K_{12} are not inhomogeneous perfect. However they are inhomogeneous eutactic. We will get a geometrical interpretation from Theorem 6.1: These lattices yield local *covering pessima*, i.e. the set of perturbations in which the covering density decreases has measure zero. In Table 4.2 we give some details about these PQFs and the Delone polytopes

A PQF belonging to the Leech lattice is neither inhomogeneous perfect nor inhomogeneous eutactic. In fact, geometrically, the Leech lattice gives a local minimum for the covering density, see [30].

name	# orbits	$\lvert \mathrm{Min}_c(Q)\rvert$	$s(\mathrm{Min}_c(Q))$	reference
E_6	1	27	4	Conway, Sloane [7]
E_7	2	56	5	CS [7]
BW_{16}	4	512	5	Dutour Sikirić, Schürmann, Vallentin [18]
O_{23}	5	94208	7	DSV [18]
Λ_{23}	709	47104	7	DSV [18]

Table 4.1. Lattices belonging to inhomogeneous strongly perfect forms.

name	# orbits	$\lvert \mathrm{Min}_c(Q)\rvert$	$s(\mathrm{Min}_c(Q))$	reference
$\mathbb{Z}^n$	1	2^n	3	Conway, Sloane [7]
D_3	2	6	3	CS [7]
D_4	1	8	3	CS [7]
D_n, $n \geq 5$	2	2^{n-1}	3	CS [7]
E_6^*	1	9	2	CS [7]
E_7^*	1	16	3	CS [7]
E_8	2	16	3	CS [7]
K_{12}	4	81	3	Dutour Sikirić, Schürmann, Vallentin [18]

Table 4.2. Lattices belonging to inhomogeneous eutactic forms.

We finish this section by posing several problems:

(i) Are there strongly perfect functions which do not define inhomogeneous strongly perfect forms?
(ii) Is a PQF of the Thompson-Smith lattice Λ_{248} inhomogeneous strongly perfect?
(iii) It would be interesting to classify strongly perfect quadratic functions in low dimensions. So far only a classification up to dimension 6 is known. It is described in the next section. In the homogeneous case, strongly perfect forms have been classified up to dimension 12 by Nebe and Venkov [25].

5. Finiteness and classification

In this section we show that there are only finitely many inequivalent perfect quadratic functions, respectively eutactic quadratic functions, in a given dimension. Here, equivalence is defined using scaling and using the action of the affine general linear group

$$\mathsf{AGL}_n(\mathbb{Z}) = \{u : \mathbb{R}^n \to \mathbb{R}^n : u(x) = v + Ax, \text{ with } v \in \mathbb{Z}^n \text{ and } A \in \mathsf{GL}_n(\mathbb{Z})\}.$$

More precisely, we say that two quadratic functions f and g are *equivalent* if there exists a positive scalar λ and $u \in \mathsf{AGL}_n(\mathbb{Z})$ so that $f(x) = \lambda g(u(x))$.

Theorem 5.1. *In any dimension there are only finitely many inequivalent perfect quadratic functions, respectively weakly eutactic quadratic functions.*

Proof. From the work of Voronoi [37, §98] (see also Deza, Laurent [13, Chapter 13.3]) it follows that, up to $\mathsf{AGL}_n(\mathbb{Z})$ equivalence, there are only finitely many Delone polytopes of quadratic functions. This implies that there are only finitely many inequivalent perfect quadratic functions.

Now we argue that every Delone polytope D determines up to equivalence at most one eutactic quadratic function. For this we define the cone

$$\Delta(D) = \{f \in \mathcal{E}_{>0} : \operatorname{Del} f = D\}.$$

Since the function μ is strictly positive on it, Lemma 3.3 and its proof show that H has at most one critical point, which is a minimum of H.

If f is weakly eutactic, then for all $g \in \Delta(D)$ we have

$$\begin{aligned}(-(\operatorname{grad} H)(f), g) &= \frac{1}{(\det Q_f)^{1/n}}\left(\operatorname{ev}_{c_f} + \frac{\mu(f)}{n} Q_f^{-1}, g\right)\\ &= \frac{1}{(\det Q_f)^{1/n}}\left(\sum_{v \in \operatorname{vert} \operatorname{Del} f} \alpha_v \operatorname{ev}_v, g\right)\\ &= 0,\end{aligned}$$

and hence f is a critical point of H. □

Perfect quadratic functions have been classified up to dimension 6; the classifications in dimension 7 and 8 seem to be complete:

Dimension 2, . . . , 5: Erdahl [19, Theorem 5.1] showed that there are no perfect quadratic functions in dimension $n = 2, \ldots, 5$.

Dimension 6: Dutour [14] showed that up to equivalence there is exactly one perfect quadratic function in dimension 6: It is defined by the Schläfli polytope 2_{21} in dimension 6 having 27 vertices (see e.g. [10, Chapter 11.8]). It is strongly perfect since the vertices of 2_{21} carry a spherical 4-design.

Dimension 7: In dimension 7 the known list of perfect quadratic functions is given in Dutour, Erdahl, Rybnikov [16, Section 7]: One is defined by the Gosset polytope 3_{21} in dimension 7 having 56 vertices (see e.g. [10, Chapter 11.8]). It is strongly perfect since the vertices of 3_{21} carry a spherical 5-design. The other one is defined by the 35-tope constructed by Erdahl, Rybnikov [21]. It is eutactic (although the strength of the design is 0), but it is not strongly perfect.

Dimension 8: In dimension 8 the known list of perfect quadratic function is given by Dutour, Erdahl, Rybnikov [16, Section 8]. There are up to equivalence (at least) 27 perfect quadratic functions in dimension 8, and 21 of them are eutactic. Among them there is no strongly perfect quadratic function.

It would be interesting to understand the asymptotics of the number of perfect quadratic functions and the number of eutactic quadratic functions. At the moment it is not even clear whether the number grows with every dimension. This appears to be extremely likely: In dimension 9 we found more than $100,000$ perfect quadratic functions.

6. Pessima and topological Morse functions

In this section we study inhomogeneous eutactic forms. First we consider inhomogeneous eutactic forms which are not inhomogeneous perfect. They can be almost local maxima for the inhomogeneous Hermite invariant. By this we mean the following: A PQF is called a *pessimum*, if it is not a local maximum of the inhomogeneous Hermite invariant, but for which almost all local perturbations decrease it. Note that there does not exist an analogue of pessima for the homogeneous Hermite invariant: There is no PQF for which almost all local perturbations increase the Hermite invariant. However, it is known (Štogrin [32]) that when a PQF is eutactic then the Hermite invariant decreases in almost every direction.

Theorem 6.1. *Let Q be an inhomogeneous eutactic PQF which is not inhomogeneous extreme. Suppose for all quadratic functions f lying in the positive Erdahl cone with $Q = Q_f$ and $\mu(Q) = \mu(f)$, the Delone polyhedron* $\operatorname{Del} f$ *is not a simplex. Then Q is a pessimum.*

Proof. Let Q' be a generic perturbation of Q so that all Delone polytopes of Q' are simplices. Let Δ be a Delone simplex contained in a Delone polytope $D = \operatorname{Del} f$ of Q. Let f' be the quadratic function with $\operatorname{Del} f = \Delta$ and $Q_{f'} = Q'$. Then we have the expansion

$$H(f') = H(f) - \sum_{v \in \operatorname{vert} D} \alpha_v (f' - f)(v) + \text{h.o.t.},$$

because f is eutactic. Since D is not a simplex, there is a $v \in \operatorname{vert} D$ so that $(f' - f)(v) > 0$. This implies that the second summand of the expansion is negative. □

This situation occurs for instance for the PQFs belonging to lattice given in Table 4.2.

As a second application we show that the inhomogeneous Hermite invariant is generally not a topological Morse function. We recall the following definition from Morse [24].

Definition 6.2. *Let M be an m-dimensional topological manifold and let f be a real valued continuous function on M.*

(i) *A point $q \in M$ is called* topologically ordinary *if there exist neighborhoods U of q and V of $0 \in \mathbb{R}^m$ and a homeomorphism $\phi : V \to U$ such that for all $x \in V$*
$$\phi(0) = q, \quad f(\phi(x)) = x_1 + f(q).$$
Otherwise, it is called topologically critical.

(ii) *A topologically critical point is called* topologically non-degenerate *of index r if there exist U, V, ϕ as above such that for all $x \in V$*
$$\phi(0) = q, \quad f(\phi(x)) = -x_1^2 - \cdots - x_r^2 + x_{r+1}^2 + \cdots + x_m^2 + f(q).$$

(iii) *A function is called* topological Morse function *if all points are either ordinary or topologically non-degenerate.*

Note that at a topological non-degenerate point the directions of decrease are homotopically equivalent to the sphere $S^{r-1} = \{x \in \mathbb{R}^r : \|x\| = 1\}$. The directions of increase are homotopically equivalent to the sphere S^{m-r-1}.

Since H is scale invariant, it is not a topological Morse function for trivial reasons; the same is true for the homogeneous Hermite invariant γ. Ash [1] showed that γ

is a topological Morse function on the cone of positive semidefinite $n \times n$-matrices where we mod out by positive scaling: $\mathcal{S}^n_{>0}/\mathbb{R}_{>0}$. As the following theorem shows, this is in general not the case for H.

Theorem 6.3. *The inhomogeneous Hermite invariant is a topological Morse function on $\mathcal{S}^n_{>0}/\mathbb{R}_{>0}$ if and only if n is at most three.*

We need the following lemma:

Lemma 6.4. *Let Q be an inhomogeneous eutactic form. Then Q is a topologically critical point for H in $\mathcal{S}^n_{>0}/\mathbb{R}_{>0}$. It is a topologically non-degenerate point if and only if there exist one Delone polytope D attaining the maximum circumradius such that for all Delone polytopes D' attaining the maximum circumradius we have*

$$\operatorname{lin} \Delta(D') \subseteq \operatorname{lin} \Delta(D).$$

Proof. Let $D_1, \ldots, D_r$ be the translation classes of Delone polytopes attaining the maximum circumradius. The argument in the proof of Theorem 6.1 shows that H increases in the direction of

$$U = \bigcup_{i=1}^{r} \operatorname{lin} \Delta(D_i)/\mathbb{R}_{>0}.$$

It decreases in all other directions. So it is a topologically critical point. If $U = \operatorname{lin} \Delta(D_i)$ for some D_i, then Q is a topologically non-degenerate point. If U is a union of subspaces which is not contained in $\operatorname{lin} \Delta(D_i)$ for one D_i, then U is not homotopically equivalent to a sphere, so Q is not a topologically non-degenerate point. □

Proof of Theorem 6.3. There is at most one critical point in the secondary cone of a fixed Delone decomposition up to the action of $\mathsf{GL}_n(\mathbb{Z})$.

If n equals two, there are two critical points: The PQF corresponding to the lattice $\mathbb{Z}^2$ and the one corresponding to the lattice A_2. They are both inhomogeneous eutactic. In both cases there is only one Delone polytope up to translations and antipodality. So both PQFs are topologically non-degenerate by the previous lemma.

If n equals three, there are five types of Delone subdivisions (due to the Russian crystallographer E.S. Fedorov, see also Vallentin [33]). In all but the generic case one can check the following facts by inspection and elementary hand calculation: For every Delone subdivision which is not a triangulation there is a inhomogeneous eutactic PQF in which the Delone polytopes attaining the maximum circumradius are equivalent up to translations and antipodality. So we can apply the previous lemma, showing that these four points are topologically non-degenerate. In the generic case, where the subdivision is a triangulation, there is a PQF (associated to the lattice A_3^*) where H attains a local minimum.

If n equals four, we consider the PQF which corresponds to the root lattice D_4. It is inhomogeneous eutactic. There are three translation classes of Delone polytopes D_1, D_2, D_3 which are all regular cross polytopes realizing the circumradius. Their linear subspaces $\operatorname{lin} \Delta(D_i)$ are not contained in each other, so by the preceding lemma the PQF is not topologically non-degenerate.

For n greater than four, we take the PQF which corresponds to the lattice $\mathsf{D}_4 \times \mathbb{Z}^{n-4}$. □

7. An infinite series of inhomogeneous extreme forms

In this section we construct a series of inhomogeneous extreme forms for dimensions $n \geq 6$. The first two PQFs in the series correspond to the lattices E_6 and E_7. These PQFs were originally introduced in [15].

For giving the construction and for its analysis it is convenient not to work with the standard lattice but with the lattice L_n which is spanned by the root lattice $(\mathsf{D}_{n-1}, 0)$ and the vector $(-1/2, (1/2)^{n-2}, 1)$. It comes with the PQF

$$Q_n[x] = \begin{cases} x_1^2 + \cdots + x_{n-1}^2 + (n-3)/4x_n^2 & \text{if } n \text{ even,} \\ x_1^2 + \cdots + x_{n-1}^2 + (n-5)/4x_n^2 & \text{if } n \text{ odd} \end{cases}$$

We denote this pair by $[L_n, Q_n]$. We have $|\operatorname{Aut}([L_n, Q_n])| = |\operatorname{Aut}(\mathsf{D}_{n-1})|$.

Theorem 7.1. *For $n \geq 6$, the lattice $[L_n, Q_n]$ is a covering maxima.*

The main step of the computation is to prove that the big Delone polytope P_n defined in the next section is the only one attaining the maximum circumradius. In order to show this we enumerate all Delone polytopes up to symmetry. We shall prove that our list is complete by a volume argument.

In the remaining part of this section will be used to give a proof of the theorem which is largely computational. The idea of the proof is based on the algorithms given in [18] which are implemented in [38].

In the proof we heavily rely on the computation of volumes of polyhedra: Let P be a non-necessarily full dimensional polytope of $\mathbb{R}^n$. By $\operatorname{vol}(P)$ we denote the volume of P for the volume form induced by the scalar product on the affine space $\operatorname{aff}(P)$ defined by P. If $v \notin \operatorname{aff}(P)$, we will then have the relation

$$\operatorname{vol}(\operatorname{conv}(P, v)) = \frac{1}{\dim(\operatorname{conv}(P, v))} \operatorname{dist}(v, \operatorname{aff}(P)) \operatorname{vol}(P), \tag{2}$$

where $\operatorname{conv}(P, v)$ denotes the convex hull of the polytope P and the point v, and where $\operatorname{dist}(v, \operatorname{aff}(P))$ denotes the Euclidean distance between v and $\operatorname{aff}(P)$. An easy consequence of this formula is that if $\operatorname{aff}(P)$ is a hyperplane of dimension $n-1$ defined by an affine equality $\phi(x) = 0$, then we have for $v, v' \notin \operatorname{aff}(P)$ the relation

$$\operatorname{vol}(\operatorname{conv}(P, v)) = \frac{|\phi(v)|}{|\phi(v')|} \operatorname{vol}(\operatorname{conv}(P, v')). \tag{3}$$

Relation (2) admits a generalization: If P, Q are a p-, q-dimensional polytopes, then the $1 + p + q$-dimensional polytope $P \times Q$ defined as

$$P \times Q = \operatorname{conv}((0, P, 0^q), (1, 0^p, Q))$$

has volume

$$\operatorname{vol}(P \times Q) = \operatorname{vol}(P) \operatorname{vol}(Q) \frac{p!q!}{(1 + p + q)!}. \tag{4}$$

In the following we use the notation

$$\frac{1}{2} H_n = \left\{ x \in \{0, 1\}^n : \sum_{i=1}^n x_i \text{ even} \right\}.$$

for the half cube.

7.1. **The big Delone polytope.** As we shall prove later, there is only one Delone polyope of $[L_n, Q_n]$ where the maximum circumradius is attained. It is the polytope P_n which is defined as follows. If n is even then P_n has the vertices

$$((1/2)^{n-1}, 1) \pm e_i, i = 1, \dots, n-1, \ \ ((1/2)^{n-1}, -1), \ \ (\frac{1}{2}H_{n-1}, 0),$$

If n is odd, then P_n has the vertices

$$((1/2)^{n-1}, \pm 1) \pm e_i, i = 1, \dots, n-1, \ \ (\frac{1}{2}H_{n-1}, 0).$$

The squared circumradius of P_n is

$$\mu_{P_n} = \begin{cases} (n-2)^2/(4(n-3)), & \text{if } n \text{ even,} \\ (n-1)/4, & \text{if } n \text{ odd.} \end{cases}$$

The center of P_n is

$$c_{P_n} = \begin{cases} ((1/2)^{n-1}, 1/(n-3)), & \text{if } n \text{ even,} \\ ((1/2)^{n-1}, 0), & \text{if } n \text{ odd.} \end{cases}$$

It is proved in [15] that P_n uniquely determines $[L_n, Q_n]$ if $n \geq 6$. So the quadratic function f_n corresponding to P_n is inhomogeneous perfect. It is also inhomogenous extreme:

Lemma 7.2. *The quadratic function f_n is inhomogeneous eutactic.*

Proof. The polytope P_n has three orbits of vertices if n is even which can be distinguished by considering the last coordinate: -1, 0, $+1$. Then, the following coefficients satisfy the eutaxy condition

$$\begin{aligned} a_{-1} &= (n-2)/(2n(n-3)^2), \\ a_0 &= ((n-2)(n^2-5n+2))/(2^{n-2}n(n-3)^2), \\ a_1 &= 2/(n(n-3)^2). \end{aligned}$$

The polytope P_n has only two orbits of vertices if n is odd which can be distinguished by considering the last coordinate: ± 1, 0. Then, the following coefficients satisfy the eutaxy condition

$$\begin{aligned} a_{\pm 1} &= 1/(4n(n-5)), \\ a_0 &= (n^2-6n+1)/(2^{n-2}n(n-5)). \end{aligned}$$

□

The lower bound on the volume of P_n will turn out to be tight.

Lemma 7.3. *The volume of P_n is at least V_n where*

$$\begin{aligned} V_n = {} & 2(n-1)\frac{1}{n(n-1)}\left(1 - \frac{2^{n-3}}{(n-2)!}\right) + 2^{n-2}2^{n-3}\frac{n-3}{n!} \\ & + \sum_{j=3}^{n-3} \frac{2^{n-2}(n-1)!}{(i+1)!2^{j-1}j!}(j! - 2^{j-1})\frac{n-j-1}{2n!} \\ & + \frac{2^{n-1}}{n!} + 2^{n-2}\frac{n-1}{2n!} + 2^{n-2}\frac{n-3}{2n!}, \end{aligned}$$

if n is even, and

$$V_n = 2(n-1)(n-2)\frac{4}{n(n-1)(n-2)}\left(1-\frac{2^{n-4}}{(n-3)!}\right) + 2^{n-1}\frac{n-1}{2n!}$$
$$+\sum_{j=3}^{n-4}\frac{2^{n-1}(n-1)!}{(i+1)!2^{j-1}j!}(j!-2^{j-1})\frac{n-j-1}{2n!}$$
$$+2^{n-1}2^{n-3}\frac{n-3}{n!}+2\frac{2^{n-1}}{n!}+2^{n-2}(n-1)\frac{n-4}{n!},$$

if n is odd.

Proof. Denote by $\mathcal{F}(P)$ the set of facets of P and by c the point $((1/2)^{n-1}, 0)$. We have

$$\mathrm{vol}(P_n) = \sum_{F\in\mathcal{F}(P_n)} \mathrm{vol}(\mathrm{conv}(F, c)).$$

Since c is invariant under the automorphism group of P_n, the above sum can be grouped by orbits of facets of P_n.

Below, we list the facets F of P_n. The first line gives the separating hyperplane, the second line contains a list of incident vertices, the third line contains the volume $\mathrm{vol}(\mathrm{conv}(F, c))$ and the last line contains the size of the orbits. We frequently make use of the transformation g defined by

$$g(x_1, x_2, \ldots, x_n) = (1 - x_1, x_2, \ldots, x_n).$$

- Facet F_1: a cross polytope
  - $\sum_{j=1}^{n-1} x_j + (n-5)/2x_n \geq 1$,
  - $g(e_j)$, $g(((1/2)^{n-1}, 1) - e_j)$ for $1 \leq j \leq n-1$,
  - $2^{n-3}/n!(n-3)$,
  - 2^{n-2}.
- Facet F_2: a cross polytope
  - $x_n \leq 1$,
  - $((1/2)^{n-1}, 1) \pm e_j$ for $1 \leq j \leq n-1$,
  - $2^{n-1}/n!$,
  - 1 if n even, 2 if n odd.
- Facet F_3: simplex
  - $\sum_{i=1}^{n-1} x_i + (n-3)/2x_n \geq 0$,
  - 0, $((1/2)^{n-1}, 1) - e_j$ for $1 \leq j \leq n-1$,
  - $(n-1)/2n!$,
  - 2^{n-2} if n even, 2^{n-1} if n odd.
- Facet F_4: only if n even
  - $2x_1 - x_n \geq 0$,
  - $((1/2)^{n-1}, 1)$, $(0, \frac{1}{2}H_{n-2}, 0)$ and $(-1/2, (1/2)^{n-2}, -1)$,
  - $1/n(n-1)\left(1 - 2^{n-3}/(n-2)!\right)$.
  - $2(n-1)$.
- Facet F_5: simplex, only if n even
  - $\sum_{i=1}^{n-1} x_i + (n-1)/2x_n \geq 1$,
  - $((1/2)^{n-1}, -1)$, $g(e_j)$ for $1 \leq j \leq n-1$,
  - $(n-3)/2n!$,
  - 2^{n-2}.
- Facet F_6: only if n odd

  - $x_1 + x_2 \geq 0$,
  - $((1/2)^{n-1}, \pm 1) - e_j$ for $j = 1, 2$, $(0, 0, \frac{1}{2}H_{n-3}, 0)$,
  - $4/(n(n-1)(n-2))\left(1 - 2^{n-4}/(n-3)!\right)$,
  - $2(n-1)(n-2)$.
- Facet F_7: simplex, only if n odd
  - $\sum_{i=1}^{n-2} x_i + (n-4)x_{n-1} \geq 1$,
  - $g(e_j)$ for $1 \leq j \leq n-2$, $((1/2)^{n-2}, -1/2, \pm 1)$,
  - $(n-4)/n!$,
  - $2^{n-2}(n-1)$.
- Facet $F_{i,j}$: for $i + j = n - 2$, $j \geq 3$ and $i \geq 1$ for n even, $i \geq 2$ for n odd
  - $\sum_{k=j+1}^{n-1} x_j + (1-i)/2x_n \geq 0$,
  - $(\frac{1}{2}H_j, 0^{i+1}, 0)$, $((1/2)^{n-1}, 1) - e_k$ for $j + 1 \leq k \leq n - 1$,
  - $\left(j! - 2^{j-1}\right)(n-j-1)/2(n!)$,
  - $(i+1)!2^{j-1}j!$.

□

7.2. **Proof of Theorem 7.1.** We only have to show that for every Delone polytope P of $[L_n, Q_n]$ which is not equivalent to P_n we have $\mu_P < \mu_{P_n}$.

We now construct the remaining classes of Delone polytopes of $[L_n, Q_n]$: If n is even we have one additional class and if n is odd we two additional classes.

- If $i+j = n-1$ and $3 \leq i \leq j$, we denote by $H_{i,j}$ the polytope with vertices
$$(\frac{1}{2}H_i, 0^{j-1}, 0), ((1/2)^i, (1/2)^j - g(\frac{1}{2}H_j), 1).$$
The size of the stabilizer is
$$|\operatorname{Stab}(H_{i,j})| = \begin{cases} 2^{i-1}i!2^{j-1}j!, & \text{if } i \neq j, \\ 2 \times 2^{i-1}i!2^{j-1}j!, & \text{if } i = j. \end{cases}$$
Using the formula for the product polytope we get
$$\operatorname{vol}(H_{i,j}) = \left(1 - \frac{2^{i-1}}{i!}\right)\left(1 - \frac{2^{j-1}}{j!}\right)\frac{i!j!}{n!} = (i! - 2^{i-1})(j! - 2^{j-1})\frac{1}{n!}.$$
We set $C = n-3$ if n is even and $C = n-5$ if n is odd. The center of $H_{i,j}$ is
$$c_{H_{i,j}} = ((1/2)^i, 0^j, \alpha), \text{ with } \alpha = \frac{C + j - i}{2C}.$$
The squared radius of the sphere around $H_{i,j}$ is
$$\mu_{H_{i,j}} = \frac{C^2 + 2C(n-1) + (j-i)^2}{16C} < \mu_{P_n}.$$
- If n is odd, then the simplex S_n with vertex set
$$0, (0^{n-1}, 2), ((1/2)^{n-1}, 1) - e_j, \text{ with } j = 1, \ldots, n-1,$$
is a Delone polytope. We have
$$\begin{aligned} |\operatorname{Stab}(S_n)| &= 2(n-1)!, \\ \operatorname{vol}(S_n) &= \frac{n-3}{n!}, \\ c_{S_n} &= ((1/(n-3))^{n-1}, 1), \\ \mu_{S_n} &= \frac{n-5}{4} + \frac{n-1}{(n-3)^2} < \mu_{P_n}. \end{aligned}$$

Now we finish the proof by a volume computation showing that our list of orbits is complete. Denote by $O(D_1), \ldots, O(D_r)$ the orbits of Delone polytope of $[L_n, Q_n]$ of representative D_i. On the one hand, we have

$$2 = \sum_{i=1}^{r} |O(D_i)| \operatorname{vol}(D_i).$$

One the other hand, we have the equality

$$2 = \sum_{i=1}^{\frac{n-2}{2}} |O(H_{i,j})| \operatorname{vol}(H_{i,j}) + 2V_n,$$

if n is even, and

$$2 = |O(S_n)| \operatorname{vol}(S_n) + \sum_{i=1}^{\frac{n-1}{2}} |O(H_{i,j})| \operatorname{vol}(H_{i,j}) + V_n,$$

if n is odd. This implies that $\operatorname{vol}(P_n) = V_n$ and that the list of orbits of Delone polytopes is complete. This finishes the proof of the theorem.

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M. Dutour Sikirić, Rudjer Bosković Institute, Bijenicka 54, 10000 Zagreb, Croatia
E-mail address: mdsikir@irb.hr

A. Schürmann, Universität Rostock, Institut für Mathematik, 18051 Rostock, Germany
E-mail address: achill.schuermann@uni-rostock.de

F. Vallentin, Delft Institute of Applied Mathematics, Technical University of Delft, P.O. Box 5031, 2600 GA Delft, The Netherlands
E-mail address: f.vallentin@tudelft.nl