

DELAUNAY POLYTOPES DERIVED FROM THE LEECH LATTICE

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ABSTRACT. Given a lattice L of \mathbb{R}^n , a polytope D is called a *Delaunay polytope* in L if the set of its vertices is $S \cap L$ where S is a sphere having no lattice points in its interior. D is called *perfect* if the only ellipsoid in \mathbb{R}^n that contains $S \cap L$ is exactly S .

For a vector v of the Leech lattice Λ_{24} we define $\Lambda_{24}(v)$ to be the lattice of vectors of Λ_{24} orthogonal to v . We studied Delaunay polytopes of $L = \Lambda_{24}(v)$ for $\|v\|^2 \leq 22$. We found some remarkable examples of Delaunay polytopes in such lattices and disproved a number of long standing conjectures. In particular, we discovered:

- (1) Perfect Delaunay polytopes of lattice width 4; previously, the largest known width was 2.
- (2) Perfect Delaunay polytopes in L , which can be extended to perfect Delaunay polytopes in superlattices of L of the same dimension.
- (3) Polytopes that are perfect Delaunay with respect to two lattices $L \subset L'$ of the same dimension.
- (4) Perfect Delaunay polytopes D for L with $|\text{Aut } L| = 6|\text{Aut } D|$: all previously known examples had $|\text{Aut } L| = |\text{Aut } D|$ or $|\text{Aut } L| = 2|\text{Aut } D|$.
- (5) Antisymmetric perfect Delaunay polytopes in L , which cannot be extended to perfect $(n+1)$ -dimensional centrally symmetric Delaunay polytopes.
- (6) Lattices, which have several orbits of non-isometric perfect Delaunay polytopes.

Among perfect Delaunay polytopes discovered by us many have large vertex-sets and sporadic simple groups as their isometry groups. Finally, we derived an upper bound for the covering radius of $\Lambda_{24}(v)^*$, which generalizes the Smith bound and we prove that it is met only by Λ_{23}^* , the best known lattice covering in \mathbb{R}^{23} .

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1. INTRODUCTION

Given an n -lattice $L \subset \mathbb{R}^n$, a sphere $S = S(c, r)$ of center $c \in \mathbb{R}^n$ and radius r is called an *empty sphere* for L if there is no $v \in L$ such that $\|v - c\| < r$. A polytope $D = D_L(c)$ in \mathbb{R}^n (not necessarily of full dimension) is called a *Delaunay polytope* in L if the set of its vertices $\text{vert } D$ is $S \cap L$ where S is an *empty sphere* for L centered at c . An n -dimensional Delaunay polytope D in L is *perfect* with respect to L (or, extreme, as in [11]) if every linear bijective transformation ϕ of \mathbb{R}^n that maps D onto a Delaunay polytope in $\phi(L)$ is a composition of a homothety and an isometry (see [11, 10, 16] for more details on the theory of perfect Delaunay polytopes). Perfect Delaunay polytopes were first studied by Erdahl in connection with his work on quantum mechanics of many electrons [19, 20].

The *norm* of a lattice vector is its squared length. A vector of L is *minimal* if it has the smallest non-zero norm. Denote by $\text{Min } L$ the set of minimal vectors of L . If $\{v_1, \dots, v_n\}$ is a basis of L and the coordinates of $\text{Min } L$ with respect to $\{v_1, \dots, v_n\}$ determine uniquely L up to isometries and homotheties, then L is called *perfect*. Perfect Delaunay polytopes are inhomogeneous analogs of perfect lattices. Perfect lattices have been studied for more than 100 years since their introduction (as perfect quadratic forms) by Korkin and Zolotarev [25] in 1873. The book by Martinet [27] contains a wealth of information about perfect lattices. Not as much is known about perfect Delaunay polytopes. Up to similarity the unit interval in \mathbb{Z}^1 and the Gosset polytope 2_{21} in E_6 are the only perfect Delaunay polytopes in dimension $n \leq 6$ [9]. We currently know a number of infinite series of perfect Delaunay polytopes [22, 23, 12], a few sporadic examples [10] related to highly symmetric lattices in dimensions 12-23, and a large number of examples in dimensions 7-9 [16]. A systematic study of thousands of 9-dimensional perfect Delaunay polytopes known prior to this paper showed certain uniqueness properties of these polytopes and their lattices. All of the known examples satisfied the following conditions:

- (1) A polytope can be perfect Delaunay only with respect to one lattice.
- (2) The vertex set of a perfect Delaunay polytope cannot be a proper subset of the vertex set of another perfect Delaunay polytope of the same dimension.
- (3) A lattice can have at most one isometry class of perfect Delaunay polytopes.
- (4) The lattice width of a perfect Delaunay polytope is 2.

- (5) An antisymmetric perfect Delaunay n -polytope uniquely determines a centrally symmetric perfect Delaunay $(n + 1)$ -polytope (see Section 4)
- (6) The isometry group of a perfect Delaunay polytope in a lattice L determines the linear isometry group of L : the latter is either isomorphic to $\text{Iso } D$ if D is centrally symmetric or is an extension of $\text{Iso } D$ with a group of size 2 if D is antisymmetric.

We set to find counterexamples for some of these properties and we have found counterexamples for all of them, which often have sporadic simple isometry groups. We have found them in lattices constructed as sections of the famous Leech lattice Λ_{24} , which plays a prominent role in geometry, algebra, and number theory (see, almost all chapters of [6]). More precisely, given $v \in \Lambda_{24}$ we define $\Lambda_{24}(v)$ to be the lattice of vectors of Λ_{24} orthogonal to v (see Section 2 for details). Other well-known lattices that are used in our constructions include the laminated lattices Λ_n [6, Chapter 6] and the lattice O_{23} of O'Connor and Pall [29], also known as the shorter Leech lattice [6, page 179]. As a byproduct of our research, we established the covering radius of Λ_{23}^* , the lattice holding the covering density record in dimension 23.

It is instructive to draw a parallel between perfect polytopes and perfect lattices for property (1) and (2) of the above list. A perfect lattice L may have the following property: there is a non-trivial centering $L \subset L'$ such that $\text{Min } L \subset \text{Min } L'$. For example, $D_8 \subset E_8$ with $\text{Min } D_8 \subsetneq \text{Min } E_8$ and $A_9 \subset A_9^2$ with $\text{Min } A_9 = \text{Min } A_9^2$ (see [8, 6]). In Section 3 we describe a perfect Delaunay polytope of $\Lambda_{23} = \Lambda_{24}(v_2)$ with 47104 vertices, which extends to a perfect Delaunay polytope with 94208 vertices in the index 2 superlattice O_{23} of Λ_{23} . Similarly, we find a perfect Delaunay polytope with 891 vertices in Λ_{22} , which remains a perfect Delaunay in an index 2 superlattice of Λ_{22} . Note that in dimension $n \leq 8$ the set of minimal vectors of a perfect lattice uniquely determines the lattice; furthermore, in dimension $n \leq 7$ such a set cannot be extended to the set of minimal vectors of a denser lattice.

The paper is organized as follows. Sections 1.1 and 1.2 introduce basic notions and terminology. Section 2 introduces the Leech lattice and the lattices $\Lambda_{24}(v)$, computes the covering radius of Λ_{23}^* and proves an upper bound on the covering radius of $\Lambda_{24}(v)^*$. Section 3 describes the obtained counterexamples to property (1)-(4), (6). Section 4 discusses the construction of perfect polytopes by lamination; finds a counterexample to property (5) and characterizes the cases for which the construction works.

1.1. Lattices. A *lattice* L is a subgroup of the vector space \mathbb{R}^n of the form $\mathbb{Z}v_1 + \cdots + \mathbb{Z}v_k$, where v_1, \dots, v_k are independent vectors. The determinant $\det L$ is defined as the k -dimensional volume of the parallelepiped

$$\{x_1v_1 + \cdots + x_kv_k : 0 \leq x_i \leq 1 \text{ for all } i\}.$$

For $\mathbb{K} = \mathbb{R}$ or \mathbb{Q} we denote by $L \otimes \mathbb{K}$ the set $\mathbb{K}v_1 + \cdots + \mathbb{K}v_k$. A lattice of rank k is called a k -dimensional lattice or simply a k -lattice. The full affine isometry group of a lattice L is denoted by $\text{Iso } L$ and its linear subgroup by $\text{Aut } L$. If L is a *sublattice* of L' , then L' is called a *superlattice* of L . The pair $L \subset L'$, where both lattices are of the same rank, is called a *centering* of L .

If $L \subset \mathbb{R}^n$ is a lattice then the dual lattice L^* is defined as follows:

$$L^* = \{x \in L \otimes \mathbb{R} \text{ such that for all } y \in L \text{ we have } \langle x, y \rangle \in \mathbb{Z}\}.$$

A lattice is called *integral* if $\langle x, y \rangle \in \mathbb{Z}$ for $x, y \in L$, i.e. if $L \subset L^*$. A lattice is called unimodular if $\det L = 1$. A lattice is *self-dual*, that is $L = L^*$, if and only if it is integral and unimodular.

If B^n is the n -dimensional unit ball centered at the origin and L is an n -dimensional lattice, then the *covering radius* $\text{cov } L$ is defined as follows:

$$\text{cov } L = \min\{\mu : L + \mu B^n \text{ covers } \mathbb{R}^n\}.$$

It is easy to see (see, e.g. [6, Section 2.1.3]) that $\text{cov } L$ is equal to the maximum circumradius of the Delaunay polytopes of L .

A $(n-1)$ -dimensional lattice L' of L is called *primitive* if $(L' \otimes \mathbb{R}) \cap L = L'$. A *lamination* of an n -dimensional lattice L is a partition of L of the form $\cup_{k \in \mathbb{Z}} (L' + kv)$, where L' is a primitive $(n-1)$ -dimensional sublattice $L' \subset L$ and v is a vector in $L \setminus L'$. Set $\Lambda_0 = \mathbb{Z}^0$. A laminated n -lattice Λ_n is defined, up to similarity, as the densest n -lattice that has a lamination $\cup_{k \in \mathbb{Z}} (L' + kv)$, with L' isometric to Λ_{n-1} [6, Chapter 6]. In general Λ_n is not unique, but it is unique in dimensions 22–24. The lattice Λ_{24} has 196560 minimal vectors and is known as *the Leech lattice*. The Leech lattice is integral and unimodular and therefore self-dual. The lattices Λ_{23} , Λ_{23}^* have 93150, respectively 4600, minimal vectors of norm 4, respectively 3. The 4600 minimal vectors of Λ_{23}^* generate an index 2 sublattice called O_{23} . The lattice O_{23} is known as the *shorter Leech lattice*, as it was constructed by John Leech from the 23-dimensional Golay code (after O'Connor and Pall's work). The lattice O_{23} is integral and unimodular and therefore self-dual. The lattice Λ_{23} is an index 2 sublattice of O_{23} ; thus, Λ_{23} is an index 4 sublattice of Λ_{23}^* . More precisely, $\Lambda_{23}^*/\Lambda_{23} = \mathbb{Z}/4\mathbb{Z}$.

1.2. Delaunay Polytopes in Lattices. Given a lattice $L \subset \mathbb{R}^n$ a point $x \in \mathbb{R}^n$ defines a (not necessarily full dimensional) *Delaunay polytope* $D_L(x)$ by

$$D_L(x) = \text{conv} \left\{ v \in L : \|x - v\| = \min_{w \in L} \|x - w\| \right\}.$$

Given a full dimensional Delaunay polytope D denote by $c(D)$ its center and by $L(D)$ the lattice it affinely generates, i.e. the lattice generated by the difference between vertices of D . If D is a Delaunay polytope with empty sphere $S(c, r)$ for $L = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_n$ (rank $L = n$) then the function

$$\begin{aligned} f_D : \mathbb{Z}^n &\rightarrow \mathbb{R} \\ x = (x_1, \dots, x_n) &\mapsto \left\| \sum_{i=1}^n x_i v_i - c \right\|^2 - r^2 \end{aligned}$$

is a polynomial of degree 2 such that $f_D(x) \geq 0$ for all $x \in \mathbb{Z}^n$ and $f_D(x) = 0$ if and only if $\sum_{i=1}^n x_i v_i$ is a vertex of D . The dimension of the cone of quadratic functions

$$C_D = \left\{ \begin{array}{l} f : f(x) \geq 0 \text{ for all } x \in \mathbb{Z}^n, \text{ and } f(x) = 0 \\ \text{if and only if } \sum x_i v_i \text{ is a vertex of } D \end{array} \right\}$$

is denoted by *perfrank* D and called the *perfection rank* of the Delaunay polytope D . D is perfect if and only if its perfection rank is 1. The isometry group of a Delaunay polytope $D = D_L(c)$ is denoted by $\text{Iso}_L D$. The subgroup $\text{Aut}_L D$ is the group of isometries of L preserving D . It can happen that $\text{Aut}_L D \neq \text{Iso}_L D$ but if $L(D) = L$ then we have equality.

For a Delaunay polytope D and a $(n - 1)$ -sublattice L' of L , the non-empty sections of $\text{vert } D$ by hyperplanes $L' + kv$ are called *laminae*. The lamination number (equal to the lattice width number plus 1, see [24]) of D is the minimum over all primitive $(n - 1)$ -sublattices L' of the number of laminae. We found perfect Delaunay polytopes with lamination number 5, while all the previously known ones had lamination number 3. [24] inquired about the possible width of Delaunay polytopes and conjectured that they cannot have large width. We expect that there exist Delaunay polytopes with arbitrarily high width.

Given a vector $v \in L \otimes \mathbb{Q}$, denote by $\text{den}(v)$ the least common denominator of its coordinate, i.e. the smallest integer $d > 0$ such that $dv \in L$. A Delaunay polytope D is either centrally-symmetric with respect to its circumcenter c , or *antisymmetric*, in which case for any $v \in \text{vert } D$ we have $2c - v \notin \text{vert } D$. Note that $\text{den}(c(D)) = 2$ if and only if D is centrally symmetric. From an antisymmetric perfect Delaunay polytope D , it is always possible to get a centrally symmetric Delaunay

polytope by stacking D and $v-D$ for a suitably chosen vector $v \notin L \otimes \mathbb{R}$. Often, by varying the vector v one can ensure that the sphere S around $\text{vert } D$ and $v - \text{vert } D$ contains other points of $L + \mathbb{Z}v$, in which case $S \cap (L + \mathbb{Z}v)$ is the vertex set of a centrally symmetric perfect Delaunay polytope. Many centrally symmetric perfect Delaunay polytopes were constructed in this way and it was open whether this method always produces a centrally symmetric perfect Delaunay polytope. We find an antisymmetric perfect Delaunay polytope for which the construction does not produce a centrally symmetric perfect Delaunay polytope and we characterize the cases where it does in Section 4.

We say that a finite, nonempty subset X in \mathbb{R}^n carries a *spherical t -design* if there is a similarity transformation mapping X to points on the unit sphere $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$ so that for the spherical measure $d\omega$ on S^{n-1} and for all polynomials $f \in \mathbb{R}[x_1, \dots, x_n]$ up to degree t we have

$$\frac{1}{|Y|} \sum_{y \in Y} f(y) = \frac{1}{\omega(S^{n-1})} \int_{S^{n-1}} f(y) d\omega(y).$$

The maximal possible t is called the *strength* of the design (see, for example, [32] for more details). A Delaunay polytope defines a 0-design on its empty sphere and it is a 1-design if and only if its circumcenter is equal to its barycenter. In Section 3 we find many perfect t -designs associated to perfect Delaunay polytopes.

The algorithms for Delaunay polytopes used in this study are described in [13] and reference therein and an implementation is available from [33]. The algorithm for enumerating index 2 sublattices and superlattices is exposed in [1].

2. THE LATTICES $\Lambda_{24}(v)$

The *Leech lattice* Λ_{24} is a remarkable 24-lattice, which can be characterized as the unique 24-dimensional self-dual lattice whose non-zero vectors have norm at least 4. In addition Λ_{24} is *even*, i.e. every $v \in \Lambda_{24}$ has even norm. The symmetry group $\text{Aut } \Lambda_{24}$ is the Conway group Co_0 of order 8315553613086720000. Let us say that a vector $v \in \Lambda_{24}$ has type n if it is of norm $2n$ and has type $n_{a,b}$ if it is also the sum of two vectors of types a and b . $\text{Aut } \Lambda_{24}$ is transitive on the vectors of following types:

$$2, 3, 4, 5, 6_{2,2}, 6_{3,2}, 7, 8_{2,2}, 8_{3,2}, 8_{4,2}, 9_{3,3}, 9_{4,2}, 10_{3,3}, 10_{4,2}, 10_{5,2}, 11_{4,3}, 11_{5,2}$$

and this exhausts the list of vectors of norm at most 22 [6, Section 10.3.3]. By v_n , respectively $v_{n;a,b}$, we denote a vector of type n , respectively $n_{a,b}$. Note that $2v_2$ is of type $8_{2,2}$.

Let v be a primitive vector of the Leech lattice Λ_{24} , i.e. one such that $\frac{v}{d} \notin \Lambda_{24}$ for all natural $d > 1$. Then, we define

$$\Lambda_{24}(v) = \{x \in \Lambda_{24} \text{ such that } \langle x, v \rangle = 0\}$$

and we have

$$\Lambda_{24}(v)^*/\Lambda_{24}(v) \simeq \mathbb{Z}/\|v\|^2\mathbb{Z}.$$

Denote by $\text{Stab}(v)$ the stabilizer of $v \in \Lambda_{24}$ under $\text{Aut } \Lambda_{24}$; the stabilizers of the 17 vectors $v_n, v_{n;a,b}$ are given in [6, Table 10.4] and many of them involve sporadic simple groups. Any $d \in \mathbb{N}$ dividing $\|v\|^2$ determines uniquely a lattice $\Lambda_{24}(v, d)$ with

$$\Lambda_{24}(v) \subset \Lambda_{24}(v, d) \subset \Lambda_{24}(v)^* \quad \text{and} \quad [\Lambda_{24}(v, d) : \Lambda_{24}(v)] = d.$$

For any such d there is a corresponding integral representation of $\text{Stab}(v)$. We have $\Lambda_{24}(v, 1) = \Lambda_{24}(v)$ and $\Lambda_{24}(v, \|v\|^2) = \Lambda_{24}(v)^*$.

The lattice $\Lambda_{24}(v_2)$ is known as the laminated lattice of dimension 23, Λ_{23} . The lattice $\Lambda_{24}(v_2, 2)$, denoted by O_{23} , is self-dual and is known as the *shorter Leech lattice*, as it was constructed by John Leech from the 23-dimensional Golay code (after O'Connor and Pall's work). The stabilizer $\text{Stab}(v_2)$ is the sporadic group Co_2 . The group $\mathbb{Z}_2 \times Co_2$ has three (equivalent over \mathbb{Q}) integral representations in $\text{GL}_{23}(\mathbb{Z})$ as $\text{Aut } \Lambda_{23}$, $\text{Aut } \Lambda_{23}^*$ and $\text{Aut } O_{23}$. Those three representations correspond to the integral 23-dimensional representations of Co_2 enumerated in [30].

Theorem 1. *Let v be a primitive vector of the Leech lattice and define*

$$r(v) = \sqrt{2 - \frac{1}{4\|v\|^2}}.$$

(i) *The covering radius of the lattice $\Lambda_{24}(v)^*$ is at most $r(v)$;*

(ii) *the only lattice $\Lambda_{24}(v)^*$ with covering radius $r(v)$ is $\Lambda_{23}^* = \Lambda_{24}(v_2)^*$.*

The Delaunay polytopes of maximal circumradius of Λ_{23}^ belong to a single orbit of Delaunay polytopes, whose representatives have 64 vertices and 2688 symmetries.*

Proof. Denote by p_v the orthogonal projection operator of Λ_{24} onto $\Lambda_{24}(v) \otimes \mathbb{R}$. It is proved in the appendix of [27] that the dual lattice $\Lambda_{24}(v)^*$ is equal to the projection $p_v(\Lambda_{24})$ of Λ_{24} . We suppose that the covering radius of $\Lambda_{24}(v)^*$ is strictly greater than $r(v)$, that is that there exists a vector $w \in \Lambda_{24}(v) \otimes \mathbb{R}$ such that for every $x \in \Lambda_{24}(v)^*$ we have $\|x - w\| > r(v)$. Define $v' = \frac{1}{\|v\|^2}v$ and $w' = w + \frac{1}{2}v'$. For every $y \in \Lambda_{24}$ set

$$h_y = y - w' = p_v(y - w') + \alpha v' = (p_v(y) - w) + \alpha v',$$

where

$$\alpha = \langle v, h_y \rangle = -\frac{1}{2} + \langle v, y \rangle \in \frac{1}{2} + \mathbb{Z}.$$

Thus we get

$$(1) \quad \begin{aligned} \|h_y\|^2 &= \|p_v(y) - w\|^2 + \alpha^2 \|v'\|^2 \\ &> r(v)^2 + \alpha^2 \|v'\|^2 \geq 2. \end{aligned}$$

The inequality $\|h_y\|^2 > 2$ contradicts the fact that the covering radius of Λ_{24} is $\sqrt{2}$ (see [5], [6, Chapter 23]).

Suppose now that D is a Delaunay polytope of $\Lambda_{24}(v)^*$ of center c and circumradius $r(v)$. Define $c' = c + \frac{1}{2}v'$. By an argument similar to (i) we get that $\|y - c'\|^2 \geq 2$ for every $y \in \Lambda_{24}$. Thus c' is the center of a Delaunay polytope D' of Λ_{24} of circumradius $\sqrt{2}$. Define $f(x) = \langle x, v \rangle$. We have $f(c') = \frac{1}{2}$ and $f(v) \in \mathbb{Z}$ for $v \in \text{vert } D' \subset \Lambda_{24}$.

If $v \in \text{vert } D'$ then we have

$$\|p_v(v) - c\|^2 = 2 - \frac{1}{\|v\|^2} \left(\frac{1}{2} - f(v) \right)^2.$$

Thus in order for D to be a Delaunay polytope of circumradius $r(v)$, it is necessary that $f(v) = 0$ or 1 for $v \in \text{vert } D'$. So D' has lamination number 2 and the vector v is defined up to a scalar multiple by the corresponding 2-lamination. For a n -dimensional polytope P a 2-lamination in two layers L_0, L_1 corresponds to a partition of $\text{vert } P$ in two subsets P_0 and P_1 . If $\mathcal{S} = \{v_1, \dots, v_{n+1}\}$ is a set of $n+1$ independent vertices of P then the possible partitions $\{P_0, P_1\}$ are determined by the intersections $\mathcal{S} \cap P_0$. Thus there are at most $2^{n+1} - 2$ 2-laminations on P and they can be enumerated by considering all subsets of an independent set \mathcal{S} of $\text{vert } P$ and checking if they correspond to a partition $\{P_0, P_1\}$. The list of 23 types $D_{\Lambda_{24}}(c_1), \dots, D_{\Lambda_{24}}(c_{23})$ of Delaunay polytopes of Λ_{24} of circumradius $\sqrt{2}$ is known (see [5], [6, Chapter 23]). Given a polytope $D_{\Lambda_{24}}(c_i)$ we enumerate its 2-laminations; determine the possible vectors v ; keep the ones such that the projection $p_v(c_i)$ determines a Delaunay polytope of $\Lambda_{24}(v)^*$ of circumradius $r(v)$. It turns out that, up to equivalence, only one such vector v satisfies the required conditions. This vector is v_2 and so Λ_{23}^* is the only lattice meeting the bound. \square

The lower bound of the above theorem was proved in [31] for the lattice Λ_{23}^* . The Delaunay polytope of Λ_{24} that determines the Delaunay polytope of Λ_{23}^* of maximal circumradius is named A_3^8 [6, Chapter 23].

The only general method for computing the covering radius of a lattice is to compute the full Delaunay tessellation. For Λ_{23} , respectively O_{23} , there are 709, respectively 5, orbits of Delaunay polytopes [13].

For Λ_{23}^* the same program yields several hundred thousands of orbits before the computation could terminate.

3. MAIN DELAUNAY POLYTOPES OF $\Lambda_{24}(v)$

The set $\text{Min } \Lambda_{24}$ consists of 196560 minimal vectors. For a given vector $v \in \Lambda_{24}$ and $\alpha \in \mathbb{Z}$, define $\text{Min}_{\alpha,v} \Lambda_{24} = \{x \in \text{Min } \Lambda_{24} : \langle x, v \rangle = \alpha\}$. The following facts are easy to check:

- (i) If $\text{Min}_{0,v} \Lambda_{24} \neq \emptyset$ then $\text{Min } \Lambda_{24}(v) = \text{Min}_{0,v} \Lambda_{24}$.
- (ii) If $\alpha \neq 0$ and $\text{Min}_{\alpha,v} \Lambda_{24}$ is of rank n , then it defines a Delaunay polytope of $\Lambda_{24}(v)$ of dimension $n-1$ (see [11, Lemma 13.2.11]).

We call Delaunay polytopes obtained by this method *main*. If a Delaunay polytope is main then its stabilizer in $\Lambda_{24}(v)$ contains $\text{Stab}(v)$. Obviously, there is a finite number of main perfect Delaunay polytopes but we are not able at this point to determine the complete list. Therefore we limit ourselves to v from the first 17 types. We denote by $D(v, N)$ the main Delaunay polytopes of $\Lambda_{24}(v)$ with N vertices since this notation does not have ambiguity in the cases considered here. In Table 3 we list the informations about the full dimensional main perfect Delaunay polytopes in the second column for the 16 vector types in first column (The 17 types, except $8_{2,2}$, which is covered by type 2) and their possible extension in lattices $\Lambda_{24}(v, d)$. For every main Delaunay D and d for which it admits an extension in $\Lambda_{24}(v, d)$, we give the number N of vertices, the denominator $den = \text{den}(c(D))$ of the circumcenter $c(D)$, the strength s of the spherical t -design and the index ind of $L(D)$ in $\Lambda_{24}(v, d)$ by the symbol “ $d: (N, den, s, ind)$ ”.

The remarkable centrally symmetric perfect Delaunay $D(v_3, 552)$ was first identified in [10], it defines 276 equiangular lines [26], it is universally optimal [4] and it gives the facet of maximal incidence of the contact polytope of Λ_{24} [18]. It was noted in [11] that a 22-dimensional antisymmetric perfect Delaunay with 275 vertices is included in $D(v_3, 552)$. This polytope is $D(v_5, 275)$ and the lattice L_{22} , which it affinely generates belongs to the \mathbb{Q} -class of lattices of the irreducible finite subgroup $(C_2 \times Mc).C_2$ of $\text{GL}_{22}(\mathbb{Z})$ [28]. The polytopes $D(v_5, 275)$ and $D(v_3, 552)$ define spherical t -design of strength 4, respectively 5 just like Gosset’s 2_{21} and 3_{21} , which are perfect Delaunay polytopes in \mathbf{E}_6 and \mathbf{E}_7 . The set $\text{Min } \Lambda_{23}(v_3)^*$ is equivalent to $D(v_3, 552)$ and the set $\text{Min } L_{22}^*$ is equivalent to $D(v_5, 275) \cup (2c(D(v_5, 275)) - D(v_5, 275))$. Similarly 3_{21} is equivalent to $\text{Min } \mathbf{E}_7^*$ and $\text{Min } \mathbf{E}_6^*$ is equivalent to $2_{21} \cup (2c(2_{21}) - 2_{21})$.

Many lattices $\Lambda_{24}(v)$ have several orbits of perfect Delaunay polytopes. No such example is known in dimension $n \leq 9$. It turns out that

for a given vector v of the 17 cases the strength of the spherical t -design is always the same for all main full dimensional Delaunay polytopes. In particular, for vectors of type 2, 3, 4, and $6_{3,2}$ the main Delaunay polytopes P define spherical t -designs for $t = 7, 5, 3$ and 2. Our proof was obtained by direct computation and it would be interesting to have a less computational proof, for example, using modular forms in the spirit of the theory of strongly perfect lattices explained in [32].

The 22-dimensional Delaunay cell $D(v_{6;2,2}, 891)$ affinely generates the 22-dimensional lattice Λ_{22} . A remarkable fact is that $|\text{Aut } \Lambda_{22}| = 6|\text{Aut } D(v_{6;2,2}, 891)|$. For all other known perfect Delaunay polytopes D in a lattice L we have $|\text{Aut } L| = |\text{Aut } D|$ if D is centrally symmetric and $|\text{Aut } L| = 2|\text{Aut } D|$ if D is antisymmetric.

The perfect main Delaunay polytopes $D(v_3, 552)$, $D(v_3, 11178)$ and $D(v_{6;2,2}, 891)$ admit index 3, 2 and 2 superlattices $\Lambda_{24}(v_3, 3)$, $\Lambda_{24}(v_3, 2)$ and $\Lambda_{24}(v_{6;2,2}, 2) \cap \Lambda_{22} \otimes \mathbb{R}$ in which they are still Delaunay polytopes with the same vertex-set. Note that in [15] we obtained Delaunay polytopes with the same property by a different method. Table 3 lists 5 perfect Delaunay polytopes $D(v, N)$ that admit superlattices $\Lambda_{24}(v, d)$ in which these polytopes are proper subsets of perfect Delaunay polytopes. The first, respectively second, phenomenon is a direct analog of the relation between the set of minimal vectors of the pair of perfect lattices $A_9 \subset A_9^2$ and $D_8 \subset E_8$.

All the found counterexamples were long suspected to exist. A technique, which we previously used to no avail for this task was to get more than 85000 perfect Delaunay polytopes in dimension 9 by using the algorithm of [17]. It seems that such counterexamples exist only in high dimension.

A Delaunay polytope D in a lattice L is called *basic* if there are $n + 1$ vertices v_0, v_1, \dots, v_n of D such that for every vertex v of D there exists $\lambda_i \in \mathbb{Z}$ such that $v = \sum_{i=0}^n \lambda_i v_i$ and $1 = \sum_{i=0}^n \lambda_i$. A non-basic Delaunay polytope is given in [14] and a lattice L with $\text{Min } L$ generating L but with no basis of minimal vectors is given in [7]. We do not know any perfect non-basic Delaunay polytopes; finding one would be extremely difficult. The corresponding homogeneous problem (whether there exists a perfect lattice which is spanned by its minimal vectors, but where there is no basis of minimal vectors) is also open.

3.1. Lamination numbers. Perfect Delaunay polytopes have lamination number at least 3 [16, Theorem 10] and it is conjectured [3] that a n -dimensional polytope, whose vertices belong to a lattice L and is free of lattice point in its interior, has lamination number at most $n + 1$. All known perfect Delaunay polytopes in dimensions 6, 7,

Type of v	Main Delaunay polytopes	
2	1: (47104, 4, 7, 1), 2: (94208, 2, 7, 1) 1: (4600, 2, 7, 1)	
3	1: (48600, 6, 5, 1) 1: (11178, 3, 5, 1), 2: (11178, 3, 5, 2) 1: (552, 2, 5, 1), 3: (552, 2, 5, 3)	
4	1: (47104, 8, 3, 1) 1: (16192, 4, 3, 1), 2: (32384, 2, 3, 1)	
5	1: (45100, 10, 1, 1)	1: (19450, 5, 1, 1)
$6_{2,2}$	1: (22518, 6, 1, 1)	
$6_{3,2}$	1: (43056, 12, 2, 1) 1: (21528, 6, 2, 1) 1: (6072, 4, 2, 1), 2: (12144, 2, 3, 1)	
7	1: (41152, 14, 0, 1) 1: (7900, 14, 0, 1)	1: (22825, 7, 0, 1)
$8_{3,2}$	1: (24576, 8, 0, 1), 2: (47104, 4, 7, 1), 4: (94208, 2, 7, 1)	
$8_{4,2}$	1: (39424, 16, 1, 1) 1: (23608, 8, 1, 1) 1: (9472, 16, 1, 1) 1: (2268, 4, 1, 1), 2: (4536, 2, 1, 1)	
$9_{3,3}$	1: (37908, 18, 1, 1) 1: (10758, 6, 1, 1)	1: (14057, 9, 1, 1) 1: (3159, 9, 1, 1)
$9_{4,2}$	1: (37743, 18, 0, 1) 1: (24035, 9, 0, 1) 1: (10879, 6, 0, 1), 3: (32384, 2, 3, 1)	
$10_{3,3}$	1: (25300, 10, 1, 1)	1: (4325, 5, 1, 1)
$10_{4,2}$	1: (25036, 10, 0, 1)	1: (3489, 5, 0, 1)
$10_{5,2}$	1: (36454, 20, 0, 1) 1: (11882, 20, 0, 1)	1: (24266, 10, 0, 1) 1: (3993, 5, 0, 1)
$11_{4,3}$	1: (35200, 22, 0, 1) 1: (12760, 22, 0, 1)	1: (24332, 11, 0, 1) 1: (4832, 11, 0, 1)
$11_{5,2}$	1: (34782, 22, 0, 1) 1: (13122, 22, 0, 1)	1: (24200, 11, 0, 1)

TABLE 1. First column gives the 17 types of vector of Λ_{24} of norm at most 22 except $8_{2,2}$. The entries “ $d: (N, den, s, ind)$ ” in second column correspond to a main Delaunay polytope D of $\Lambda_{24}(v, d)$ with N vertices, denominator of circumcenter $den = den(c(D))$, strength s of t -design and index ind of $L(D)$ in $\Lambda_{24}(v, d)$

8 have lamination number 3. We will give examples of perfect Delaunay polytopes with lamination number 5. Note that there is no general efficient method known for determining the lamination number of a given polytope.

Theorem 2. *The polytopes $D(v_3, 48600)$ and $D(v_3, 11178)$ have lamination number 5.*

Proof. Let us assume that $l(D) \leq 4$ for $D = D(v_3, 48600)$ or $D(v_3, 11178)$. This means that we can find a 22-dimensional sublattice L' and four vectors w_1, w_2, w_3, w_4 such that the layers $L_i = w_i + L'$ cover vert D .

We checked with a computer that $L(D) = \Lambda_{24}(v_3)$, i.e., the difference vectors of vert D generate $\Lambda_{24}(v_3)$. There exists a linear function f on $\Lambda_{24}(v_3)$ such that $L' = \ker f$ and $f(\Lambda_{24}(v_3)) = \mathbb{Z}$. We define an index 2 sublattice

$$L'_2 = \{w \in \Lambda_{24}(v_3) \text{ such that } f(w) \in 2\mathbb{Z}\}$$

of $\Lambda_{24}(v_3)$ and take $w \in \Lambda_{24}(v_3)$ such that $f(w) = 1$. It is not possible for L'_2 or $w + L'_2$ to contain all four layers $w_i + L'$ since if it were so, then D would not be generating. So, $L'_2 \cap D$ or $(w + L'_2) \cap D$ contains at most 2 layers. If one of them contains just one layer, then it is of dimension at most 22. By enumerating all index 2 sublattices of $\Lambda_{24}(v_3)$ we found that $L'_2 \cap D$ and $(w + L'_2) \cap D$ are always 23-dimensional. So, $L'_2 \cap D$ and $(w + L'_2) \cap D$ are contained in two layers and thus have lamination number 2. We enumerated their 2-laminations by using the same method as in Theorem 1 and found that each 2-lamination of $L'_2 \cap D$, respectively $(w + L'_2) \cap D$, induces a lamination of $(w + L'_2) \cap D$, respectively $L'_2 \cap D$ with at least three layers. So, the lamination number of D is 5. \square

4. CONSTRUCTION OF PERFECT DELAUNAY POLYTOPES BY LAMINATION

In [11, Lemma 15.3.7], [23], [16], [21] the following construction of centrally symmetric Delaunay $(n + 1)$ -polytopes from Delaunay n -polytopes is discussed: take a Delaunay n -polytope D of circumcenter c in an n -dimensional lattice $L \subset \mathbb{R}^{n+1}$ and take its inverted copy $e_{n+1} - D$ in $L + e_{n+1}$ for a vector $e_{n+1} \in \mathbb{R}^{n+1}$. In order for vert D and $e_{n+1} - \text{vert } D$ to lie on a common sphere S it is necessary and sufficient that $c - (e_{n+1} - c)$ is orthogonal to $L \otimes \mathbb{R}$. So, up to isometry, e_{n+1} is determined by the distance between the layer L and the layer $L + e_{n+1}$. Thus for all $\delta \geq 0$ we define the lattice $L(\delta) = L + \mathbb{Z}e_{n+1}$ with δ the square Euclidean distance between the layers L and $L + e_{n+1}$ and $2c - e_{n+1}$ orthogonal to L .

The theorem below clarifies for which Delaunay polytopes D some other vectors of $L(\delta)$ can lie on S and for which Delaunay D this cannot happen.

Theorem 3. *Let D be a Delaunay polytope in a n -dimensional lattice L of center c . For $i \in \mathbb{Z}$, define $D_i = D_L((1 - 2i)c)$ and denote by r_i the common distance between $(1 - 2i)c$ and vertices of D_i . Either:*

- (i) *For all i , $r_i \geq r_0$. Then $L(0)$ is an index 2 superlattice of L such that $D' = D_{L(0)}(c)$ is a centrally symmetric Delaunay n -polytope containing $D \cup (2c - D)$ with $\text{perfrank } D' \leq \text{perfrank } D$.*
- (ii) *Or there exists i such that $r_i < r_0$. Then there exists $\delta_s > 0$ such that $D' = D_{L(\delta_s)}(c')$ with $c' = \frac{1}{2}e_{n+1}$ is a centrally symmetric Delaunay $(n+1)$ -polytope containing $D \cup (2c' - D)$ with $\text{perfrank } D' \leq \text{perfrank } D$.*

Proof. Define

$$r_i(\delta) = \sqrt{r_i + \delta \left(i - \frac{1}{2}\right)^2}$$

and $c' = \frac{1}{2}e_{n+1}$. The sphere circumscribing D and $e_{n+1} - D$ is $S(c', r_0(\delta))$ and we have $r_0(\delta) = r_1(\delta)$. For $i \in \mathbb{Z}$, the set of closest points in layer $L + ie_{n+1}$ to c' is

$$S_i(c) = ie_{n+1} + \text{vert } D_L((1 - 2i)c)$$

and the common distance to c' is $r_i(\delta)$. If there exists an index i such that $r_i < r_0$ then there exists $\delta_i > 0$ such that $r_i(\delta_i) = r_0(\delta_i)$ and $S_i(c)$ is outside of $S(c', r_0(\delta))$ if and only if $\delta \geq \delta_i$. If one takes $\delta_s = \max_{i \in \mathbb{Z}} \delta_i$ then $S(c', r_0(\delta))$ is an empty sphere if and only if $\delta \geq \delta_s$ and $D' = S(c', r_0(\delta)) \cap L(\delta)$ has more than two layers if and only if $\delta = \delta_s$. In that case $L(\delta_s)$ is determined by L and thus $\text{perfrank } D' \leq \text{perfrank } D$.

On the other hand, if for all i $r_i \geq r_0$ then $\delta_s = 0$ and $L(0)$ is actually an n -dimensional superlattice of L . We have $\text{perfrank } D' \leq \text{perfrank } D$ since D' has more vertices than D . \square

If a Delaunay polytope falls into case (i) then we say that this Delaunay polytope is of the *first type* and otherwise it is of the *second type*. Given a Delaunay polytope D of the first type we can define for $\delta > 0$ a polyhedron

$$D_{cyl} = \text{conv} \bigcup_{i \in \mathbb{Z} \text{ s.t. } r_i = r_0} (ie_{n+1} + \text{vert } D_i),$$

whose vertices lie in $L(\delta)$ and belong to a cylinder, empty of lattice points. It is proved in [20] that this infinite lattice polyhedron is in

fact arithmetically equivalent to the product $\text{vert } D' \times \mathbb{Z}$ where D' is the n -dimensional Delaunay polytope of case (i).

Corollary 4. *Take D a Delaunay polytope of a lattice L of center c .*

(i) *If $\text{den}(c) = 2$ or 4 then D is of first type.*

(ii) *If $\text{den}(c)$ is odd then D is of second type.*

Proof. If $\text{den}(c) = 2$ then D is centrally symmetric and thus of the first type. If $\text{den}(c) = 4$, then D is asymmetric and therefore $-D$ is also a Delaunay polytope. Thus when $\text{den}(c) = 4$ there is a Delaunay polytope centered at $3c$; $r_i = r_0$ for all $i \in \mathbb{Z}$ and by Theorem 3 D is of first type. If $\text{den}(c)$ is odd then there exists an index i such that $(1 - 2i)c \in L$ and thus $r_i = 0$. So, by Theorem 3 D is of second type. \square

Let us give two examples, from [11] of the situation where $\text{den}(c(D))$ are odd. The perfect Delaunay polytopes 2_{21} and $D(v_5, 275)$ have $\text{den}(c(2_{21})) = 3$, respectively $\text{den}(D(v_5, 275)) = 5$. Thus they are of second type and the higher dimensional centrally symmetric Delaunay polytope they define are 3_{21} and $D(v_3, 552)$.

It was an interesting open question whether there exist Delaunay polytopes of first type, which are antisymmetric. This question was bypassed in [11, Lemma 15.3.7] by assuming that D is antisymmetric and of second type and it was left open in [23]. The polytopes $D(v_2, 47104)$, $D(v_4, 16192)$ and $D(v_{6,3,2}, 6072)$ have $\text{den}(c(D)) = 4$ and thus are of first type and antisymmetric.

Using the method of [17], we obtained 85000 perfect Delaunay polytopes in dimension 9. All the ones of first type were centrally symmetric. All the centrally symmetric ones were obtained by the construction of Theorem 3 but we think that this is not the case in a large enough dimension.

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