# WYTHOFF POLYTOPES AND LOW-DIMENSIONAL HOMOLOGY OF MATHIEU GROUPS

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ABSTRACT. We describe two methods for computing the low-dimensional integral homology of the Mathieu simple groups and use them to make computations such as $H_5(M_{23}, \mathbb{Z}) = \mathbb{Z}_7$ and $H_3(M_{24}, \mathbb{Z}) = \mathbb{Z}_{12}$. One method works via Sylow subgroups. The other method uses a Wythoff polytope and perturbation techniques to produce an explicit free $\mathbb{Z}M_n$-resolution. Both methods apply in principle to arbitrary finite groups.

## 1. INTRODUCTION

We describe two methods for computing the integral homology for the Mathieu simple groups presented on Table 1. The first homology $H_1(G, \mathbb{Z})$ is trivial for any simple group and so is omitted from the table (see [3] for an exposition of relevant facts on group homology). The second homology of Mathieu groups is well-known [16]. A computer method for the second homology of a permutation group was illustrated on the Mathieu groups $M_{21}$ and $M_{22}$ in [15]. The mod $p$ cohomology $H^*(G, \mathbb{F}_p)$ is now known for all Mathieu groups except $M_{24}$ [21, 1, 2, 17]. With the help of the Bockstein spectral sequence it is, in principle, possible to obtain integral homology from mod $p$ cohomology ($p$ ranging over the prime divisors of the group order), though the details can be difficult. For example, the calculation of $H_n(M_{23}, \mathbb{Z})$ was obtained in this way for $1 \le n \le 6$ by Milgram [17] and provided the first example of a non-trivial finite group with trivial integral homology in dimensions $\le 3$. It seems that the mod $p$ cohomology of $M_{24}$ is not known for all primes $p$ (see [14] for the case $p = 3$) and so we can assign the status of a new theorem to the following result.

**Theorem 1.** *$H_3(M_{24}, \mathbb{Z}) = \mathbb{Z}_{12}$ and $H_4(M_{24}, \mathbb{Z}) = 0$.*

This result (and other table entries) can be obtained from the HAP homological algebra package [10] for the GAP computational algebra system [12] using (variants of) the following command.

```
gap> GroupHomology(MathieuGroup(24),3);
gap> [ 4, 3 ]
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The algorithm underlying this command is explained in Section 2. The current implementation is unable to determine the integers $a, b$ in Table 1 though it does establish the ranges $0 \le a \le 53$, $0 \le b \le 1$.

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| $G$ | $H_2(G,\mathbb{Z})$ | $H_3(G,\mathbb{Z})$ | $H_4(G,\mathbb{Z})$ | $H_5(G,\mathbb{Z})$ |
|---|---|---|---|---|
| $M_{11}$ | 0 | $\mathbb{Z}_8$ | 0 | $\mathbb{Z}_2$ |
| $M_{12}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_6 \oplus \mathbb{Z}_8$ | $\mathbb{Z}_3$ | $(\mathbb{Z}_2)^3$ |
| $M_{21}$ | $\mathbb{Z}_4 \oplus \mathbb{Z}_{12}$ | $\mathbb{Z}_5$ | 0 | $(\mathbb{Z}_2)^4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_7$ |
| $M_{22}$ | $\mathbb{Z}_{12}$ | 0 | 0 | $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_7$ |
| $M_{23}$ | 0 | 0 | 0 | $\mathbb{Z}_7$ |
| $M_{24}$ | 0 | $\mathbb{Z}_{12}$ | 0 | $(\mathbb{Z}_2)^a \oplus (\mathbb{Z}_4)^b \oplus \mathbb{Z}_7$ |

TABLE 1. Low dimensional homology of Mathieu groups with $0 \le a \le 53$ and $0 \le b \le 1$.

Abelian invariants of a (co)homology group are the easiest cohomological information to access. More difficult information would be, for example, explicit cocycles $G^n \to A$ corresponding to cohomology classes in $H^n(G, A)$. Explicit cocycles are constructed in HAP using the induced chain map $B_*^G \to R_*^G$ from the bar resolution $B_*^G$ to an explicit small free $\mathbb{Z}G$-resolution $R_*^G$ of $\mathbb{Z}$. In Sections 3-5 we explain how the Wythoff polytope construction can be used to produce such a resolution $R_*^G$. This resolution provides an alternative computation of $H_3(M_{24}, \mathbb{Z})$.

In Section 6 we determine the $p$-part $H_n(M_m, \mathbb{Z})_{(p)}$ of the integral homology of the Mathieu groups for $n \ge 1$ and primes $p \ge 5$. For $p \in \{5, 7, 11, 23\}$ the $p$-part is either trivial or $\mathbb{Z}_p$; it is trivial for all other primes $p \ge 5$. Table 2 lists the values of $n$ for which the $p$-part is non-trivial.

Although the paper focuses on Mathieu groups, the techniques are applicable in principle to arbitrary finite groups. In some cases the Wythoff polytopal method is a significantly faster method for computing the homology groups.

## 2. Algorithm underlying the HAP function

Given a group $G$, a *free $\mathbb{Z}G$-resolution* of the trivial module $\mathbb{Z}$ is an exact sequence

$$0 \leftarrow \mathbb{Z} \leftarrow R_0^G \leftarrow R_1^G \leftarrow \cdots \leftarrow R_k^G \leftarrow \dots$$

of free $\mathbb{Z}G$-modules $R_i^G$. A previous paper [9] describes an algorithm for computing free $\mathbb{Z}G$-resolutions for finite $G$. This has now been implemented as part of the HAP package. It takes as input a finite group $G$ and a positive integer $n$. It returns:

- The rank of the $k$th module $R_k^G$ in a free $\mathbb{Z}G$-resolution $R_*^G$ $(0 \le k \le n)$.
- The image of the $i$th free $\mathbb{Z}G$-generator of $R_k^G$ under the boundary homomorphism $d_k\colon R_k^G \to R_{k-1}^G$ $(1 \le k \le n)$.
- The image of the $i$th free $\mathbb{Z}$-generator of $R_k^G$ under a contracting homotopy $h_k\colon R_k^G \to R_{k+1}^G$ $(0 \le k \le n-1)$.

The contracting homotopies $h_k$ satisfy, by definition, $h_k d_{k+1} + d_{k+2} h_{k+1} = 1$ and need to be specified on a set of free Abelian group generators of $R_k$ since they are not $G$-equivariant.

| $H_n(M_m,\mathbb{Z})_{(p)}=\mathbb{Z}_p$ | $m=11$ | $m=12$ | $m=21$ | $m=22$ | $m=23$ | $m=24$ |
|---|---|---|---|---|---|---|
| | $n=$ | $n=$ | $n=$ | $n=$ | $n=$ | $n=$ |
| $p=5$ | $8k-1$ | $8k-1$ | $4k-1$ | $8k-1$ | $8k-1$ | $8k-1$ |
| $p=7$ | – | – | $6k-1$ | $6k-1$ | $6k-1$ | $6k-1$ |
| $p=11$ | $10k-1$ | $10k-1$ | – | $10k-1$ | $10k-1$ | $20k-1$ |
| $p=23$ | – | – | – | – | $22k-1$ | $22k-1$ |

Table 2. Values of $n$ expressed in term of $k \geq 1$ such that $H_n(M_m,\mathbb{Z})_{(p)}=\mathbb{Z}_p$

The homotopy can be used to make constructive the following frequent element of choice.

*For $x \in \ker(d_k\colon R^G_k \to R^G_{k-1})$ choose an element $\tilde{x} \in R^G_{k+1}$ such that $d_{k+1}(\tilde{x}) = x$.*

One sets $\tilde{x} = h_k(x)$. In particular, for any group homomorphism $\phi\colon G \to G'$, the homotopy allows one to define an induced $\phi$-equivariant chain map $\phi_*\colon R^G_* \to R^{G'}_*$.

The algorithm in [9] can only handle fairly small groups. For example, the HAP implementation takes 20 seconds on a 2.66GHz Intel PC with 2G of memory to compute eight terms of a free $\mathbb{Z}G$-resolution $R^G_*$ for the symmetric group $G = S_5$; the $\mathbb{Z}G$-rank of $R^G_8$ is 115. However, for any group $G$ there is a surjection

$$H_n(\mathrm{Syl}_p,\mathbb{Z}) \to H_n(G,\mathbb{Z})_{(p)}$$

from the homology of a Sylow $p$-subgroup $\mathrm{Syl}_p = \mathrm{Syl}_p(G)$ onto the $p$-part of the homology of $G$. For a Sylow $p$-subgroup $P$ there is a description of the kernel of the surjection $H_n(P,Z) \to H_n(G,Z)_{(p)}$ due to Cartan and Eilenberg [4]. It is generated by elements

$$\phi_K(a) - \phi_{xKx^{-1}}(a)$$

where $x$ ranges over the double coset representatives of $P$ in $G$, $K = P \cap xPx^{-1}$, the homomorphisms $\phi_K$, $\phi_{x^{-1}Kx}\colon H_n(K,\mathbb{Z}) \to H_n(P,\mathbb{Z})$ are induced by the inclusion $K \to P, k \mapsto k$ and the conjugated inclusion $K \to P, k \mapsto x^{-1}kx$, and $a$ ranges over the generators of $H_n(K,\mathbb{Z})$. Thus, the homology of a large finite group $G$ can be computed from free resolutions (with specified contracting homotopy) for each of its Sylow subgroups. Our implementation of the algorithm in [9] can be used to produce six terms of free $\mathbb{Z}(\mathrm{Syl}_p)$-resolutions for all Sylow subgroups $\mathrm{Syl}_p$ of all Mathieu groups except $M_{24}$. The Sylow subgroup $\mathrm{Syl}_2(M_{24})$ has order 1024 and requires a specific application of a general technique.

To explain the technique suppose that $G$ is a group, possibly infinite, for which we have some $\mathbb{Z}G$-resolution of $\mathbb{Z}$

$$C_*\colon \cdots \to C_n \to C_{n-1} \to \cdots \to C_0 \to \mathbb{Z}.$$

but that $C_*$ is not free. Suppose that for each $m$ we have a free $\mathbb{Z}G$-resolution of the module $C_m$

$$D_{m*}\colon \to D_{m,n} \to D_{m,n-1} \to \cdots \to D_{m,0} \to C_m.$$

| $p$ | $H_1(\mathrm{Syl}_p, \mathbb{Z})$ | $H_2(\mathrm{Syl}_p, \mathbb{Z})$ | $H_3(\mathrm{Syl}_p, \mathbb{Z})$ | $H_4(\mathrm{Syl}_p, \mathbb{Z})$ | $H_5(\mathrm{Syl}_p, \mathbb{Z})$ |
|---|---|---|---|---|---|
| 2 | $(\mathbb{Z}_2)^4$ | $(\mathbb{Z}_2)^8$ | $(\mathbb{Z}_2)^{11} \oplus (\mathbb{Z}_4)^6$ | $(\mathbb{Z}_2)^{32}$ | $(\mathbb{Z}_2)^{52} \oplus \mathbb{Z}_4$ |
| 3 | $(\mathbb{Z}_3)^2$ | $(\mathbb{Z}_3)^2$ | $(\mathbb{Z}_3)^4$ | $(\mathbb{Z}_3)^3$ | $(\mathbb{Z}_3)^4 \oplus \mathbb{Z}_9$ |

TABLE 3. Low dimensional homology of Sylow subgroups of $M_{24}$ for $p = 2, 3$

**Theorem 2.** [20] *There is a free $\mathbb{Z}G$-resolution $R_*^G \to \mathbb{Z}$ with*

$$R_n^G = \bigoplus_{p+q=n} D_{p,q}$$

The proof of this theorem of C.T.C. Wall can be made constructive by using contracting homotopies on the resolutions $D_{m*}$. Furthermore, a contracting homotopy on $R_*^G$ can be constructed by a formula involving contracting homotopies on the $D_{m*}$ and on $C_*$. Details are given in [11].

Suppose now that $N$ is a normal subgroup of $G$ and that $C_*$ is a free $\mathbb{Z}(G/N)$-resolution. Then, regarding $C_*$ as a $\mathbb{Z}G$-resolution, each free $\mathbb{Z}G$-generator of $C_m$ is stabilized by $N$. Any free $\mathbb{Z}N$-resolution of $\mathbb{Z}$ can be used to construct a free $\mathbb{Z}G$-resolution $D_{m*}$ of $C_m$. Thus, using Theorem 2, we can construct a free $\mathbb{Z}G$-resolution $R_*^G$ from a free $\mathbb{Z}N$-resolution $R_*^N$ and free $\mathbb{Z}(G/N)$-resolution $R_*^{G/N}$. The constructed resolution is often referred to as a *twisted tensor product* and denoted by $R_*^G = R_*^N \tilde{\otimes} R_*^{G/N}$.

This twisted tensor product has been implemented in HAP and can be used to provide free resolutions for the Sylow subgroup $\mathrm{Syl}_p(M_{24})$. Since $|M_{24}| = 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$ the non-cyclic Sylow subgroups occur only for $p = 2, 3$. Their low-dimensional integral homology can be computed using HAP and is given in Table 3.

In degrees $n = 5$ the current version of HAP fails to determine the image of $H_n(\mathrm{Syl}_2, \mathbb{Z})$ in $H_n(M_{24}, \mathbb{Z})$. It succeeds in constructing the image as a finitely presented group but fails to determine the group from this presentation. This failure should be resolved in a future release of HAP.

The remainder of the paper is aimed at constructing small free resolutions for large groups such as $M_{24}$.

## 3. Orbit polytopes

Suppose that a finite group $G$ acts linearly on $\mathbb{R}^n$. For a vector $v \in \mathbb{R}^n$ we consider the convex hull

$$P = P(G, v) = \mathrm{Conv}(v^g : g \in G)$$

of the orbit of $v$ under the action of $G$. The polytope $P$ has a natural cell structure with respect to which we can consider the cellular chain complex $C_*(P)$. The action of $G$ on $\mathbb{R}^n$ induces an action of $G$ on $C_*(P)$ and we can view $C_*(P)$ as a chain complex of $\mathbb{Z}G$-modules. Since $P$ is contractible we have $H_i(C_*(P)) = 0$ for all $i \geq 1$ and $H_0(C_*(P)) = \mathbb{Z}$. Furthermore, if the polytope is of dimension $m$ then $H_0(C_*(P)) \cong \mathbb{Z} \cong C_m(P)$. So there is a homomorphism $C_0(P) \to C_{m-1}(P)$ which can be used to splice together infinitely many copies of $C_*(P)$ to form an infinite $\mathbb{Z}G$-resolution

$$\cdots \to C_1 \to C_0 \to C_{m-1} \to \cdots \to C_2 \to C_1 \to C_0 \to \mathbb{Z}$$

of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$. In principle one can use Theorem 2 to convert $C_*$ to a free $\mathbb{Z}G$-resolution. Precise details are given in [11]. To put this idea into practice one requires:

(1) The face lattice of the orbit polytope $P(G, v)$.
(2) For each orbit of cell $e$ in $P(G, v)$, the subgroup $\mathrm{Stab}(G, e) \le G$ of elements that stabilize $e$ globally.
(3) A free $\mathbb{Z}\,\mathrm{Stab}(G, e)$-resolution $R_*^{\mathrm{Stab}(G,e)}$ for each stabilizer $\mathrm{Stab}(G, e)$.

Assuming that the stabilizer groups $\mathrm{Stab}(G, e)$ are reasonably small, resolutions $R_*^{\mathrm{Stab}(G,e)}$ are readily obtained from HAP's implementation of the algorithm in [9]. Thus, to convert $C_*$ to a free $\mathbb{Z}G$-resolution, we must focus on requirements (1) and (2).

One could use computational geometry software such as Polymake [13] to determine the combinatorial structure of $P(G, v)$ for small groups $G$. For instance, any permutation group $G \le S_n$ acts on $\mathbb{R}^n$ by $\pi(x_1, \ldots, x_n) = (x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(n)})$ for $\pi \in G$. In particular, the Mathieu group $M_{10}$ of order 720, generated by $\pi_1 = (1, 9, 6, 7, 5)(2, 10, 3, 8, 4)$ and $\pi_2 = (1, 10, 7, 8)(2, 9, 4, 6)$, acts on $\mathbb{R}^{10}$. For the vector $v = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10)$ the polytope $P(M_{10}, v)$ is 9-dimensional with 720 vertices each of degree 632. The polytope thus has 227520 edges.

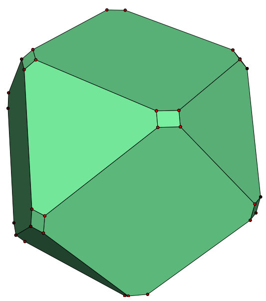
## 4. Orbit polytopes of finite reflection groups

Let $W$ be a finite reflection group generated by a simple system of Euclidean reflections $S = \{s_1, \ldots, s_n\}$. For each reflection $s \in S$ let $H_s$ denote the corresponding reflecting hyperplane and $\Delta$ the fundamental simplex for $S$. The Coxeter-Dynkin reduced diagram is the graph on $S$ with two reflections adjacent if they do not commute. Fix a subset $\emptyset \subsetneq V \subseteq S$. The *type* $T = t(v) \subset S$ of a point $v \in \Delta$ is the set of $s \in S$ such that $v \notin H_s$. Choose a point $v$ of type $V$. Let $P(W; V, v)$ denote the $n$-dimensional polytope formed by the convex hull of the orbit of $v$ under the action of $W$.

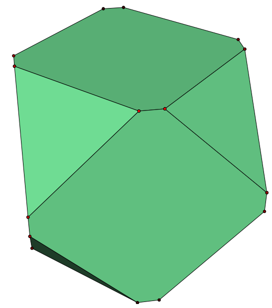
As an example, consider the 3-dimensional reflection group $W = \mathsf{B}_3$ generated by reflections $s_1, s_2, s_3$ where $(s_1s_2)^3 = 1$, $(s_1s_3)^2 = 1$ and $(s_2s_3)^4 = 1$. For $V = \{s_1, s_2, s_3\}$ and vector $v \in \mathbb{R}^3$ in general position but close to the mirrors $H_{s_1}$ and $H_{s_3}$ the polytope $P(W; V, v)$ is pictured in Figure 1.a). For $V = \{s_2, s_3\}$ and $v \in H_{s_1}$ the polytope $P(W; V, v)$ is pictured in Figure 1.b).

**Proposition 3.** *The combinatorial type of $P(W; V, v)$ is independent of the choice of $v$.*

*Proof.* For $V = S$ the polytope obtained is the well-known permutahedron, whose face-lattice is independent of $v$. The stabilizer of a face of $P(W; V, v)$ is a parabolic subgroup and this establishes an isomorphism between the face lattice of $P(W; V, v)$ and the lattice of parabolic subgroups of $W$. Furthermore, the 1-skeleton of $P(W; S, v)$ is the Cayley graph $\mathrm{Cay}(W, S)$ of $W$ with respect to the generating set $S$. Observe that in $\mathrm{Cay}(W, S)$ the length



a) The polytope $P(\mathsf{B}_3; \{s_1, s_2, s_3\}, v)$



b) The polytope $P(\mathsf{B}_3; \{s_2, s_3\}, v)$

FIGURE 1. Two Wythoff polytopes constructed from $D_3$

of any edge labelled by generator $s \in S$ decreases as $v$ is moved towards the hyperplane $H_s$, and that the edge ceases to exist when $v$ moves into $H_s$. Denote by $\mathrm{Cay}_V(W, S)$ the obtained reduced Cayley graph. The group $W$ acts transitively on its vertex set. The set of vectors $v$ of type $t(v) = S$ has measure one in the set of all vectors. Thus any face $F$ of $P(W; V, v)$ is obtained as the limit of some face $G$ of the permutahedron and the set of vertices contained in $F$ is the limit of the vertices contained in $G$. Thus the graph $\mathrm{Cay}_V(W, S)$ determines the face-lattice and this proves the required result. □

We are going to describe explicitly the face lattice $P_{W,V}$ of the polytope $P(W; V, v)$ for any $v$ of type $V$. Our description is of course equivalent to the classic one given in [5, 6, 22] and reproduces the one of [19]. Since we assume that $W$ is finite, the Coxeter-Dynkin reduced diagram of $W$ is a tree and given any two vertices $u$ and $v$ of it we denote by $[u, v]$ the unique path from $u$ to $v$.

For two subsets $U, U' \in S$ we say that $U'$ *blocks* $U$ (from $V$) if for all $u \in U$ and $v \in V$ there is a $u' \in U'$, such that $u' \in [u, v]$. This defines a binary relation on subsets of $S$, which we will denote by $U' \leq U$. We also write $U' \sim U$ if $U' \leq U$ and $U \leq U'$, and we write $U' < U$ if $U' \leq U$ and $U \not\leq U'$.

It is easy to see that $\leq$ is reflexive and transitive, which implies that $\sim$ is an equivalence relation. Let $[U]$ denote the equivalence class containing $U$. It can be shown that if $U \sim U'$ then $U \cap U' \sim U \sim U \cup U'$. This yields that every equivalence class $X$ contains a unique smallest (under inclusion) subset $m(X)$ and unique largest subset $M(X)$. The subsets $m(X)$ will be called the *essential* subsets of $S$ (with respect to $V$). Let $E(V)$ be the set of all essential subsets of $S$. Clearly, the above relation $<$ is a partial order on $E(V)$. Also, $V \in E(V)$ and $V$ is the smallest element of $E(V)$ with respect to $<$.

The faces $F$ of $P(W; V, v)$ are indexed by their isobarycenters $g(F)$. The stabilizer of $F$ is the stabilizer of $g(F)$, that is the parabolic subgroup of $W$ generated by $S - t(g(F))$. The type of such an isobarycenter is an essential subset of $S$ and all essential subsets are realized as isobarycenters of faces. The rank of an essential subset is the dimension of the corresponding face. Given two faces $F, F'$ of $P(W; V, v)$, $F \subset F'$ if and only if we have the type inequality $t(g(F)) < t(g(F'))$ and $\{g(F), g(F')\}$ is contained in at least one image $g(\Delta)$ with $g \in W$ of the fundamental simplex $\Delta$.

We can use the above formalism to obtain the combinatorial structure of the orbit polytope $P(M_{24}, v)$ where the Mathieu group acts on $\mathbb{R}^{24}$ by permuting basis vectors, and $v = (1, 2, 3, 4, 5, 0, \dots, 0) \in \mathbb{R}^{24}$. Since $M_{24}$ is a 5-transitive permutation group we have

$$P(M_{24}, v) = P(S_{24}, v).$$

The symmetric group $S_{24}$ is a finite reflection group with simple generating system $S = \{s_i = (i, i+1)\colon 1 \leq i \leq 23\}$. The vector $v$ lies in those mirrors $H_{s_i}$ for $6 \leq i \leq 23$. So $P(M_{24}, v) = P(S_{24}, V, v)$ for $V = \{s_1, \dots, s_5\}$.

Our proof of Proposition 3 implies that the polytope $P(M_{24}, v)$ has $|S_{24}/\langle s_i\colon 6 \leq i \leq 23\rangle| = 5100480$ vertices. The essential subsets of rank 1 defining edges are $V - \{s_k\}$ for $1 \leq k \leq 4$ and $(V - \{s_5\}) \cup \{s_6\}$. So, the number of edges is

$$|S_{24}/\langle s_5, s_i : 7 \leq i \leq 23\rangle| + \sum_{1 \leq k \leq 4} |S_{24}/\langle s_k, s_i : 6 \leq i \leq 23\rangle| = 58655520.$$

Each vertex of the polytope has the same degree $d$ say. Thus the number of edges is $d \times 5100480/2 = 58655520$ from which $d = 23$. Since $P(M_{24}, v)$ is of dimension 23, this shows that it is simple.

Each vertex of $P(M_{24}, v)$ has stabilizer group $\mathrm{Stab}(M_{24}, v) = M_{24} \cap \langle s_i : 6 \leq i \leq 23\rangle \cong (C_2 \times C_2 \times C_2 \times C_2) : C_3$ of order 48. Under $M_{24}$, for $1 \leq k \leq 4$, there is only one orbit of edges of type $V - \{s_k\}$; they have stabilizer $\mathrm{Stab}(M_{24}, v) : C_2$ of order 96. Under $M_{24}$ there are two orbits of edges of type $(V - \{s_5\}) \cup \{s_6\}$, one with stabilizer $S_3$, the other with stabilizer a 2-group of order 32.

The formalism of essential subsets is a useful tool to determine the face lattice of $P(W; V, v)$ for a Coxeter group $W$ and provides ready access to the lattice for homology computations. The equality between the polytopes $P(M_{24}, v)$ and $P(S_{24}, v)$ was essential for being able to apply this formalism and thus get a reasonably simple description of the face lattice.

For an arbitrary vector $v$ and group $G$ we cannot expect to have a simple combinatorial description of the face lattice of $P(G, v)$ and we need to use specific computational techniques. If $G$ is large, then we cannot expect to be able to store the vertex set of $P(G, v)$. Fortunately, by the group action, the full face lattice is encoded in the set $S(v)$ of vertices adjacent to $v$. This set $S(v)$ can be computed iteratively by using the Poincaré polyhedron theorem (see [18, 7] for some example of such computations). Once the list of neighbours is known the face-lattice follows easily.

After one has obtained the low dimensional faces of $P(M_{24}, v)$ and their stabilizer groups, we can use Theorem 2 to compute the initial terms of a free $\mathbb{Z}M_{24}$-resolution of $\mathbb{Z}$.

## 5. Wythoff construction for polytopes

The Wythoff construction can also be defined for partially ordered sets. A *flag* in a poset is an arbitrary completely ordered subset. We say that a connected poset $\mathcal{K}$ is a *d-dimensional complex* (or, simply, a *d-complex*) if every maximal flag in $\mathcal{K}$ has size $d + 1$.

| $G$ | $P$ | $V$ | Free rank of resolution in degrees $0, 1, 2, \dots$ |
|---|---|---|---|
| $M_{22}$ | $\alpha_{21}$ | $\{0,1,2\}$ | 1, 7, 33, 113, 301, 694 |
| $M_{23}$ | $\alpha_{22}$ | $\{0,1,2,3,4\}$ | 2, 20, 116, 451, 1334, 3279 |
| $M_{24}$ | $\alpha_{23}$ | $\{0,1,2,3,4\}$ | 1, 9, 50, 204, 649 |

TABLE 4. Rank of resolutions of $M_{22}$, $M_{23}$, $M_{24}$ obtained from the Wythoff construction

In a $d$-complex $\mathcal{K}$ every element $x$ can be uniquely assigned a number $\dim(x) \in \{0, \dots, d\}$, called the *dimension* of $x$, in such a way, that the minimal elements of $\mathcal{K}$ have dimension zero and $\dim(y) = \dim(x) + 1$ whenever $x < y$ and there is no $z$ with $x < z < y$. The elements of a complex $\mathcal{K}$ are called *faces*, or *$k$-faces* if the dimension of the face needs to be specified. Furthermore, 0-faces are called *vertices* and $d$-faces (maximal faces) are called *facets*. If $x < y$ and $\dim(x) = k$, we will say that *$x$ is a $k$-face of $y$*.

For a flag $f \subset \mathcal{K}$ define its *type* as the set $t(f) = \{\dim(F) \quad : \quad F \in f\}$. Clearly, $t(f)$ is a subset of $S = \{0, \dots, d\}$ and, conversely, every subset of $S$ is the type of some flag. Let $\Omega$ be the set of all nonempty subsets of $S$ and fix an arbitrary $V \in \Omega$. For two subsets $U, U' \in \Omega$ we say that $U'$ *blocks* $U$ (from $V$) if for all $u \in U$ and $v \in V$ there is a $u' \in U'$ and $u \le u' \le v$ or $v \le u' \le u$. With this notion of blocking we can define the notion of essential subset of $S$ and the inequality $<$ in the same way as for Coxeter groups.

The construction of $P(\mathcal{K}; V)$ mimics the one of $P(W; D, v)$ above for Coxeter groups. The *Wythoff complex* $P(\mathcal{K}; V)$ consists of all flags $F$ such that $t(F)$ is essential. For two such flags $F$ and $F'$, we have $F' < F$ whenever $t(F') < t(F)$ and $F'$ is compatible with $F$, that is, $F \cup F'$ is a flag. It can be shown that $P(\mathcal{K}, V)$ is again a $d$-complex.

The face lattice $\mathcal{K}(P)$ of a $(d+1)$-dimensional polytope $P$ is a $d$-complex, which is a CW-complex topologically equivalent to a sphere. It is proved in [19] that the topological type of $P(\mathcal{K}; V)$ is the same as the one of $\mathcal{K}$. This version of the Wythoff construction when applied to a regular polytope gives a face lattice which is isomorphic to the one obtained by applying the Wythoff construction to the corresponding Coxeter group. The complex $P(\mathcal{K}(P), \{0\})$ is equal to $\mathcal{K}(P)$ and $P(\mathcal{K}(P), \{d\})$ is the complex of the polytope dual to $P$. In general $P(\mathcal{K}(P), V)$ is not a polytope since the notion of convexity is not well preserved by the Wythoff construction without any regularity assumption.

The topological invariance means that if a group $G$ acts on a polytope $P$ then we can apply the orbit polytope construction to $P(\mathcal{K}(P), V)$ for a chosen $V$ in order to compute $H_i(G, \mathbb{Z})$.

In the case of $M_{24}$, we take as polytope the 23-dimensional simplex $\alpha_{23}$ and we build the Wythoff polytope $P(\alpha_{23}; \{0, 1, 2, 3, 4\})$. In Table 4 we give the results obtained for the larger Mathieu groups. The method applies to any finite group acting on $n$ points by using the simplex $\alpha_{n-1}$. We do not need $G$ to act transitively. All programs are available from [8].

## 6. Homology at $p = 5, 7, 11, 23$

Suppose that a group $G$ has Sylow $p$-subgroup $P = C_p$ of prime order. The Cartan-Eilenberg double coset formula implies that the surjection

$$\pi_n \colon H_n(P, \mathbb{Z}) \to H_n(G, \mathbb{Z})_{(p)}$$

has kernel generated by the elements

$$H_n(\phi_g)(a) - a$$

for $g \in N_G(P), a \in H_n(P, \mathbb{Z})$ and $\phi_g \colon P \to P, p \mapsto gpg^{-1}$. Here $N_G(P)$ is the normalizer of $P$ in $G$.

Using the isomorphism $H_{n-1}(P, \mathbb{Z}) \cong H^n(P, \mathbb{Z})$ and the cohomology ring structure $H^*(P, \mathbb{Z}) \cong \mathbb{Z}_p[x^2]$, we see that a group homomorphism $\phi \colon P \to P, p \mapsto p^m$ induces a homology homomorphism $H_{2k-1}(\phi) \colon H_{2k-1}(P, \mathbb{Z}) \to H_{2k-1}(P, \mathbb{Z}), a \mapsto a^{m^k}$.

For $p \in \{5, 7, 11, 23\}$ the Mathieu groups have Sylow $p$-subgroups which are either trivial or of prime order. One can use GAP to determine their normalizers. It is thus a routine exercise to determine the $p$-part of the integral homology of the Mathieu groups, the results of which are given in the Introduction.

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