PERFECT, STRONGLY EUTACTIC LATTICES ARE PERIODIC EXTREME

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ABSTRACT. We introduce a parameter space for periodic point sets, given as a union of m translates of a point lattice. In it we investigate the behavior of the sphere packing density function and derive sufficient conditions for local optimality. Using these criteria we prove that perfect, strongly eutactic lattices cannot be locally improved to yield a denser periodic sphere packing. This in particular implies that the densest known lattice sphere packings in dimension $d \leq 8$ and $d = 24$ cannot locally be modified to yield a periodic sphere packing with greater density.

1. INTRODUCTION

The classical and widely studied *sphere packing problem* asks for a non-overlapping arrangement of equally sized spheres in a Euclidean space, such that the fraction of space covered by spheres is maximized. The problem arose from the arithmetical study of positive definite quadratic forms. By the works Thue [Thu10] and Hales [Hal05] the optimal arrangements of spheres are known up to dimension 3. We refer to [GL87], [CS99], [Mar03] and [Sch08] for details and further reading.

For reasons related to the historical roots of the sphere packing problem, special attention has been on *(point) lattices* as the discrete set of sphere centers. In dimension 2 the *hexagonal lattice* and in dimension 3 the *face-centered-cubic lattice* yield optimal sphere packings. For the restriction of the sphere packing problem to lattices, the optimal configurations are known up to dimension 8 and in dimension 24 (see Table 1). Here, solutions are given by fascinating objects, the so-called *root lattices* and the *Leech Lattice*. We refer to [CS99], [Mar03] and [NS] for further information on these exceptional objects.

A major **open problem** in sphere packings is to find a dimension in which optimal arrangements are not given by a lattice. In dimension 10 there exists a non-lattice sphere packing, which is conjectured to have a higher density than any lattice sphere packing (see [LS70]). As shown in Table 1, below dimension 24 similar sphere packings have been found in dimensions 11, 13, 18, 20 and 22. All of them are *periodic*, that is, a finite union of translates of a lattice sphere packing. By a well known conjecture, attributed by Gruber [Gru07] to Zassenhaus, optimal

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sphere packings can always be attained by periodic sphere packings. It is known that their density comes arbitrarily close to the optimal value (see [GL87]).

A natural idea to resolve the above mentioned open problem is to "locally modify" one of the optimal known lattice sphere packings, i.e. in dimensions $d = 4, \ldots, 8$, to obtain a better non-lattice sphere packing. In this paper we show that such modifications are not possible, if ones stays within the set of periodic sphere packings (see Corollary 11). We more generally show in Theorem 10 that such modifications are not possible for *perfect, strongly eutactic lattices*.

The paper is organized as follows. In Section 2 we recall some necessary background on lattices and positive definite quadratic forms. In Section 3 we introduce the so-called Ryshkov polyhedron, and based on it we give a geometrical interpretation of Voronoi's characterization of locally optimal lattice sphere packings. This viewpoint allows a natural generalization to study local optimal periodic sphere packings. For their study we introduce a parameter space in Section 4. In Section 5 we give characterizations of local optimal periodic sphere packings with up to m lattices translates. Based on these general characterizations we obtain the main result of this paper in Section 6: We show that perfect, strongly eutactic lattices are *periodic extreme* (see Definition 8), meaning they cannot locally be modified to yield a better periodic sphere packing.

2. Background on lattices and quadratic forms

A (full rank) *lattice* L in $\mathbb{R}^d$ is a discrete subgroup $L = \mathbb{Z}\boldsymbol{a}_1 + \ldots + \mathbb{Z}\boldsymbol{a}_d$ generated by d linear independent (column) vectors $\boldsymbol{a}_i \in \mathbb{R}^d$. We say that these vectors form a *basis* of L and associate it with the matrix $A = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_d) \in \mathsf{GL}_d(\mathbb{R})$. We write $L = A\mathbb{Z}^d$. It is well known that L is generated in this way precisely by the matrices AU with $U \in \mathsf{GL}_d(\mathbb{Z})$. We refer to [GL87] for details and more background on lattices. Given a lattice L and *translational vectors* $\boldsymbol{t}_i$, for say $i = 1, \ldots, m$, the discrete set

$$\Lambda = \bigcup_{i=1}^{m} \boldsymbol{t}_i + L \tag{1}$$

is called a *periodic (point) set*.

The *sphere packing radius* $\lambda(\Lambda)$ of a discrete set Λ in the Euclidian space $\mathbb{R}^d$ (with norm $\|\cdot\|$) is defined as the infimum of half the distances between distinct points:

$$\lambda(\Lambda) = \frac{1}{2} \inf_{\boldsymbol{x},\boldsymbol{y}\in\Lambda, \boldsymbol{x}\neq\boldsymbol{y}} \|\boldsymbol{x} - \boldsymbol{y}\|.$$

The sphere packing radius is the largest possible radius λ such that solid spheres of radius λ around points of Λ do nowhere overlap. Denoting the solid unit sphere by B^d, the *sphere packing* defined by Λ is the union of non-overlapping spheres

$$\bigcup_{\boldsymbol{x}\in\Lambda} \boldsymbol{x} + \lambda(\Lambda) B^d.$$

d	point set	$\delta/\operatorname{vol} B^d$	author(s)
2	A_2	$0.2886\ldots$	Lagrange, 1773, [Lag73]
3	$\mathsf{A}_3 = \mathsf{D}_3, *$	$0.1767\ldots$	Gau, 1840, [Gau40]
4	D_4	0.125	Korkine & Zolotareff, 1877, [KZ77]
5	$\mathsf{D}_5, *$	$0.0883\ldots$	Korkine & Zolotareff, 1877, [KZ77]
6	$\mathsf{E}_6, *$	$0.0721\ldots$	Blichfeldt, 1935, [Bli35]
7	$\mathsf{E}_7, *$	0.0625	Blichfeldt, 1935, [Bli35]
8	E_8	0.0625	Blichfeldt, 1935, [Bli35]
9	$\mathsf{\Lambda}_9, *$	$0.0441\ldots$	
10	P_{10c}	$0.0390\ldots$	Leech & Sloane, 1970, [LS70]
11	P_{11a}	$0.0351\ldots$	Leech & Sloane, 1970, [LS70]
12	K_{12}	$0.0370\ldots$	
13	P_{13a}	$0.0351\ldots$	Leech & Sloane, 1970, [LS70]
14	$\mathsf{\Lambda}_{14}, *$	$0.0360\ldots$	
15	$\mathsf{\Lambda}_{15}, *$	$0.0441\ldots$	
16	$\mathsf{\Lambda}_{16}, *$	0.0625	
17	$\mathsf{\Lambda}_{17}, *$	0.0625	
18	V_{18}	$0.0750\ldots$	Bierbrauer & Edel, 1998, [BE00]
19	$\mathsf{\Lambda}_{19}, *$	$0.0883\ldots$	
20	V_{20}	$0.1315\ldots$	Vardy, 1995, [Var95]
21	$\mathsf{\Lambda}_{21}, *$	$0.1767\ldots$	
22	V_{22}	$0.3325\ldots$	Conway & Sloane, 1996, [CS96]
23	$\mathsf{\Lambda}_{23}$	0.5	
24	$\mathsf{\Lambda}_{24}$	1	Cohn & Kumar, 2004, [CK08]

Table 1. Point sets defining best known sphere packings up to dimension 24. In dimensions $d \leq 8$ and $d = 24$ the corresponding authors solved the lattice sphere packing problem. The other mentioned authors found the listed, densest known periodic sphere packings. The stars $*$ indicate that an equally dense, periodic non-lattice sphere packing is known.

Its *density* $\delta(\Lambda)$ is, loosely spoken, defined as the fraction of space covered by spheres. We can make this definition more precise by considering a cube $C = \{\boldsymbol{x} \in \mathbb{R}^d : |x_i| \leq 1/2\}$ and setting

$$\delta(\Lambda) = \lambda(\Lambda)^d \operatorname{vol} B^d \cdot \liminf_{\lambda \to \infty} \frac{\operatorname{card}(\Lambda \cap \lambda C)}{\operatorname{vol} \lambda C}.$$

It can be shown that the value of δ is the same for any other centrally symmetric convex bodies C (see [GL87]). For general discrete sets, it may be difficult to compute the density, respectively the limes inferior in the definition. In case of a lattice the limes inferior can simply be replaced by $1/\det L$, where $\det L = |\det A|$ is the *determinant* of the lattice $L = A\mathbb{Z}^d$. Note that the determinant of L is independent of the particular choice of the basis A. For periodic sets Λ as in (1) we get the estimate

$$\delta(\Lambda) \leq \frac{m\lambda(\Lambda)^d \operatorname{vol} B^d}{\det L}$$

with equality if and only if the lattice translates $\boldsymbol{t}_i + L$ are pairwise disjoint.

Among similarity classes of lattices, hence in the space $O_d(\mathbb{R})\backslash \mathsf{GL}_d(\mathbb{R})/\mathsf{GL}_d(\mathbb{Z})$, there exist only finitely many local maxima of δ up to scaling. In order to characterize and to work with them, i.e. enumerate them, it is convenient to use the language of real *positive definite quadratic forms* (PQFs for short). These are simply identified with the set $\mathcal{S}^d_{>0}$ of real symmetric, positive definite matrices. Given a matrix $Q \in \mathcal{S}^d_{>0}$, we set $Q[\boldsymbol{x}] = \boldsymbol{x}^t Q \boldsymbol{x}$ for $\boldsymbol{x} \in \mathbb{R}^d$, defining a corresponding PQF. Note that every matrix $Q \in \mathcal{S}^d_{>0}$ can be decomposed into $Q = A^t A$ with $A \in \mathsf{GL}_d(\mathbb{R})$ and therefore $\mathcal{S}^d_{>0}$ can be identified with the space $O_d(\mathbb{R})\backslash \mathsf{GL}_d(\mathbb{R})$ of lattice bases up to orthogonal transformations. Two PQFs (respectively matrices) Q and Q' are called *arithmetically equivalent* (or *integrally equivalent*) if there exists a matrix $U \in \mathsf{GL}_d(\mathbb{Z})$ with $Q' = U^t Q U$. Thus arithmetical equivalence classes of PQFs are in one-to-one correspondence with similarity classes of lattices.

The *arithmetical minimum* $\lambda(Q)$ of a PQF Q is defined by

$$\lambda(Q) = \min_{\boldsymbol{x} \in \mathbb{Z}^d \setminus \{\boldsymbol{0}\}} Q[\boldsymbol{x}].$$

If $L = A\mathbb{Z}^d$ with $A \in \mathsf{GL}_d(\mathbb{R})$ satisfying $Q = A^t A$ is a corresponding lattice, there is an immediate relation to the packing radius of L: We have $\lambda(Q) = (2\lambda(L))^2$ and therefore

$$\delta(L) = \mathcal{H}(Q)^{d/2} \frac{\operatorname{vol} B^d}{2^d},$$

where

$$\mathcal{H}(Q) = \frac{\lambda(Q)}{(\det Q)^{1/d}}$$

is the so-called *Hermite invariant* of Q. Note that $\mathcal{H}(\cdot)$ is invariant with respect to scalings. A classical problem in the arithmetic theory of quadratic forms is the determination of the *Hermite constant*

$$\mathcal{H}_d = \sup_{Q \in \mathcal{S}^d_{>0}} \mathcal{H}(Q).$$

By the relation described above, it corresponds to determining the supremum of possible lattice sphere packing densities. Local maxima of the Hermite invariant on $\mathcal{S}^d_{>0}$ and corresponding lattices are called *extreme*.

3. VORONOI'S CHARACTERIZATION OF EXTREME FORMS

The Ryshkov polyhedron. Since the Hermite invariant is invariant with respect to scaling, a natural approach of maximizing it, is to consider all forms with a fixed arithmetical minimum, say 1, and minimize the determinant among them. We may even relax the condition on the arithmetical minimum and only require that it is at least 1. In other words, we have

$$\mathcal{H}_d = 1/\inf_{\mathcal{R}} (\det Q)^{1/d},$$

where

$$\mathcal{R} = \left\{ Q \in \mathcal{S}^d_{>0} : \lambda(Q) \geq 1 \right\}. \tag{2}$$

We refer to $\mathcal{R}$ as *Ryshkov polyhedron*, as it was Ryshkov [Rys70] who noticed that this view on Hermite's constant allows a simplified description of Voronoi's theory.

We denote by $\mathcal{S}^d$ the space of real symmetric matrices, respectively of real quadratic forms in d variables. It is a Euclidian vector space of dimension $\binom{d+1}{2}$ with the usual inner product defined by

$$\langle Q, Q' \rangle = \sum_{i,j=1}^{d} q_{ij} q'_{ij} = \operatorname{trace}(Q \cdot Q').$$

Because of the fundamental identity

$$Q[\boldsymbol{x}] = \langle Q, \boldsymbol{x}\boldsymbol{x}^t \rangle,$$

quadratic forms $Q \in \mathcal{S}^d$ attaining a fixed value on a given $\boldsymbol{x} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$ lie all in a *hyperplane* (*affine subspace of co-dimension* 1). Thus Ryshkov polyhedra $\mathcal{R}$ are intersections of infinitely many *halfspaces*:

$$\mathcal{R} = \{ Q \in \mathcal{S}^d_{>0} : \langle Q, \boldsymbol{x}\boldsymbol{x}^t \rangle \geq \lambda \text{ for all } \boldsymbol{x} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}\}. \tag{3}$$

It can be shown that $\mathcal{R}$ is "locally like a polyhedron", meaning that any intersection with a *polytope* (convex hull of finitely many vertices) is itself a polytope. For a proof we refer to [Sch08]. As a consequence $\mathcal{R}$ has *vertices*, *edges*, *facets* and in general k-*dimensional faces* (k-*faces*). For details on terminology and basic properties of polytopes we refer to [Zie97].

Perfect forms. The vertices Q of the Ryshkov polyhedron are *perfect forms*. Such forms are characterized by the fact that they are determined uniquely by their arithmetical minimum (here 1) and its representatives

$$\operatorname{Min} Q = \{ \boldsymbol{x} \in \mathbb{Z}^d : Q[\boldsymbol{x}] = \lambda(Q) \}.$$

A corresponding lattice is called perfect too. The following proposition due to Minkowski implies that the Hermite constant can only be attained among perfect forms, respectively, the maximal lattice sphere packing density can only be attained by perfect lattices.

Proposition 1 (Minkowski [Min05])**.** $(\det Q)^{1/d}$ *is a strictly concave function on* $\mathcal{S}^d_{>0}$.

For a proof see for example [GL87]. Note, that in contrast to $(\det Q)^{1/d}$, the function $\det Q$ is not a concave function on $\mathcal{S}^d_{>0}$ (cf. [Nel74]). However Minkowski's theorem implies that the set

$$\{ Q \in \mathcal{S}^d_{>0} : \det Q \geq D \} \tag{4}$$

is strictly convex for $D > 0$.

Another property of perfect forms which we use later is the following.

Proposition 2. *If* $Q \in \mathcal{S}^d$ *is perfect, then* $\operatorname{Min} Q$ *spans* $\mathbb{R}^d$.

The existence of d linear independent vectors in $\operatorname{Min} Q$ for a perfect form Q follows from the observation that the rank-1 forms $\boldsymbol{x}\boldsymbol{x}^t$ with $\boldsymbol{x} \in \operatorname{Min} Q$ have to span $\mathcal{S}^d$, since they uniquely determine Q through the linear equations $\langle Q, \boldsymbol{x}\boldsymbol{x}^t \rangle = \lambda(Q)$. If however $\operatorname{Min} Q$ does not span $\mathbb{R}^d$ then these rank-1 forms can maximally span a $\binom{d}{2}$-dimensional subspace of $\mathcal{S}^d$.

Finiteness up to equivalence. The arithmetical equivalence operation $Q \mapsto U^t Q U$ of $\mathsf{GL}_d(\mathbb{Z})$ on $\mathcal{S}^d_{>0}$ leaves $\lambda(Q)$, $\operatorname{Min} Q$ and also $\mathcal{R}$ invariant. In fact, $\mathsf{GL}_d(\mathbb{Z})$ acts on the sets of faces of a given dimension, thus in particular on the sets of vertices, edges and facets of $\mathcal{R}$. The following theorem shows that the Ryshkov polyhedron $\mathcal{R}$ contains only finitely many arithmetically inequivalent vertices. By Proposition 1 this implies in particular that $\mathcal{H}_d$ is actually attained, namely by some perfect forms. For a proof we refer to [Sch08].

Theorem 3 (Voronoi 1907)**.** *Up to arithmetical equivalence and scaling there exist only finitely many perfect forms in a given dimension $d \geq 1$.*

Thus the classification of perfect forms in a given dimension, respectively the enumeration of vertices of the Ryshkov polyhedron up to arithmetical equivalence, yields the Hermite constant. Perfect forms have been classified up to dimension 8 (see [DSV07] and [Sch08]).

Characterization of extreme forms. From dimension 6 onwards not every perfect form is extreme (see [Mar03]). In order to characterize extreme forms among perfect forms the notion of *eutaxy* is used: A PQF Q is called *eutactic*, if its inverse Q^{-1} is contained in the (relative) interior $\operatorname{relint} \mathcal{V}(Q)$ of its *Voronoi domain*

$$\mathcal{V}(Q) = \operatorname{cone}\{\boldsymbol{x}\boldsymbol{x}^t : \boldsymbol{x} \in \operatorname{Min} Q\}.$$

Here $\operatorname{cone} M$ denotes the *conic hull*

$$\left\{ \sum_{i=1}^{n} \alpha_i \boldsymbol{x}_i : m \in \mathbb{N} \text{ and } \boldsymbol{x}_i \in M, \alpha_i \geq 0 \text{ for } i = 1, \dots, n \right\}$$

of a set M. Note that the Voronoi domain is full-dimensional (of dimension $\binom{d+1}{2}$) if and only if Q is perfect. Note also that the rank-1 forms $\boldsymbol{x}\boldsymbol{x}^t$ give inequalities $\langle Q, \boldsymbol{x}\boldsymbol{x}^t \rangle \geq 1$ defining the Ryshkov polyhedron and by this the Voronoi domain of Q is equal to the *normal cone*

$$\{N \in \mathcal{S}^d : \langle N, Q/\lambda(Q) \rangle \leq \langle N, Q' \rangle \text{ for all } Q' \in \mathcal{R}\} \tag{5}$$

of $\mathcal{R}$ at $Q/\lambda(Q)$.

Algebraically the eutaxy condition $Q^{-1} \in \operatorname{relint} \mathcal{V}(Q)$ is equivalent to the existence of positive $\alpha_{\boldsymbol{x}}$ with

$$Q^{-1} = \sum_{\boldsymbol{x} \in \operatorname{Min} Q} \alpha_{\boldsymbol{x}} \boldsymbol{x}\boldsymbol{x}^t. \tag{6}$$

Thus computationally eutaxy of Q can be tested by solving the *linear program*

$$\max \alpha_{\min} \quad \text{s.t. } \alpha_{\boldsymbol{x}} \geq \alpha_{\min} \text{ and (6) holds.} \tag{7}$$

The form Q is eutactic, if and only if the maximum is greater 0.

Voronoi [Vor07] showed that perfectness, together with eutaxy implies extremality and vice versa:

Theorem 4 (Voronoi, [Vor07])**.** *A PQF $Q \in \mathcal{S}^d_{>0}$ is extreme if and only if Q is perfect and eutactic.*

We here give a proof providing a geometrical viewpoint that turns out to be quite useful for the intended generalization discussed in the following sections.

Proof. The function $\det Q$ is a positive real valued polynomial on $\mathcal{S}^d$, depending on the $\binom{d+1}{2}$ different coefficients q_{ij} of Q. Using the expansion theorem we obtain

$$\det Q = \sum_{i=1}^{d} q^{\#}_{ji} q_{ij}$$

for any fixed column index $j \in \{1, \dots, d\}$. Here, $q^{\#}_{ij} = (-1)^{i+j} \det Q_{ij}$ (with Q_{ij} the minor matrix of Q, obtained by removing row i and column j) denote the coefficients of the *adjoint form* $Q^{\#} = (\det Q) Q^{-1} \in \mathcal{S}^d_{>0}$ of Q. Thus

$$\operatorname{grad} \det Q = (\det Q) Q^{-1} \tag{8}$$

and the tangent hyperplane T in Q of the smooth *determinant-*$\det Q$*-surface*

$$S = \{Q' \in \mathcal{S}^d_{>0} : \det Q' = \det Q\}$$

is given by

$$T = \{Q' \in \mathcal{S}^d : \langle Q^{-1}, Q' \rangle = \langle Q^{-1}, Q \rangle\}.$$

Or in other words, Q^{-1} is a normal vector of the tangent plane T of S at Q. By Proposition 1 and the observation that (4) is convex, we know that S is contained in the halfspace

$$\{Q' \in \mathcal{S}^d : \langle Q^{-1}, Q' - Q \rangle \geq 0\}, \tag{9}$$

with Q being the unique intersection point of S and T.

As a consequence, a perfect form Q attains a local minimum of $\det Q$ (hence is extreme) if and only if the halfspace (9) contains the Ryshkov polyhedron $\mathcal{R}$, and its boundary meets $\mathcal{R}$ only in Q. This is easily seen to be equivalent to the condition that the normal cone (Voronoi domain) $\mathcal{V}(Q)$ of $\mathcal{R}$ at Q contains Q^{-1} in its interior. □

Note that eutaxy alone does not suffice for extremality. However, there exist only finitely many eutactic forms in every dimension and they can (in principle) be enumerated too (see [Mar03]). Nevertheless, this seems computationally more difficult than the enumeration of perfect forms (see [Sto75], [BM96], [Bat01], [EGS02]).

4. PARAMETER SPACES FOR PERIODIC SETS

We want to study the more general situation of periodic sphere packings. Recall from (1) that a periodic set with m lattice translates (an m-*periodic set*) in $\mathbb{R}^d$ is of the form

$$\Lambda' = \bigcup_{i=1}^{m} \boldsymbol{t}'_i + L, \tag{10}$$

with a lattice $L \subset \mathbb{R}^d$ and translation vectors $\boldsymbol{t}'_i \in \mathbb{R}^d$, $i = 1, \ldots, m$.

We want to work with a parameter space for m-periodic sets similar to $\mathcal{S}^d_{>0}$ for lattices. For this, we consider Λ' as a linear image $\Lambda' = A\Lambda_{\boldsymbol{t}}$ of a *standard periodic set*

$$\Lambda_{\boldsymbol{t}} = \bigcup_{i=1}^{m} \boldsymbol{t}_i + \mathbb{Z}^d. \tag{11}$$

Here, $A \in \mathsf{GL}_d(\mathbb{R})$ satisfies in particular $L = A\mathbb{Z}^d$. Since we are only interested in properties of periodic sets up to isometries, we encode Λ' by $Q = A^tA \in \mathcal{S}^d_{>0}$, together with the m translation vectors $\boldsymbol{t}_1, \ldots, \boldsymbol{t}_m$. Since every property of periodic sets we deal with here is invariant up to translations, we may assume without loss of generality that $\boldsymbol{t}_m = \boldsymbol{0}$. Thus we consider the parameter space

$$\mathcal{S}^{d,m}_{>0} = \mathcal{S}^d_{>0} \times \mathbb{R}^{d\times(m-1)} \tag{12}$$

for m-periodic sets (up to isometries). We hereby in particular generalize the space $\mathcal{S}^{d,1}_{>0} = \mathcal{S}^d_{>0}$ in a natural way. We call the elements of $\mathcal{S}^{d,m}_{>0}$ *periodic forms* and denote them usually by $X = (Q, \boldsymbol{t})$, where $Q \in \mathcal{S}^d_{>0}$ and

$$\boldsymbol{t} = (\boldsymbol{t}_1, \ldots, \boldsymbol{t}_{m-1}) \in \mathbb{R}^{d\times(m-1)}$$

is a real valued matrix containing $m-1$ columns with vectors $\boldsymbol{t}_i \in \mathbb{R}^d$. One should keep in mind, that although we omit $\boldsymbol{t}_m = \boldsymbol{0}$, we implicitly keep it as a translation vector. Note that a periodic set Λ' as in (10) has many *representations* by periodic forms. In particular, m may vary and we have different choices for A. The parameter space $\mathcal{S}^{d,m}_{>0}$ is contained in the space

$$\mathcal{S}^{d,m} = \mathcal{S}^d \times \mathbb{R}^{d\times(m-1)}. \tag{13}$$

Latter can be turned into a Euclidean space with inner product $\langle\cdot,\cdot\rangle$, defined for $X = (Q, \boldsymbol{t})$ and $X' = (Q', \boldsymbol{t}')$ by

$$\langle X, X'\rangle = \langle Q, Q'\rangle + \sum_{i=1}^{m-1} \boldsymbol{t}_i^t \boldsymbol{t}'_i.$$

Note, for the sake of simplicity we use the same symbol for the inner products on all spaces $\mathcal{S}^{d,m}$.

We extend the definition of the arithmetical minimum λ, by defining the *generalized arithmetical minimum*

$$\lambda(X) = \min\{Q[\boldsymbol{t}_i - \boldsymbol{t}_j - \boldsymbol{v}] : 1 \le i, j \le m \text{ and } \boldsymbol{v} \in \mathbb{Z}^d, \text{ with } \boldsymbol{v} \neq \boldsymbol{0} \text{ if } i = j\}$$

for the periodic form $X = (Q, \boldsymbol{t}) \in \mathcal{S}^{d,m}_{>0}$. Note that we have $\lambda(X) = 0$ in the case of intersecting lattice translates $(\boldsymbol{t}_i + \mathbb{Z}^d) \cap (\boldsymbol{t}_j + \mathbb{Z}^d) \neq \emptyset$ with $i \neq j$. The set of *representations of the generalized arithmetical minimum* $\operatorname{Min} X$ is the set of all $\boldsymbol{w} = \boldsymbol{t}_i - \boldsymbol{t}_j - \boldsymbol{v}$ attaining $\lambda(X)$. Computationally, $\operatorname{Min} X$ and $\lambda(X)$ can be obtained by solving a sequence of *closest vector problems* (CVPs), one for each pair i, j with $i \neq j$. In addition one shortest vector problem (SVP) has to be solved, taking care of the cases where $i = j$. Implementations of algorithms solving CVPs and SVPs are provided for example in MAGMA [MAG] or GAP [GAP].

In order to define the sphere packing density function $\delta : \mathcal{S}^{d,m}_{>0} \to \mathbb{R}$ we set $\det X = \det Q$ for periodic forms $X = (Q, \boldsymbol{t})$. Then

$$\delta(X) = \left(\frac{\lambda(X)}{(\det X)^{1/d}} \right)^{\frac{d}{2}} m \operatorname{vol} B^d / 2^d. \tag{14}$$

In analogy to the lattice case, we call a periodic form $X \in \mathcal{S}^{d,m}_{>0}$ *m-extreme*, if it attains a local maximum of δ within $\mathcal{S}^{d,m}_{>0}$.

The relation (14) shows that the supremum of δ among m-periodic sphere packings is up to some power and a constant factor equal to the "Hermite like constant"

$$\sup_{X \in \mathcal{S}^{d,m}_{>0}} \lambda(X)/(\det X)^{1/d} = 1/ \inf_{X \in \mathcal{R}_m} (\det X)^{1/d},$$

where the set $\mathcal{R}_m$ on the right side is the *(generalized) Ryshkov set*

$$\mathcal{R}_m = \left\{ X \in \mathcal{S}^{d,m}_{>0} : \lambda(X) \geq 1 \right\}. \tag{15}$$

The condition $\lambda(X) \geq 1$ gives infinitely many linear inequalities

$$p_{\boldsymbol{v}}(X) = Q[\boldsymbol{v}] = \langle X, (\boldsymbol{v}\boldsymbol{v}^t, 0) \rangle \geq 1$$

for $\boldsymbol{v} \in \mathbb{Z}^d \setminus \{\boldsymbol{0}\}$, as in the case $m = 1$. For $m > 1$ we additionally have the infinitely many polynomial inequalities

$$p_{i,j,\boldsymbol{v}}(X) = Q[\boldsymbol{t}_i - \boldsymbol{t}_j - \boldsymbol{v}] \geq 1, \tag{16}$$

where $i, j \in \{1, \dots, m\}$ with $i \neq j$ and $\boldsymbol{v} \in \mathbb{Z}^d$. These polynomials are of degree 3 in the parameters q_{kl}, t_{kl} of X. Note that they are linear for a fixed $\boldsymbol{t}$. Observe also that $p_{i,m,\boldsymbol{v}}$ and $p_{m,j,\boldsymbol{v}}$ are special due to our assumption $\boldsymbol{t}_m = \boldsymbol{0}$ and that there is a symmetry $p_{i,j,\boldsymbol{v}} = p_{j,i,-\boldsymbol{v}}$ by which we may restrict our attention to polynomials with $i \leq j$. In case of equality $i = j$ we have the linear function $p_{i,j,\boldsymbol{v}} = p_{\boldsymbol{v}}$.

5. LOCAL ANALYSIS OF PERIODIC SPHERE PACKINGS

Characterizing local optima. Before we generalize perfectness and eutaxy to a notion of *m-perfectness* and *m-eutaxy* (in order to obtain a sufficient condition for a periodic form to be m-extreme from it) we discuss a rather general setting: Assume we want to optimize a smooth function on a *basic closed semialgebraic set*, that is, on a region which is described by finitely many (non-strict) polynomial inequalities. Let E denote a Euclidean space with inner product $\langle \cdot, \cdot \rangle$. Further, let $f : E \to \mathbb{R}$ be *smooth* (infinitely differentiable) and $g_1, \dots, g_k$ be (real valued) polynomials on E. Assume, we want to determine whether or not we have a local minimum of f at X_0 on

$$G = \{X \in E : g_i(X) \geq 0 \text{ for } i = 1, \dots, k\}. \tag{17}$$

For simplicity, we further assume $(\operatorname{grad} f)(X_0) \neq 0$ and $g_i(X_0) = 0$, as well as $(\operatorname{grad} g_i)(X_0) \neq 0$, for $i = 1, \dots, k$. Then, in a sufficiently small neighborhood of X_0, the polynomials f and g_i can be approximated arbitrarily close by corresponding affine functions. For example f is approximated by the beginning of its *Taylor series*

$$f(X_0) + \langle(\operatorname{grad} f)(X_0), X - X_0\rangle.$$

From this one easily derives the following well known criterion (cf. for example [JS03]) for an isolated local minimum of f at X_0, depending on the normal cone

$$\mathcal{V}(X_0) = \operatorname{cone}\{(\operatorname{grad} g_i)(X_0) : i = 1, \dots, k\}.$$

The function f attains an isolated local minimum on G, if

$$(\operatorname{grad} f)(X_0) \in \operatorname{int} \mathcal{V}(X_0), \tag{18}$$

and f does not attain a local minimum, if

$$(\operatorname{grad} f)(X_0) \notin \mathcal{V}(X_0). \tag{19}$$

The behavior in case of $(\operatorname{grad} f)(X_0) \in \operatorname{bd}\operatorname{cone} \mathcal{V}(X_0)$ depends on the involved functions f and g_i and has to be treated, depending on the specific problem.

In case of the lattice sphere packing problem, we have $E = \mathcal{S}^d$, $f = \det^{1/d}$ and for $Q_0 \in \mathcal{S}^d_{>0}$ we set $g_i(Q) = Q[\boldsymbol{v}_i] - \lambda(Q_0)$ with $(\operatorname{grad} g_i)(Q) = \boldsymbol{v}_i\boldsymbol{v}_i^t$ for each pair $\pm\boldsymbol{v}_i$ in $\operatorname{Min} Q_0$. By Theorem 4 we have a local minimum of $f(Q) = (\det Q)^{1/d}$ at Q_0 on G (as in (17)) if and only if Q_0 is perfect and eutactic, respectively if $\mathcal{V}(Q_0)$ is full-dimensional and $(\operatorname{grad} f)(Q_0) \in \operatorname{int} \mathcal{V}(Q_0)$. Here, $(\operatorname{grad} f)(Q_0)$ is a positive multiple of Q_0^{-1}. Thus in this special case (due to Proposition 1 we do not have a local minimum of f in case $(\operatorname{grad} f)(Q_0) \in \operatorname{bd}\operatorname{cone} \mathcal{V}(Q_0)$.

Let us consider the case of periodic sets, hence of $E = \mathcal{S}^{d,m}$ with $m > 1$. We want to know if a periodic form $X_0 \in \mathcal{S}^{d,m}_{>0}$ attains a local minimum of $f = \det^{1/d}$. We may assume $\lambda(X_0) > 0$. The set $\operatorname{Min} X_0$ is finite and moreover, for $X = (Q, \boldsymbol{t})$ in a small neighborhood of $X_0 = (Q_0, \boldsymbol{t}^0)$, every $\boldsymbol{t}_i - \boldsymbol{t}_j - \boldsymbol{v} \in \operatorname{Min} X$ corresponds to a $\boldsymbol{t}_i^0 - \boldsymbol{t}_j^0 - \boldsymbol{v} \in \operatorname{Min} X_0$. Thus locally at X_0, the generalized Ryshkov set $\mathcal{R}_m$ is given by the basic closed semialgebraic set G defined by the inequalities $p_{i,j,\boldsymbol{v}}(X) - \lambda(X_0) \geq 0$, one for each pair $\pm(\boldsymbol{t}_i^0 - \boldsymbol{t}_j^0 - \boldsymbol{v})$ in $\operatorname{Min} X_0$. As explained in Section 4, we may assume $1 \leq i \leq j \leq m$ and $\boldsymbol{t}_j^0 = \boldsymbol{0}$ if $j = m$. An elementary calculation yields

$$(\operatorname{grad} p_{i,j,\boldsymbol{v}})(X) = (\boldsymbol{w}\boldsymbol{w}^t, \boldsymbol{0}, \dots, \boldsymbol{0}, 2Q\boldsymbol{w}, \boldsymbol{0}, \dots, \boldsymbol{0}, -2Q\boldsymbol{w}, \boldsymbol{0}, \dots, \boldsymbol{0}), \tag{20}$$

where we set $X = (Q, \boldsymbol{t})$ and use $\boldsymbol{w}$ to abbreviate $\boldsymbol{t}_i - \boldsymbol{t}_j - \boldsymbol{v}$. This is to be understood as a vector in $\mathcal{S}^{d,m} = \mathcal{S}^d \times \mathbb{R}^{d\times(m-1)}$, with its "$\mathcal{S}^d$-component" being the rank-1 form $\boldsymbol{w}\boldsymbol{w}^t$ and its "translational-component" containing the zero-vector $\boldsymbol{0}$ in all, but the ith and jth column. In case $j = m$, the jth column is omitted and in case $i = j$ the corresponding column is $\mathbf{0}$. For $(\operatorname{grad} f)(X)$ we obtain a positive multiple of $(Q^{-1}, \mathbf{0})$.

A sufficient condition for local m-periodic sphere packing optima. Generalizing the notion of perfectness, we say a periodic form $X = (Q, \boldsymbol{t}) \in \mathcal{S}^{d,m}_{>0}$ (and a corresponding periodic set represented by X) is *m-perfect* if the *generalized Voronoi domain*

$$\mathcal{V}(X) = \operatorname{cone}\{(\operatorname{grad} p_{i,j,\boldsymbol{v}})(X) : \boldsymbol{t}_i - \boldsymbol{t}_j - \boldsymbol{v} \in \operatorname{Min} X \text{ for some } \boldsymbol{v} \in \mathbb{Z}^d\} \tag{21}$$

is full dimensional, that is, if $\dim \mathcal{V}(X) = \dim \mathcal{S}^{d,m} = \binom{d+1}{2} + (m-1)d$. Generalizing the notion of eutaxy, we say that X (and a corresponding periodic set) is *m-eutactic* if

$$(Q^{-1}, \mathbf{0}) \in \operatorname{relint} \mathcal{V}(X).$$

So the general discussion at the beginning of this section yields the following sufficient condition for a periodic form X to be *isolated m-extreme*, that is, for X having the property that any sufficiently small change which preserves $\lambda(X)$, necessarily lowers $\delta(X)$.

Theorem 5. *If a periodic form $X \in \mathcal{S}^{d,m}_{>0}$ is m-perfect and m-eutactic, then X is isolated m-extreme.*

Note that the theorem gives a computational tool to certify isolated m-extremeness of a given periodic form $X = (Q, \boldsymbol{t}) \in \mathcal{S}^{d,m}_{>0}$: First, we compute $\operatorname{Min} X$ and use equation (20) to obtain generators of the generalized Voronoi domain $\mathcal{V}(X)$. Next, we can computationally test whether $(Q^{-1}, \mathbf{0})$ is in $\mathcal{V}(X)$ or not (for example by solving a linear program similar to (7)). In case we can show $(Q^{-1}, \mathbf{0}) \in \operatorname{int} \mathcal{V}(X)$, the periodic form X represents an isolated m-extreme periodic set. If we can show $(Q^{-1}, \mathbf{0}) \notin \mathcal{V}(X)$, the periodic form X does not represent an m-extreme periodic set. In this situation, we can even find a "direction" $N \in \mathcal{S}^{d,m}$, for which we can improve the sphere packing density of the periodic form X, that is, such that $\delta(X + \epsilon N) > \delta(X)$ for all sufficiently small $\epsilon > 0$.

Remark 6. Let $X \in \mathcal{S}^{d,m}_{>0}$ with $(Q^{-1}, \mathbf{0}) \notin \mathcal{V}(X)$. Then we can improve the sphere packing density of X in direction N given by the nearest point to $-(Q^{-1}, \mathbf{0})$ in the polyhedral cone

$$\mathcal{P}(X) = \{N \in \mathcal{S}^{d,m} : \langle V, N\rangle \geq 0 \text{ for all } V \in \mathcal{V}(X)\}. \tag{22}$$

Note that the cone $\mathcal{P}(X)$ is dual to the generalized Voronoi domain $\mathcal{V}(X)$ and (added to X) gives locally a linear approximation of the generalized Ryshkov set $\mathcal{R}_m$.

Fluid diamond packings. For general m we are confronted with a difficulty which does not show up in the lattice case $m = 1$: There may be non-isolated m-extreme sets, which are not m-perfect. The *fluid diamond packings* in dimension 9, described by Conway and Sloane in [CS95], give such an example.

Example. The *root lattice* D_d can be defined by

$$\mathsf{D}_d = \{\boldsymbol{x} \in \mathbb{Z}^d : \sum_{i=1}^{d} x_i \equiv 0 \mod 2\}.$$

The *fluid diamond packings* are 2-periodic sets

$$\mathsf{D}_9\langle \boldsymbol{t} \rangle = \mathsf{D}_9 \cup (\mathsf{D}_9 + \boldsymbol{t})$$

with $\boldsymbol{t} \in \mathbb{R}^9$ such that the minimal distance among elements is equal to the minimum distance $\sqrt{2}$ of D_9 itself. We may choose for example $\boldsymbol{t}_\alpha = (\frac{1}{2}, \dots, \frac{1}{2}, \alpha)^t$ with any $\alpha \in \mathbb{R}$. For integers α we obtain the densest known packing lattice $\Lambda_9 = \mathsf{D}_9\langle \boldsymbol{t}_\alpha \rangle$ in dimension 9, showing that it is part of a family of uncountably many, equally dense 2-periodic sets.

The sets $\mathsf{D}_9\langle \boldsymbol{t}_\alpha \rangle$ give examples of non-isolated 2-extreme sets, which are 2-eutactic, but not 2-perfect. In order to see this, let us consider a representation $X_\alpha \in \mathcal{S}^{9,2}_{>0}$ for $\mathsf{D}_9\langle \boldsymbol{t}_\alpha \rangle$. We choose a basis A of D_9. Then $X_\alpha = (Q, A^{-1}\boldsymbol{t}_\alpha)$ with $Q = A^t A$ is a representation of $\mathsf{D}_9\langle \boldsymbol{t}_\alpha \rangle$.

For non-integral α we find $\operatorname{Min} X_\alpha = \operatorname{Min} Q$ (using `MAGMA` for example). It follows (for example by Lemma 9 below) that X_α is 2-eutactic, but not 2-perfect. For integral α we find

$$\operatorname{Min} X_\alpha = \operatorname{Min} Q \cup \{(x_1, \dots, x_8, 0)^t \in \{0,1\}^9 : \sum_{i=1}^{8} x_i \equiv 0 \mod 2\}.$$

Thus the vectors in $\operatorname{Min} X_\alpha \setminus \operatorname{Min} Q$ span only an 8-dimensional space. Therefore X_α is not 2-perfect. Nevertheless, a corresponding calculation shows that X_α is 2-eutactic, as in the case of non-integral α.

In order to see that X_α is non-isolated 2-extreme, we can apply Proposition 7 below. One easily checks that in case of integral α (hence for the lattice Λ_9) we have only one degree of freedom for a local change of $\boldsymbol{t}_\alpha$ giving an equally dense sphere packing. In case of non-integral α we have nine degrees of freedom for such a modification.

□

Non-isolated m-extreme sets as in this example can only occur for periodic forms $X \in \mathcal{S}^{d,m}_{>0}$ if $(Q^{-1}, \mathbf{0}) \in \operatorname{bd} \mathcal{V}(X)$ (which is for example always the case when X is m-eutactic, but not m-perfect). In this case, it is in general not clear what an infinitesimal change of X in a direction $N \in \mathcal{S}^{d,m}$ leads to (already assuming it is orthogonal to $(Q^{-1}, \mathbf{0})$ as well as in the boundary of the set $\mathcal{P}(X)$ in (22)). If $\mathcal{F}(X)$ denotes the unique face of $\mathcal{V}(X)$ containing $(Q^{-1}, \mathbf{0})$ in its relative interior, then this "set of uncertainty" is equal to the face of $\mathcal{P}(X)$ dual to $\mathcal{F}(X)$, that is, equal to

$$\mathcal{U}(X) = \{N \in \mathcal{P}(X) : \langle V, N \rangle = 0 \text{ for all } V \in \mathcal{F}(X)\}. \tag{23}$$

Or in other words, the set $\mathcal{U}(X)$ is the intersection of $\mathcal{P}(X)$ with the hyperplane orthogonal to $(Q^{-1}, \mathbf{0})$. Note that it is possible to determine $\mathcal{F}(X)$ (and hence a description of $\mathcal{U}(X)$ by linear inequalities) computationally, using linear programming techniques.

Purely translational changes. Below we give an additional sufficient condition for m-extremeness. For this we consider the case when all directions in $\mathcal{U}(X)$ are "*purely translational changes*" $N = (0, \boldsymbol{t}^N) \in \mathcal{S}^{d,m}$. A vivid interpretation of a purely translational change can be given by thinking of the corresponding modification of a periodic sphere packing. The spheres of each lattice translate are jointly moved. If in such a local change all contacts among spheres are lost, we can increase their radius and obtain a new sphere packing with larger density. If some contacts among spheres are preserved however, the sphere packing density remains the same. Latter case is captured in the following proposition, which gives an easily testable criterion for m-extremeness. We apply this proposition in Section 6, where we consider potential local improvements of best known packing lattices to periodic non-lattice sets.

Proposition 7. *For a periodic form* $X = (Q, \boldsymbol{t}) \in \mathcal{S}^{d,m}_{>0}$ *with* $(Q^{-1}, \boldsymbol{0}) \in \operatorname{bd} \mathcal{V}(X)$*, let* $\mathcal{U}(X)$ *be contained in*

$$\{(\boldsymbol{0}, \boldsymbol{t}^N) \in \mathcal{S}^{d,m} : \boldsymbol{t}_i^N = \boldsymbol{t}_j^N \text{ for at least one } \boldsymbol{t}_i - \boldsymbol{t}_j - \boldsymbol{v} \in \operatorname{Min} X \text{ with } \boldsymbol{v} \in \mathbb{Z}^d\}.$$

Then X *is (possibly non-isolated)* m*-extreme.*

Note, if X is m-eutactic (possibly not m-perfect), the set $\mathcal{U}(X)$ is the *orthogonal complement* $\mathcal{V}(X)^{\perp}$ of the linear hull of $\mathcal{V}(X)$. Note also that Proposition 7 includes in particular the special case where some $\boldsymbol{v} \in \mathbb{Z}^d$ are in $\operatorname{Min} X$ (and therefore $\boldsymbol{t}_i = \boldsymbol{t}_j = \boldsymbol{0}$ for $i = j = m$). This situation occurs for the 2-periodic, fluid diamond packings in the example above.

From the sphere packing interpretation of the proposition its assertion is clear. Nevertheless, we give a proof below, based on a local analysis in $\mathcal{S}^{d,m}_{>0}$. More than actually needed for the proof, we analyze how δ changes locally at a periodic form $X \in \mathcal{S}^{d,m}_{>0}$ in a direction $N \in \mathcal{U}(X)$. As a byproduct, we obtain tools allowing a computational analysis of possible local optimality for a given periodic form, not necessarily covered by the proposition. These can for example be used in a numerical search for good periodic sphere packings.

Proof of Proposition 7. The generalized Voronoi domain $\mathcal{V}(X)$ is spanned by gradients $(\operatorname{grad} p_{i,j,\boldsymbol{v}})(X)$ (as given in (20)), one for each pair of vectors $\pm\boldsymbol{w} \in \operatorname{Min} X$. The assumption that a direction $N = (Q^N, \boldsymbol{t}^N)$ is in $\mathcal{U}(X)$ for a periodic form $X = (Q, \boldsymbol{t})$, implies $\langle Q^{-1}, Q^N \rangle = 0$. Moreover, for the unique maximal face $\mathcal{F}(X)$ of $\mathcal{V}(X)$ with $(Q^{-1}, \boldsymbol{0}) \in \operatorname{relint} \mathcal{F}(X)$, the condition that N is orthogonal to some $(\operatorname{grad} p_{i,j,\boldsymbol{v}})(X)$ in $\mathcal{F}(X)$ translates into

$$\langle (\operatorname{grad} p_{i,j,\boldsymbol{v}})(X), N \rangle = Q^N[\boldsymbol{w}] + 2(\boldsymbol{t}_i^N - \boldsymbol{t}_j^N)^t Q \boldsymbol{w} = 0, \tag{24}$$

with $\boldsymbol{w} = (\boldsymbol{t}_i - \boldsymbol{t}_j - \boldsymbol{v})$. Recall that in the special case $i = j$ (and for $m = 1$ anyway) $p_{i,j,\boldsymbol{v}}$ is linear and (24) reduces to the condition $Q^N[\boldsymbol{w}] = 0$; if then N satisfies this linear condition, $p_{i,j,\boldsymbol{v}}(X + \epsilon N)$ is a constant function in ϵ.

In case $p_{i,j,\boldsymbol{v}}(X+\epsilon N)$ is a cubic polynomial in ϵ we need to use higher order information in order to judge its behavior. An elementary calculation yields for the *Hessian*

$$(\operatorname{hess} p_{i,j,\boldsymbol{v}})(X)[N] = 2Q[\boldsymbol{t}_i^N - \boldsymbol{t}_j^N] + 4(\boldsymbol{t}_i^N - \boldsymbol{t}_j^N)^t Q^N \boldsymbol{w}. \tag{25}$$

Now how does δ change at X in direction N, assuming it is in the set of uncertainty $\mathcal{U}(X)$? Among the polynomials $p_{i,j,\boldsymbol{v}}$ with N satisfying (24), the fastest decreasing polynomial in direction N determines $\lambda(X+\epsilon N)$ for small enough ϵ. Thus for the local change of δ in direction N, we may restrict our attention to a polynomial $p_{i,j,\boldsymbol{v}}$ with the smallest value (25) of its Hessian.

By Proposition 1 we know that $\det^{1/d}$ decreases strictly at X in a direction $N \in \mathcal{U}(X)$ if and only if $Q^N \neq 0$.

In case of a purely translational change with $Q^N = 0$, the function $\det^{1/d}$ remains constant. On the other hand, because of (25) and since Q is positive definite, we have $(\operatorname{hess} p_{i,j,\boldsymbol{v}})(X)[N] \geq 0$, with equality if and only if $\boldsymbol{t}_i^N - \boldsymbol{t}_j^N = \boldsymbol{0}$. Latter implies that $p_{i,j,\boldsymbol{v}}(X+\epsilon N)$ is a constant function of ϵ. Thus for purely translational changes $N = (0, \boldsymbol{t}^N) \in \mathcal{U}(X)$, the density function $\delta(X+\epsilon N)$ is constant for small enough $\epsilon \geq 0$, if $\boldsymbol{t}_i^N = \boldsymbol{t}_j^N$ for some pair (i,j) with $\boldsymbol{t}_i - \boldsymbol{t}_j - \boldsymbol{v} \in \operatorname{Min} X$ (for a suitable $\boldsymbol{v} \in \mathbb{Z}^d$). This proves the proposition. □

Note that our argumentation in the proof also shows that $\delta(X+\epsilon N)$ increases for small $\epsilon > 0$, in case of a purely translational change $N = (0, \boldsymbol{t}^N) \in \mathcal{U}(X)$ with $\boldsymbol{t}_i^N \neq \boldsymbol{t}_j^N$ for all pairs (i,j) with $\boldsymbol{t}_i - \boldsymbol{t}_j - \boldsymbol{v} \in \operatorname{Min} X$ (for some $\boldsymbol{v} \in \mathbb{Z}^d$). This case corresponds to a modification of a periodic sphere packing, in which all contacts among spheres are lost.

6. Periodic extreme sets

A given periodic set has many representations by periodic forms, in spaces $\mathcal{S}^{d,m}_{>0}$ with varying m. For example, by choosing some sublattice of $\mathbb{Z}^d$, we can add additional translational parts.

Now it could happen that a periodic set Λ with a given representation $X \in \mathcal{S}^{d,m}_{>0}$ is m-extreme, whereas a second representation $X' \in \mathcal{S}^{d,m'}$ is not m'-extreme. However, it may also happen that the packing density of no representation of Λ can locally be improved.

Definition 8. *A periodic set is* periodic extreme*, if it is m-extreme for all possible representations $X \in \mathcal{S}^{d,m}_{>0}$.*

Our main result is Theorem 10 below, giving a sufficient condition for a lattice to be periodic extreme. For its statement we need the notion of *strong eutaxy* for lattices, respectively PQFs: A form $Q \in \mathcal{S}^d_{>0}$ (and a corresponding lattice) is called *strongly eutactic*, if

$$Q^{-1} = \alpha \sum_{\boldsymbol{x} \in \operatorname{Min} Q} \boldsymbol{x}\boldsymbol{x}^t \tag{26}$$

for some $\alpha > 0$, hence if the coefficients in the eutaxy condition (6) are all equal. It is well known that a PQF Q is strongly eutactic, if and only if the vectors in $\operatorname{Min} Q$ form a so-called *spherical* 2*-design* with respect to Q (see [Ven01], [Mar03]).

Lemma 9. *Any representation $X \in \mathcal{S}^{d,m}_{>0}$ of a strongly eutactic lattice (respectively PQF) is m-eutactic.*

Proof. Let $Q \in \mathcal{S}^{d}_{>0}$ be strongly eutactic, satisfying (26) for some $\alpha > 0$. Let $X = (Q^X, \boldsymbol{t}^X) \in \mathcal{S}^{d,m}_{>0}$ be some representation of a strongly eutactic PQF Q, e.g. with $m > 1$.

For a fixed $\boldsymbol{w} \in \operatorname{Min} X$ we define an abstract graph, whose vertices are the indices in $\{1, \dots, m\}$. Two vertices i and j are connected by an edge, whenever there is some $\boldsymbol{v} \in \mathbb{Z}^d$ such that $\boldsymbol{w} = \boldsymbol{t}_i^X - \boldsymbol{t}_j^X - \boldsymbol{v}$. Since the periodic form X represents a lattice, we find that the graph is a disjoint union of cycles. Or, in other words, $\boldsymbol{w}$ induces a partition $(I_1, \dots, I_k)$ of $\{1, \dots, m\}$.

Let I be an index set of this partition (containing the indices of a fixed cycle of the defined graph). Summing over all triples $(i, j, \boldsymbol{v})$ with $i, j \in I$ and $\boldsymbol{v} \in \mathbb{Z}^d$ such that $\boldsymbol{w} = \boldsymbol{t}_i^X - \boldsymbol{t}_j^X - \boldsymbol{v} \in \operatorname{Min} X$, we find (using (20)):

$$\sum_{\substack{(i,j,\boldsymbol{v}) \in I^2 \times \mathbb{Z}^d: \\ \boldsymbol{w} = \boldsymbol{t}_i^X - \boldsymbol{t}_j^X - \boldsymbol{v} \in \operatorname{Min} X}} (\operatorname{grad} p_{i,j,\boldsymbol{v}})(X) = 2|I|(\boldsymbol{w}\boldsymbol{w}^t, \boldsymbol{0}).$$

The factor 2 comes from the symmetry $\operatorname{grad} p_{i,j,\boldsymbol{v}} = \operatorname{grad} p_{j,i,-\boldsymbol{v}}$. Summation over all index sets I of the partition yields

$$\sum_{\substack{(i,j,\boldsymbol{v}) \in \{1,\dots,m\}^2 \times \mathbb{Z}^d: \\ \boldsymbol{w} = \boldsymbol{t}_i^X - \boldsymbol{t}_j^X - \boldsymbol{v} \in \operatorname{Min} X}} (\operatorname{grad} p_{i,j,\boldsymbol{v}})(X) = 2m(\boldsymbol{w}\boldsymbol{w}^t, \boldsymbol{0}). \tag{27}$$

As a consequence we find by the strong eutaxy condition (26) that

$$(Q^{-1}, \boldsymbol{0}) \; = \; (\alpha/2m) \sum_{\substack{(i,j,\boldsymbol{v}) \in \{1,\dots,m\}^2 \times \mathbb{Z}^d: \\ \boldsymbol{w} = \boldsymbol{t}_i^X - \boldsymbol{t}_j^X - \boldsymbol{v} \in \operatorname{Min} X}} (\operatorname{grad} p_{i,j,\boldsymbol{v}})(X),$$

with a suitable $\alpha > 0$. Thus X is m-eutactic. □

Not all PQFs (or lattices) which are strongly eutactic have to be perfect. But if a strongly eutactic PQF is in addition also perfect, then the following theorem shows that this is sufficient for it to be periodic extreme. Note that this applies in particular to so called *strongly perfect lattices and PQFs* (see [Neb02], [Mar03]).

Theorem 10. *A perfect, strongly eutactic lattice (respectively PQF) is periodic extreme.*

Proof. Let $Q \in \mathcal{S}^{d}_{>0}$ be perfect and strongly eutactic. Hence the vectors in $\operatorname{Min} Q$ span $\mathbb{R}^d$ (by Proposition 2) and satisfy (26) for some $\alpha > 0$. Let $X = (Q^X, \boldsymbol{t}^X) \in \mathcal{S}^{d,m}_{>0}$ be a representation of Q, e.g. with $m > 1$. By Lemma 9, X is m-eutactic. If X is m-perfect as well, we know by Theorem 5 that X is also m-extreme.

So let us assume that X is not m-perfect, hence the generalized Voronoi domain $\mathcal{V}(X)$ is not full dimensional. We want to apply Proposition 7. For this we choose

$$N = (Q^N, \boldsymbol{t}^N) \in \mathcal{U}(X) = \mathcal{V}(X)^\perp \quad \text{with} \quad N \neq 0.$$

(Recall the definition of $\mathcal{U}(X)$ from (23) and that $\mathcal{U}(X) = \mathcal{V}(X)^\perp$ in case X is m-eutactic.) By this assumption we have in particular

$$\langle N, (\operatorname{grad} p_{i,j,\boldsymbol{v}})(X)\rangle = 0$$

for all triples $(i, j, \boldsymbol{v})$ with $\boldsymbol{w} = \boldsymbol{t}_i^X - \boldsymbol{t}_j^X - \boldsymbol{v} \in \operatorname{Min} X$. Using equation (27), which we obtained in the proof of Lemma 9, we get $\langle N, (\boldsymbol{w}\boldsymbol{w}^t, \boldsymbol{0})\rangle = Q^N[\boldsymbol{w}] = 0$ for every fixed $\boldsymbol{w} \in \operatorname{Min} X$.

By Proposition 2 there exist d linear independent $\boldsymbol{w}$ in $\operatorname{Min} X$, which implies $Q^N = 0$. Using (24), we obtain

$$0 = \langle N, (\operatorname{grad} p_{i,j,\boldsymbol{v}})(X)\rangle = 2(\boldsymbol{t}_i^N - \boldsymbol{t}_j^N)^t Q\boldsymbol{w}. \tag{28}$$

In case $\boldsymbol{t}_i^N - \boldsymbol{t}_j^N = \boldsymbol{0}$ for some pair (i, j) we can apply Proposition 7. Note that this includes in particular the case $i = j = m$ ($\boldsymbol{t}_i^N = \boldsymbol{t}_j^N = \boldsymbol{0}$) if $\boldsymbol{v} \in \mathbb{Z}^d \cap \operatorname{Min} X$. So we may assume that such $\boldsymbol{v}$ do not exist.

We define an abstract graph with vertices in $\{1, \dots, m\}$: (i, j) is an edge, whenever there is some $\boldsymbol{v} \in \mathbb{Z}^d$ such that $\boldsymbol{t}_i^X - \boldsymbol{t}_j^X - \boldsymbol{v} \in \operatorname{Min} X$. Let I be the set of vertices (indices) i, connected by a path with m. Then by the assumption $\mathbb{Z}^d \cap \operatorname{Min} X = \emptyset$ the cardinality $|I|$ of I is greater than 1. Since there exist d linearly independent vectors in $\operatorname{Min} X$ (by Proposition 2), the linear equations $\langle N, (\operatorname{grad} p_{i,j,\boldsymbol{v}})(X)\rangle = 0$ with $i, j \in I$ have rank $d(|I| - 1)$. Thus by (28) and $\boldsymbol{t}_m^N = \boldsymbol{0}$, we deduce $\boldsymbol{t}_i^N = \boldsymbol{0}$ for all $i \in I$; hence we can again apply Proposition 7, proving the theorem. □

Note that the set I used in the last paragraph of the proof can be a strict subset of $\{1, \dots, m\}$. This is the case, if the graph we defined is not connected. Each connected component corresponds to a union of translates, which can jointly, locally be changed, without changing $\lambda(X)$, respectively $\delta(X)$. For example the fluid diamond packings described in the example of Section 5 have this property (although Λ_9 is not strictly eutactic).

The root lattices A_d, D_d and E_d, as well as the Leech lattice are known to be perfect and strongly eutactic (cf. [Mar03]). Thus as an immediate consequence of Theorem 10, we find that these lattices, which are known to solve the lattice sphere packing problem in dimensions $d \leq 8$ and $d = 24$ (see Table 1), cannot locally be improved to a periodic non-lattice set with greater sphere packing density.

Corollary 11. *The lattices A_d, for $d \geq 2$, D_d, for $d \geq 3$, and E_d for $d = 6, 7, 8$, as well as the Leech lattice are periodic extreme.*

A result similar to Corollary 11 for the root lattices A_d and D_d was obtained by Bezdek, Bezdek and Connelly [BBC98, Remark 2.9.1]. Their method is different and based on *Delone subdivisions*. At this point it is not clear whether or not their method can be used to obtain a more general result similar to Theorem 10.

We also checked whether or not Theorem 10 can be applied to other dimensions d below 24. For these dimensions the so-called *laminated lattices* Λ_d and *sections* K_d *of the Leech lattice* give the densest known lattice sphere packings. The lattices K_d are different from Λ_d (and at the same time give the densest known lattice sphere packings) only in dimensions $d = 11, 12, 13$. For these d, the lattice K_d is strongly eutactic only for $d = 12$, where K_d is also known as *Coxeter-Todd lattice*. The laminated lattices Λ_d give the densest known packing lattices in dimensions $d = 9, 10$ and $d = 14, \ldots, 24$ (for $d = 18, \ldots, 24$ they coincide with K_d). Among those values for d, the laminated lattices Λ_d are strongly eutactic if and only if $d = 15, 16$ or $d \geq 20$. Concluding, we cannot exclude that densest known lattice sphere packings in dimensions $d \in \{9, 10, 11, 13, 14, 17, 18, 19\}$ can locally be improved to better periodic sphere packings. Further analysis is required here.

References

[Bat01] C. Batut, *Classification of quintic eutactic forms*, Math. Comp. **70** (2001), 395–417.

[BBC98] A. Bezdek, K. Bezdek, and R. Connelly, *Finite and uniform stability of sphere packings*, Discrete Comput. Geom. **20** (1998), 111–130.

[BE00] J. Bierbrauer and Y. Edel, *Dense sphere packings from new codes*, J. Algebr. Comb. **11** (2000), 95–100.

[Bli35] H.F. Blichfeldt, *The minimum values of positive quadratic forms in six, seven and eight variables*, Math. Z. **39** (1935), 1–15.

[BM96] A.M. Bergé and J. Martinet, *Sur la classification des réseaux eutactiques*, J. London Math. Soc. (2) **53** (1996), 417–432.

[CK08] H. Cohn and A. Kumar, *Optimality and uniqueness of the Leech lattice among lattices*, Ann. Math. (2008), to appear, Preprint at arXiv:math.MG/0403263.

[CS95] J.H. Conway and N.J.A. Sloane, *What are all the best sphere packings in low dimensions?*, Discrete Comput. Geom. **13** (1995), 383–403.

[CS96] ———, *The antipode construction for sphere packings*, Invent. Math. **123** (1996), 309–313.

[CS99] ———, *Sphere packings, lattices and groups*, Springer, New York, 1999, 3rd ed.

[DSV07] M. Dutour Sikirić, A. Schürmann, and F. Vallentin, *Classification of eight dimensional perfect forms*, Electron. Res. Announc. Amer. Math. Soc. **13** (2007), 21–32.

[EGS02] P. Elbaz-Vincent, H. Gangl, and C. Soulé, *Quelques calculs de la cohomologie de* $\mathrm{GL}_N(\mathbb{Z})$ *et de la* K*-théorie de* $\mathbb{Z}$, C. R. Math. Acad. Sci. Paris **335** (2002), 321–324.

[GAP] *The GAP group, GAP — Groups, Algorithms, Programming - a system for computational discrete algebra*, ver.4.4., http://www.gap-system.org/.

[Gau40] C.F. Gauss, *Untersuchungen über die Eigenschaften der positiven ternären quadratischen Formen von Ludwig August Seeber*, J. Reine. Angew. Math. **20** (1840), 312–320.

[GL87] P.M. Gruber and C.G. Lekkerkerker, *Geometry of numbers*, North–Holland, Amsterdam, 1987.

[Gru07] P.M. Gruber, *Convex and discrete geometry*, Springer, Berlin, 2007.

[Hal05] T.C. Hales, *A proof of the Kepler conjecture*, Ann. Math **162** (2005), 1065–1185.

[JS03] F. Jarre and J. Stoer, *Optimierung*, Springer, Berlin, 2003.

[KZ77] A.N. Korkin and E.I. Zolotarev, *Sur les formes quadratiques positives*, Math. Ann. **11** (1877), 242–292.

[Lag73] J.L. Lagrange, *Recherches d'arithmétique*, Nouv. Mém. Acad. Berlin (1773), 265–312, in Oeuvres de Lagrange III, 695–795.

[LS70] J. Leech and N.J.A. Sloane, *New sphere packings in dimensions* 9 - 15, Bull. Am. Math. Soc. **76** (1970), 1006–1010.

[MAG] *Computational Algebra Group, University of Sydney, MAGMA — high performance software for Algebra, Number Theory, and Geometry*, ver.2.13. http://magma.maths.usyd.edu.au/.

[Mar03] J. Martinet, *Perfect lattices in Euclidean spaces*, Springer, Berlin, 2003.

[Min05] H. Minkowski, *Diskontinuitätsbereich für arithmetische Äquivalenz*, J. Reine Angew. Math. **129** (1905), 220–274, Reprint in *Gesammelte Abhandlungen*, Band II, Teubner, Leipzig, 1911.

[Neb02] G. Nebe, *Gitter und Modulformen*, Jahresber. Deutsch. Math.-Verein. **104** (2002), no. 3, 125–144.

[Nel74] C.E. Nelson, *The reduction of positive definite quinary quadratic forms*, Aequationes Math. **11** (1974), 163–168.

[NS] G. Nebe and N.J.A. Sloane, *A catalogue of lattices*, published electronically at http://www.research.att.com/ njas/lattices/.

[Rys70] S.S. Ryshkov, *The polyhedron $\mu(m)$ and certain extremal problems of the geometry of numbers*, Soviet Math. Dokl. **11** (1970), 1240–1244, translation from Dokl. Akad. Nauk SSSR 194, 514–517 (1970).

[Sch08] A. Schürmann, *Computational geometry of positive definite quadratic forms*, 2008, to appear in AMS University Lecture Series.

[Sto75] M.I. Stogrin, *Locally quasidensest lattice packings of spheres*, Soviet Math. Dokl. **15** (1975), 1288–1292, translation from Dokl. Akad. Nauk SSSR 218, 62–65 (1974).

[Thu10] A. Thue, *Über die dichteste Zusammenstellung von kongruenten Kreisen in einer Ebene*, Norske Vid. Selsk. Skr. **1** (1910), 1–9.

[Var95] A. Vardy, *A new sphere packing in 20 dimensions*, Invent. Math. **121** (1995), 119–133.

[Ven01] B.B. Venkov, *Réseaux et designs sphériques*, Réseaux euclidiens, designs sphériques et formes modulaires, Monogr. Enseign. Math., vol. 37, Enseignement Math., Geneva, 2001, pp. 10–86.

[Vor07] G.F. Voronoi, *Nouvelles applications des paramètres continus à la théorie des formes quadratiques. Premier Mémoire. Sur quelques propriétés des formes quadratiques positives parfaites*, J. Reine Angew. Math. **133** (1907), 97–178.

[Zie97] G.M. Ziegler, *Lectures on polytopes*, Springer, New York, 1997.

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