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A unifying theory of a posteriori error control for discontinuous Galerkin FEM

Carsten Carstensen · Thirupathi Gudi · Max Jensen

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Abstract A unified a posteriori error analysis is derived in extension of Carstensen (Numer Math 100:617–637, 2005) and Carstensen and Hu (J Numer Math 107(3):473–502, 2007) for a wide range of discontinuous Galerkin (dG) finite element methods (FEM), applied to the Laplace, Stokes, and Lamé equations. Two abstract assumptions (A1) and (A2) guarantee the reliability of explicit residual-based computable error estimators. The edge jumps are recast via lifting operators to make arguments already established for nonconforming finite element methods available. The resulting reliable error estimate is applied to 16 representative dG FEMs from the literature. The estimate recovers known results as well as provides new bounds to a number of schemes.

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C. Carstensen (✉)
Department of Mathematics, Humboldt University Berlin, Unter den Linden 6, 10099 Berlin, Germany
e-mail: cc@math.hu-berlin.de

T. Gudi
Center for Computation and Technology, Johnston Hall, Louisiana State University,
Baton Rouge 70803, USA
e-mail: tgudi@cct.lsu.edu

M. Jensen
Department of Mathematical Sciences, University of Durham, Science Laboratories, South Road,
Durham DH1 3LE, UK
e-mail: m.p.j.jensen@durham.ac.uk

1 Unified mixed approach to error control

This section introduces a primal mixed formulation used in [15] to cover applications which have an abstract saddle-point structure such as the Laplace, Stokes, and Lamé equations. Throughout this paper, let V and L be Hilbert spaces (Sobolev and Lebesgue spaces) and set $X := L \times V$ with dual $X^* := L^* \times V^*$. The primal variable $u \in V$ (e.g., the displacement field or velocity) is accompanied by a dual variable $p \in L$ (e.g., the flux or stress). It is assumed that the linear operator $A : X \to X^*$, defined by

$$(A(p,u))(q,v) := a(p,q) + b(p,v) + b(q,u) \quad \text{for all } u, v \in V \text{ and } p, q \in L, \tag{1.1}$$

is bounded, bijective, and has a continuous inverse. This can be ensured under well-established conditions [4,11] on the bilinear form $a : L \times L \to \mathbb{R}$ and the linear form $b : L \times V \to \mathbb{R}$. Given any $f \in L^*$ and $g \in V^*$, there consequently exists a unique $(p,u) \in X$ such that

$$a(p,q) + b(q,u) = f(q) \quad \text{for all } q \in L, \tag{1.2}$$

$$b(p,v) = g(v) \quad \text{for all } v \in V. \tag{1.3}$$

Given any approximation $(p_h, \tilde{u}_h) \in X$ to (p,u) define the two residuals

$$\mathcal{R}es_L(q) := f(q) - a(p_h, q) - b(q, \tilde{u}_h) \quad \text{for all } q \in L, \tag{1.4}$$

$$\mathcal{R}es_V(v) := g(v) - b(p_h, v) \quad \text{for all } v \in V. \tag{1.5}$$

The residual $\mathcal{R}es_V \in V^*$ is called *equilibrium residual*, while $\mathcal{R}es_L \in L^*$ is the *consistency residual*. Here and throughout the text, $\tilde{u}_h$ is a function in V and not necessarily a discrete function; the subindex h in $\tilde{u}_h$ refers to the fact that $\tilde{u}_h$ might be closely related to u_h and, in principal, is at our disposal. Since $A : X \to X^*$ is an isomorphism, it follows that

$$\|p - p_h\|_L + \|u - \tilde{u}_h\|_V \approx \|\mathcal{R}es_L\|_{L^*} + \|\mathcal{R}es_V\|_{V^*}. \tag{1.6}$$

We note that the inequality $a \lesssim b$ replaces $a \leq Cb$ if C is a mesh-size independent constant which depends only on the domain Ω and the shape of the finite elements. Moreover, $a \approx b$ abbreviates $a \lesssim b \lesssim a$.

One obstacle in the application of (1.6) to the dG FEM is that p_h is only required to have Lebesgue regularity and hence jumps or the resulting mesh-depending norms are not meaningful for p_h. The main observation of this paper is that lifting operators provide a proper flux reconstruction to apply (1.4) and (1.5) and to avoid all mesh-dependent norms from the beginning.

Theorem 3.1 below states abstract and precise estimates of $\|\mathcal{R}es_V\|_V^*$ and $\|\mathcal{R}es_L\|_L^*$ in terms of the two estimators $\eta + \zeta$ for the equilibration and consistency error sources in continuation of [15] for dG FEMs.

We highlight that the error analysis also covers inconsistent methods such as [8,13]. The focus of this paper is on a universal view of the a posteriori error analysis and not on generic constants as, e.g., in [2].

2 Preliminaries

This section fixes the notation and outlines known results employed in the subsequent sections.

2.1 Function spaces

Let $V = H^1_0(\Omega; \mathbb{R}^m)$ and $L = L^2(\Omega; \mathbb{R}^{m\times n})$ denote standard Sobolev and Lebesgue spaces on a bounded polyhedral domain Ω in $\mathbb{R}^n$. Suppose that $\mathcal{T}$ is a decomposition of $\bar{\Omega}$ into triangles or rectangles if $n = 2$ and into tetrahedrons or parallelepipeds if $n = 3$. An $(n-1)$-dimensional face of an element T is called facet. Let $\mathcal{E}_\Omega$ be the set of all interior facets, let $\mathcal{E}_{\partial\Omega}$ be the set of all boundary facets, and let $\mathcal{E} := \mathcal{E}_\Omega \cup \mathcal{E}_{\partial\Omega}$. The diameter of an element T is denoted by h_T; the diameter of a facet is denoted by h_E.

The spaces of polynomials of total or partial degree less or equal than k are denoted by $P_k(T)$ and $Q_k(T)$, respectively. If T is a triangle or tetrahedron set $\mathcal{P}_k(T) := P_k(T)$, otherwise $\mathcal{P}_k(T) := Q_k(T)$. Define

$$
\begin{aligned}
V_h &:= V_h(\mathcal{T}; \mathbb{R}^m) := \Big\{v \in L^2(\Omega; \mathbb{R}^m) : \text{for all } T \in \mathcal{T},\, v|_T \in \mathcal{P}_k(T; \mathbb{R}^m)\Big\},\\
L_h &:= L_h(\mathcal{T}; \mathbb{R}^{m\times n}) := \Big\{w \in L^2(\Omega; \mathbb{R}^{m\times n}) : \text{for all } T \in \mathcal{T},\, w|_T \in \mathcal{P}_k(T; \mathbb{R}^{m\times n})\Big\},\\
Q_h &:= Q_h(\mathcal{T}; \mathbb{R}) := \Big\{q \in L^2(\Omega) : \text{for all } T \in \mathcal{T},\, w|_T \in \mathcal{P}_{k-1}(T)\Big\},\\
V_h^c &:= V_h^c(\mathcal{T}; \mathbb{R}^m) := V_h(\mathcal{T}; \mathbb{R}^m) \cap V.
\end{aligned}
$$

The piecewise action of a differential operator with respect to a decomposition $\mathcal{T}$ is denoted with the subscript $\mathcal{T}$, e.g., ∇, D, div become $\nabla_\mathcal{T}$, $D_\mathcal{T}$, $\mathrm{div}_\mathcal{T}$.

2.2 Jump and averages

Throughout the paper $\cdot$ and $:$ denote the scalar products in $\mathbb{R}^m$ and $\mathbb{R}^{m\times n}$. Let $E = \partial T^+ \cap \partial T^-$ be an interior facet shared by two elements T^+ and T^- which have the outward unit normals ν^+ and ν^-, respectively. Given an element v_h of the product space $\Pi_{T\in\mathcal{T}} L^2(\partial T; \mathbb{R}^m)$ set $v_h^\pm := v_h|_{\partial T^\pm}$. The *jump* and *average* operators is defined as

$$
[v_h] := v_h^+ \otimes \nu^+ + v_h^- \otimes \nu^- \quad \text{and} \quad \{v_h\} := \frac{1}{2}(v_h^+ + v_h^-),
$$

respectively. Given $\tau_h \in \Pi_T L^2(\partial T; \mathbb{R}^{m\times n})$ with $\tau_h^\pm := \tau_h|_{T^\pm}$, set

$$[\tau_h] := \tau_h^+ \nu^+ + \tau_h^- \nu^- \quad \text{and} \quad \{\tau_h\} := \frac{1}{2}(\tau_h^+ + \tau_h^-). \tag{2.1}$$

Similarly, given $q_h \in \Pi_T L^2(\partial T; \mathbb{R})$ with $q_h^\pm := q_h|_{T^\pm}$, set

$$[q_h] := q_h^+ \nu^+ + q_h^- \nu^- \quad \text{and} \quad \{q_h\} := \frac{1}{2}(q_h^+ + q_h^-). \tag{2.2}$$

For $v \in \mathbb{R}^m$ and $w \in \mathbb{R}^n$, define $v \otimes w \in \mathbb{R}^{m\times n}$ by

$$(v \otimes w)_{ij} = v_i w_j. \tag{2.3}$$

On boundary facets $E \in \mathcal{E}_{\partial\Omega}$ with unit outward normal ν, the *jump* of $v \in L^2(\partial T; \mathbb{R}^m)$ and *average* of $w \in L^2(\partial T; \mathbb{R}^n)$ are defined by

$$[v] := v \otimes \nu \quad \text{and} \quad \{w\} := w. \tag{2.4}$$

2.3 Lifting operators

Given a facet $E \in \mathcal{E}$, the local lifting operators $r_E : L^2(E; \mathbb{R}^{m\times n}) \to L_h$ and $\ell_E : L^2(E; \mathbb{R}^m) \to L_h$ are characterised by

$$\begin{aligned} \int_\Omega r_E(q) : r_h \, dx &= \int_E q : \{r_h\} \, ds \quad \text{for all } r_h \in L_h, \\ \int_\Omega \ell_E(v) : r_h \, dx &= \int_E v \cdot [r_h] \, ds \quad \text{for all } r_h \in L_h. \end{aligned} \tag{2.5}$$

The global lifting operators $r : L^2(\mathcal{E}; \mathbb{R}^{m\times n}) \to L_h$ and $\ell : L^2(\mathcal{E}_\Omega; \mathbb{R}^m) \to L_h$ are

$$r := \sum_{E\in\mathcal{E}} r_E \quad \text{and} \quad \ell := \sum_{E\in\mathcal{E}_\Omega} \ell_E. \tag{2.6}$$

We also use the lifting operators $r_E^* : L^2(E; \mathbb{R}) \to Q_h$ and $\ell_E^* : L^2(E; \mathbb{R}^n) \to Q_h$ defined by

$$\begin{aligned} \int_\Omega r_E^*(\phi) \cdot q \, dx &= \int_E \phi \cdot \{q\} \, ds \quad \text{for all } q \in Q_h, \\ \int_\Omega \ell_E^*(v) \cdot q \, dx &= \int_E v \cdot [q] \, ds \quad \text{for all } q \in Q_h. \end{aligned} \tag{2.7}$$

The corresponding global lifting operators are described by

$$r^* := \sum_{E\in\mathcal{E}} r_E^* \quad \text{and} \quad \ell^* := \sum_{E\in\mathcal{E}_\Omega} \ell_E^*.$$

Lemma 2.1 [31, Lemma 7.4]. *Let $v_h, w_h \in V_h(\mathcal{T}, \mathbb{R}^n)$ and let the polynomial degree k be greater or equal to* 1. *Then every vector-valued function $\beta \in \Pi_{T\in\mathcal{T}} L^2(\partial T;\mathbb{R}^n)$ which is constant on all $E \in \mathcal{E}$ satisfies*

$$\begin{aligned}
\|r([v_h])\|^2_{L^2(\Omega)} &\lesssim \sum_{E\in\mathcal{E}} \frac{k^2}{h_E}\|[v_h]\|^2_{L^2(E)},\\
\|\ell([v_h]\cdot\beta)\|^2_{L^2(\Omega)} &\lesssim \sum_{E\in\mathcal{E}_\Omega} \frac{k^2}{h_E}\|[v_h]\|^2_{L^2(E)},\\
\|r^*([w_h])\|^2_{L^2(\Omega)} &\lesssim \sum_{E\in\mathcal{E}} \frac{k^2}{h_E}\|[w_h]\|^2_{L^2(E)},\\
\|\ell^*([w_h]\beta)\|^2_{L^2(\Omega)} &\lesssim \sum_{E\in\mathcal{E}_\Omega} \frac{k^2}{h_E}\|[w_h]\|^2_{L^2(E)}.
\end{aligned}$$

3 Abstract reliability a posteriori error analysis

In this section an abstract framework for the explicit error estimates is developed, which extends ideas of the equilibrium and consistency analysis in [15,19].

A central observation is that dG schemes for quite diverse differential equations share a common abstract structure based on which a unified a posteriori error analysis can be carried out. Stating the differential equation in form of (1.2) and (1.3), the bilinear form b is generally of the type

$$b(q, v) = (q, Bv)_L$$

whereby B is a differential operator with domain V and co-domain L. Element-wise application of B defines the broken differential operator $B_h : V_h + V \to L$. For example if $B = \nabla$, then $B_h v = \nabla_h v$ is obtained by taking the gradient of v element-by-element. Such broken differential operators B_h naturally appear in the construction of dG methods, where (1.2) is approximated by an equation

$$a(p_h, q) + (B_h u_h, q)_L + (F u_h, q)_L = f(q).$$

Here F subsums the flux functions of the method in form of the lifting operators from the previous subsection; the precise form of F specifies the dG method and is given for the applications below in terms of various jump terms. Since B_h is defined on the sum of V_h and V it naturally induces a semi-norm which allows to control discrete as well as continuous functions:

$$|v|_B := \|B_h v\|_L, \quad v \in V_h + V.$$

Various techniques in the literature to derive residual-based error bounds can be traced back to the control of a recovery operator R in the $\|\cdot\|_B$ semi-norm. More specifically, these techniques are based an operator $R : V_h \to V$ which satisfies

$$|Rv_h - v_h|_B^2 \lesssim \sum_{E \in \mathcal{E}} k^2/h_E \, \|[v_h]\|_E^2. \tag{3.1}$$

Selecting $\tilde{u}_h := Ru_h$, the consistency residual satisfies

$$\begin{aligned}
\|\mathcal{R}es_L\|_{L^*} &= \sup_{\|q\|_L=1} \big(a(p_h, q) + b(\tilde{u}_h, q) - f(q)\big) \\
&= \sup_{\|q\|_L=1} \big((B\tilde{u}_h, q)_L - (B_h u_h, q)_L - (Fu_h, q)_L\big) \\
&\le \|B_h(u_h - \tilde{u}_h)\|_L + \|Fu_h\|_L \\
&\lesssim \Big(\sum_{E \in \mathcal{E}} k^2/h_E \, \|[u_h]\|_E^2\Big)^{1/2} + \|Fu_h\|_L.
\end{aligned}$$

As demonstrated in the previous section, lifting operators can be controlled by inter-elemental jumps, leading to the bound

$$\|\mathcal{R}es_L\|_{L^*}^2 \lesssim \sum_{E \in \mathcal{E}} k^2/h_E \, \|[u_h]\|_E^2 =: \zeta^2. \tag{3.2}$$

Observe that Ru_h does not need to be computed to obtain (3.2). Recovery operators R are for instance constructed and discussed in [19].

A unified error analysis requires to identify key properties all relevant consistent and inconsistent dG methods share. Such a key property is that dG methods are in general consistent in the conforming part of (1.3). In other words methods such as [13,8] are inconsistent in the non-conforming part only as far as (1.3) is concerned. This observation is used in our analysis in form of the following two conditions (A1) and (A2).

(A1) There exists a Clément-type operator $J : V \to V_h^c$ which satisfies the bound

$$\|k/h_T\,(v - Jv)\|_{L^2(\Omega)} + \Big(\sum_{E \in \mathcal{E}} k/h_E \, \|v - Jv\|_E^2\Big)^{1/2} + \|v - Jv\|_V \lesssim \|v\|_V.$$

(A2) The conforming finite element space V_h^c is a subset of the kernel of the linear functional $\mathcal{R}es_V$, that is $V_h^c \subset \ker \mathcal{R}es_V$.

An immediate consequence of (A2) is that

$$\|\mathcal{R}es_V\|_{V^*} = \sup_{\|v\|_V=1} \big(g(v - Jv) - (p_h, B(v - Jv))_L\big)$$
$$= \sup_{\|v\|_V=1} \big(g(v - Jv) - (B^* p_h, v - Jv)_L\big) \tag{3.3}$$

The adjoint operator B^* can be computed via integration-by-parts. To make this more concrete we shall focus on second-order differential equations and make for the sake of simplicity the assumption that the differential operators have constant coefficients. The general case follows directly. For dG schemes the equilibrium residual then takes the form of the functional

$$\mathcal{R}es_V(v) := \int_\Omega g \cdot v\, dx - \int_\Omega p_h : Dv\, dx \quad \text{for all } v \in V,$$

where $g \in L^2(\Omega; \mathbb{R}^m)$ and $p_h \in L^2(\Omega; \mathbb{R}^{m\times n})$ are appropriately selected functions. In other words $B = \nabla$.

Under the assumptions (A1) and (A2), it is proved in [19, Theorem 2.1] that

$$\eta^2 := \sum_{T\in\mathcal{T}} \frac{h_T^2}{k^2} \|g + \operatorname{div} p_h\|^2_{L^2(T)} + \sum_{E\in\mathcal{E}_\Omega} \frac{h_E}{k} \|[p_h]\|^2_{L^2(E)} \tag{3.4}$$

is reliable in the sense that $\|\mathcal{R}es_V\|_{V^*} \lesssim \eta$. Together with Equation (3.2), we obtain the reliability of the estimator $\eta + \zeta$.

Theorem 3.1 *Under the assumptions (A1) and (A2), the residual-based error estimator is reliable in the sense that*

$$\|p - p_h\|_L \lesssim \eta + \zeta.$$

Remark 3.1 Assumption (A1) is satisfied for shape-regular triangulations, following from the well-established construction [14,17,20,22] of approximation operators for conforming first-order finite elements. The dependence on the polynomial degree k is detailed in [29]. The constant in (A1) may depend on the anisotropy of the mesh and the type and number of hanging nodes. The focus of this paper is not on those fine aspects of degenerated meshes and hence (A1) serves us throughout this paper well as an underlying but rather general condition on the meshes.

4 Application to the Laplace operator

This section concerns the residual-based a posteriori error control of the dG FEMs listed in Table 1 for the Poisson problem on $\Omega \subset \mathbb{R}^n$.

Table 1 Selected dG schemes quoted in form of (4.2) and (4.3)

Method [Ref.]	$\hat{u}_T$	$\hat{p}_T$	c_1	c_2
Bassi and Rebay [10]	$\{u_h\}$	$\{p_h\}$	-1	0
Brezzi et al. [7]	$\{u_h\}$	$\{p_h\} - \alpha_r([u_h])$	-1	0
LDG [23]	$\{u_h\} - \beta \cdot [u_h]$	$\{p_h\} + \beta[p_h] - \alpha_j([u_h])$	-1	-1
IP [24]	$\{u_h\}$	$\{Du_h\} - \alpha_j([u_h])$	-1	0
Bassi et al. [12]	$\{u_h\}$	$\{Du_h\} - \alpha_r([u_h])$	-1	0
Baumann and Oden [9]	$\{u_h\} + \nu_T \cdot [u_h]$	$\{Du_h\}$	1	0
NIPG [30]	$\{u_h\} + \nu_T \cdot [u_h]$	$\{Du_h\} - \alpha_j([u_h])$	1	0
Babuska and Zlamal [13]	$(u_h\|_T)\|_{\partial T}$	$-\alpha_j([u_h])h_E^{-1}$	0	0
Brezzi et al. [8]	$(u_h\|_T)\|_{\partial T}$	$-\alpha_r([u_h])$	0	0

4.1 Model Poisson problem

Given $g \in L^2(\Omega)$, let $u \in V := H_0^1(\Omega)$, $m = 1$, be the unique solution to the *Poisson Problem*

$$\Delta u + g = 0 \text{ in } \Omega \quad \text{and} \quad u = 0 \text{ on } \partial\Omega. \tag{4.1}$$

The operator $A : X \to X^*$ is defined in (1.1) with $v \in V$, $p, q \in L := L^2(\Omega; \mathbb{R}^n)$ and

$$a(p, q) = \int_\Omega p \cdot q \, dx \quad \text{and} \quad b(p, v) = -\int_\Omega p \cdot \nabla v \, dx.$$

The operator A is bounded, linear, and bijective [15] and the flux $p := \nabla u \in L$ and $u \in V$ satisfy

$$(A(p, u))(q, v) = -\int_\Omega g v \, dx \quad \text{for all } (q, v) \in X = L \times V.$$

4.2 Unified dG formulation

With the local numerical flux functions $\hat{u}_T$ and $\hat{p}_T$ from Table 1, the unified dG formulation for the Laplace problem [1] with V_h and L_h from Sect. 2.1 reads: Find $u_h \in V_h$ and $p_h \in L_h$ such that, for all $w \in L_h$ and all $v \in V_h$,

$$a(p_h, w) = -\int_\Omega u_h \operatorname{div}_{\mathcal{T}} w \, dx + \sum_{T \in \mathcal{T}} \int_{\partial T} \hat{u}_T (\nu_T \cdot w) \, ds, \tag{4.2}$$

$$\int_\Omega p_h \cdot D_{\mathcal{T}} v \, dx = \int_\Omega g v \, dx + \sum_{T \in \mathcal{T}} \int_{\partial T} (\hat{p}_T \cdot \nu_T) v \, ds. \tag{4.3}$$

The local flux functions elementwise define the global numerical fluxes $\hat{u}$ and $\hat{p}$. The solution (u_h, p_h) of (4.2) and (4.3) satisfies [1]

$$p_h = D_{\mathcal{T}} u_h + r([\hat{u} - u_h]) + \ell(\{\hat{u} - u_h\}).$$

The substitution of p_h in (4.3) results in a linear system of equations for $u_h \in V_h$ of which we suppose that it has a unique solution.

Remark 4.1 In Table 1 and (4.2) and (4.3), $\beta \in L^2(E; \mathbb{R}^n)$ is a vector-valued function which is constant along each $E \in \mathcal{E}_\Omega$. The jumps satisfy $\alpha_j([u_h]) = \alpha_E h_E^{-1}[u_h]$ and $\alpha_r([u_h]) = \alpha_E \{r_E([u_h])\}$ on $E \in \mathcal{E}$ with $\alpha_E > 0$.

Remark 4.2 The methods of Bassi and Rebay [10] and of Baumann and Oden [9] for $k = 1$ are not stable. However, our a posteriori error analysis includes these methods.

4.3 Unified a posteriori error analysis

It is remarkable that Theorem 3.1 covers all schemes of Table 1 in one strike. In fact, the a posteriori analysis of Theorem 4.1 recovers the results [5,6,28] for the IP, NIPG and LDG methods, and yields new error estimates for the remaining methods of Table 1.

Theorem 4.1 *Suppose $u \in V$ and $p \in L$ solve the Poisson problem as stated in Subsection 4.1 while $u_h \in V_h$ and $p_h \in L_h$ solve* (4.2) *and* (4.3). *Recall η and ζ from* (3.2)–(3.4). *Then, there holds*

$$\|p - p_h\| \lesssim \eta + \zeta.$$

Proof Given $p_h \in L$ from (4.2) and (4.3) and any $\tilde{u}_h \in V$, then (1.6) holds for

$$\mathcal{R}es_L(q) := \int_\Omega q \cdot (D\tilde{u}_h - p_h)\, dx \in L^*, \tag{4.4}$$

$$\mathcal{R}es_V(v) := -\int_\Omega g v\, dx + \int_\Omega p_h \cdot Dv\, dx \in V^*. \tag{4.5}$$

As observed in [15], the consistency residuum $\mathcal{R}es_L$ from (4.4) has the norm

$$\|\mathcal{R}es_L\|_{L^*} = \|p_h - D\tilde{u}_h\|_L.$$

Notice that, on each $E \in \mathcal{E}_\Omega$, the jump $[\hat{p}] = 0$ vanishes for all $\hat{p}$ of Table 1. Moreover, (4.3) and (4.5) lead to $\mathcal{R}es_V(v_h) = 0$ for all $v_h \in V_h^c$. This verifies (A2).

Table 1 shows that $[\hat{u} - u_h] = c_1[u_h]$ for some $c_1 \in \{-1,\ 0,\ 1\}$ and $\{\hat{u} - u_h\} = c_2[u_h] \cdot \beta$ with $c_2 \in \{-1,\ 0\}$. For the LDG method $c_2 = -1$ and otherwise $c_2 = 0$. In conclusion,

$$p_h = D_{\mathcal{T}} u_h + c_1\, r([u_h]) + c_2\, \ell([u_h] \cdot \beta).$$

This and the triangle inequality show that

$$\|p_h - D\tilde{u}_h\|_L \lesssim \|D_{\mathcal{T}} u_h - D\tilde{u}_h\|_{L^2(\Omega)} + \|r([u_h])\|_{L^2(\Omega)} + \|\ell([u_h]\cdot\beta)\|_{L^2(\Omega)}.$$

Consequently, (1.6), Lemma 2.1 as well as Theorem 3.1 imply the a posteriori error estimate $\|p - p_h\|_L \lesssim \eta + \zeta$. □

5 Application to the Stokes problem

5.1 Stokes model problem

The unsymmetrical formulation of the Stokes problem with $m = n$ reads: Given $g \in L^2(\Omega;\mathbb{R}^n)$, seek $u \in V := H_0^1(\Omega;\mathbb{R}^n)$ and $p \in L_0^2(\Omega) := \{q \in L^2(\Omega) : \int_\Omega q\,dx = 0\} \equiv L^2(\Omega)/\mathbb{R}$ such that for all $(v, q) \in H_0^1(\Omega;\mathbb{R}^n) \times L_0^2(\Omega)$,

$$\mu \int_\Omega Du : Dv\,dx - \int_\Omega p \operatorname{div} v\,dx - \int_\Omega q \operatorname{div} u\,dx = \int_\Omega g \cdot v\,dx. \tag{5.1}$$

The unique existence of a solution u to (5.1) is well known [4]. The deviatoric part of a square matrix $F \in \mathbb{R}^{n\times n}$ is $\operatorname{dev} F = F - (\operatorname{tr}(F)/n)\mathbf{I}$ using the trace $\operatorname{tr}(F) = F_{11} + F_{22} + \cdots + F_{nn}$. Setting $L := \{\tau \in L^2(\Omega;\mathbb{R}^{n\times n}) : \int_\Omega \operatorname{tr}(\tau)\,dx = 0\}$ and

$$\begin{aligned} a(\sigma,\tau) &:= \int_\Omega \frac{1}{\mu} \operatorname{dev}\sigma : \operatorname{dev}\tau\,dx \quad \text{for all } \sigma,\tau \in L,\\ b(\sigma,v) &:= -\int_\Omega \sigma : Dv\,dx \quad \text{for} (\sigma,v) \in X := L \times V, \end{aligned} \tag{5.2}$$

it is known that the operator $A : X \to X^*$, defined for $(\sigma, u) \in X$ by (1.1) is linear, bounded, and bijective [4].

5.2 Unified dG formulation

Given the flux functions $\hat{u}_{T,\sigma}$, $\hat{u}_{T,p}$, and $\hat{\sigma}_T$ from Table 2, the unified dG formulation for (5.1) reads: Find $(u_h, \sigma_h, p_h) \in V_h \times L_h \times Q_h^0$ such that, for all $(v, w, q) \in$

Table 2 Selected dG schemes quoted in form of (5.3) and (5.5)

Method [Ref.]	$\hat{u}_{T,\sigma}$	$\hat{u}_{T,p}$	$\hat{\sigma}_T$	c_1	c_2
Bassi and Rebay [10]	$\{u_h\}$	$\{u_h\}$	$\{\sigma_h\}$	-1	0
IP [31]	$\{u_h\}$	$\{u_h\}$	$\{Du_h\} - \alpha_j([u_h])$	-1	0
Bassi et al. [12]	$\{u_h\}$	$\{u_h\}$	$\{Du_h\} - \alpha_r([u_h])$	-1	0
NIPG [32]	$\{u_h\} + [u_h] \cdot \nu_T$	$\{u_h\}$	$\{Du_h\} - \alpha_j([u_h])$	1	0
LDG [18]	$\{u_h\} + [u_h] \cdot \beta$	$\{u_h\} + D_{11}[p_h]$ $+D_{12}[u_h]$	$\{\sigma_h\} - [\sigma_h] \otimes \beta$ $-[p_h] \otimes \beta - D_{12} \cdot [p_h] - \alpha_j([u_h])$	-1	1

The parameters β, D_{11}, D_{12} are defined according to [18]

$V_h \times L_h \times Q_h^0$, where $Q_h^0 = \{q_h \in Q_h : \int_\Omega q_h\, dx = 0\}$,

$$\int_\Omega \sigma_h : w\, dx = -\mu \int_\Omega u_h \cdot (D_{\mathcal{T}} \cdot w)\, dx + \mu \sum_{T \in \mathcal{T}} \int_{\partial T} \hat{u}_{T,\sigma} \cdot (w \cdot \nu_T)\, ds - \int_\Omega p_h \operatorname{tr}(w)\, dx, \tag{5.3}$$

$$\int_\Omega \sigma_h : D_{\mathcal{T}} v\, dx = \int_\Omega g \cdot v\, dx + \sum_{T \in \mathcal{T}} \int_{\partial T} \hat{\sigma}_T : (v \otimes \nu_T)\, ds, \tag{5.4}$$

$$\int_\Omega u_h \cdot \nabla_{\mathcal{T}} q\, dx = \sum_{T \in \mathcal{T}} \int_{\partial T} (\hat{u}_{T,p} \cdot \nu_T)\, q\, ds \tag{5.5}$$

are satisfied.

Remark 5.1 The original form of the LDG method [18] is written in terms of the variable $s_h := \sigma_h + p_h \mathbf{I}$.

Remark 5.2 Independently of the fact that the method of Bassi and Rebay [10] is unstable, our a posteriori error analysis includes this method.

The local numerical flux functions $\hat{u}_{T,\sigma}$, $\hat{u}_{T,p}$ and $\hat{\sigma}_T$ define, respectively, the global fluxes $\hat{u}_\sigma$, $\hat{u}_p$ and $\hat{\sigma}$ by elementwise application.

Proposition 5.1 *Given the solution* (u_h, σ_h, p_h) *of* (5.3)–(5.5) *the identity*

$$\sigma_h = \mu\, D_{\mathcal{T}} u_h - p_h \mathbf{I} + \mu\, r([\hat{u}_\sigma - u_h]) + \mu\, \ell(\{\hat{u}_\sigma - u_h\}).$$

holds.

Proof The proof follows the once from [1] and, hence, is omitted. □

5.3 Unified a posteriori error control

The a posteriori analysis of Theorem 5.1 recovers the results of [26] for the IP method and provides new error estimates for the remaining methods of Table 2.

Theorem 5.1 *Suppose that u and $\sigma := \mu Du - p\mathbf{I}$ solve the Stokes problem as stated in Sect.* 5.1, *while u_h and σ_h solve* (5.3) *and* (5.4). *Recall ζ and η from* (3.2) *and* (3.4). *Then, it follows that*

$$\|\sigma - \sigma_h\|_L \lesssim \eta + \zeta + \|\operatorname{div}_{\mathcal{T}} u_h\|_{L^2(\Omega)} \lesssim \eta + \zeta.$$

Proof Given the unique discrete solution σ_h, define the *linear functional* $\mathcal{R}es_V \in V^*$ by

$$\mathcal{R}es_V(v) = \int_\Omega (g \cdot v - \sigma_h : Dv)\, dx \quad \text{for } v \in V. \tag{5.6}$$

The argument of [15,19] shows, for any $\tilde{u}_h \in V$, that the residual functional $\mathcal{R}es_L$,

$$\mathcal{R}es_L(w) = -a(\sigma_h, w) - b(w, \tilde{u}_h) \quad \text{for all } w \in L \tag{5.7}$$

has the norm

$$\|\mathcal{R}es_L\|_{L^*} = \|D(\tilde{u}_h) - \frac{1}{\mu} \operatorname{dev} \sigma_h\|_L. \tag{5.8}$$

Observe that on each of $E \in \mathcal{E}_\Omega$, that the jump $[\hat{\sigma}] = 0$ for all $\hat{\sigma}$ in the Table 2. Thus, the identities (5.4) and (5.6) lead to $\mathcal{R}es_V(v_h) = 0$ for all $v_h \in V_h^c$. Therefore assumption (A2) is satisfied.

Table 2 shows that $[\hat{u}_\sigma - u_h] = c_1\, [u_h]$ with $c_1 \in \{-1,\ 0,\ 1\}$; for the NIPG method $c_1 = 1$ and for all other methods $c_1 = -1$. Similarly, $\{\hat{u}_\sigma - u_h\} = c_2\, [u_h] \cdot \beta$ with $c_2 \in \{0, 1\}$; for the LDG method $c_2 = 1$ and for all other methods $c_2 = 0$. All these lead to

$$\sigma_h = \mu\, D_{\mathcal{T}} u_h - p_h \mathbf{I} + c_1\, \mu\, r([u_h]) + c_2\, \mu\, \ell([u_h] \cdot \beta). \tag{5.9}$$

Therefore, given $\tilde{u}_h \in V$ with $\sigma := \mu Du - p\mathbf{I}$, there holds

$$\|\sigma - \sigma_h\|_L + \|u - \tilde{u}_h\|_V \lesssim \|D(\tilde{u}_h) - D_{\mathcal{T}} u_h\|_L + \|\operatorname{div}_{\mathcal{T}} u_h\| \\ + \|r([u_h])\| + \|\ell([u_h] \cdot \beta)\| + \|\mathcal{R}es_V\|_{V^*}.$$

Using (3.1) it follows $\|D(\tilde{u}_h) - D_{\mathcal{T}} u_h\|_L^2 \lesssim \sum_{E \in \mathcal{E}} k^2/h_E\, \|[v_h]\|_E^2 = \zeta^2$. From (5.5) one derives as before

$$\operatorname{div}_{\mathcal{T}} u_h = r([u_h - \hat{u}]) + \ell(\{u_h - \hat{u}\}),$$

and hence

$$\| \operatorname{div}_{\mathcal{T}} u_h \|_{L^2(\Omega)} \lesssim \zeta.$$

Then $\|\mathcal{R}es_V\|_{V^*} \lesssim \eta$ completes the proof. □

6 Application to linear elasticity

This section is devoted to the Navier–Lamé equation and its discontinuous Galerkin discretizations. The unified a posteriori analysis recovers the results of [27,33] for the IP method and makes new error estimates for LDG method available.

6.1 Model problem in linear elasticity

The weak formulation of the linear elasticity model reads: Given $g \in L^2(\Omega; \mathbb{R}^n)$, find $u \in V := H_0^1(\Omega; \mathbb{R}^n)$ such that

$$\int_\Omega \sigma : \varepsilon(v)\, dx = \int_\Omega g \cdot v\, dx \quad \text{and} \quad \sigma = \mathbb{C}\varepsilon(u) \quad \text{for all } v \in V.$$

Therein, $\varepsilon(v) := (\nabla v + (\nabla v)^T)/2$ and for $F \in \mathbb{R}^{n\times n}$,

$$\mathbb{C}F := \lambda \operatorname{tr}(F)\mathbf{I} + 2\mu\, F \quad \text{and} \quad \mathbb{C}^{-1}F := \frac{1}{2\mu}F - \frac{\lambda}{2\mu(n\lambda + 2\mu)} \operatorname{tr}(F)\mathbf{I},$$

where $\lambda, \mu > 0$ are given quantities.

The operator $A : X = L \times V \to X^*$ is defined in form of (1.1) with

$$a(\sigma, \tau) := \int_\Omega (\mathbb{C}^{-1}\sigma) : \tau\, dx$$

for

$$\sigma, \tau \in L := L_0^2(\Omega; \mathbb{R}^{n\times n}_{\text{sym}}) := \left\{ w \in L^2(\Omega; \mathbb{R}^{n\times n}_{\text{sym}}) : \int_\Omega \operatorname{tr}(w)\, dx = 0 \right\};$$

A is linear, bounded, and bijective with λ-independent operator norms of A and A^{-1} [3,16].

Table 3 Selected dG schemes quoted in form of (6.1)–(6.3) for linear elasticity with parameters α_j, β, κ_p, d from [21,25]

Method	$\hat{u}_{T,\varepsilon}$	$\hat{u}_{T,p}$	$\hat{\varepsilon}_T$	$\hat{p}_T$	c_1	c_2
IP [25]	$\{u_h\}$	$\{u_h\}$	$\{\varepsilon(u_h)\}$ $-\alpha_j([u_h])$	$-\lambda\{\operatorname{div} u_h\}$ $+\lambda\alpha_j([u_h])$	0	0
LDG [21]	$\{u_h\} + [u_h]\cdot\beta$	$\{u_h\} + \mathsf{d}\cdot[u_h]$ $+2\mu\,\kappa_p\,[p_h]$	$\{\varepsilon_h\} - [\varepsilon_h]\otimes\beta$ $-\alpha_j([u_h])$	$\{p_h\} - \mathsf{d}\cdot[p_h]$	-1	1

6.2 Unified dG formulation

Given the numerical fluxes $\hat{u}_{T,\varepsilon}$, $\hat{u}_{T,p}$, $\hat{\sigma}_T$ and $\hat{p}_T$ from Table 3, the unified dG formulation of the linear elasticity problem reads: Find

$$(\varepsilon_h, u_h, p_h) \in L_h(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}}) \times V_h \times Q_h^0 \subset L_0^2(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}}) \times H_0^1(\mathcal{T};\mathbb{R}^n) \times L_0^2(\Omega)$$

such that, for all $(\tau, v, q) \in L_h(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}}) \times V_h \times Q_h^0$, it holds that

$$\int_\Omega \varepsilon_h : \tau\,dx = -\int_\Omega u_h \cdot \operatorname{div}_{\mathcal{T}} \tau\,dx + \sum_{T\in\mathcal{T}} \int_{\partial T} \hat{u}_{T,\varepsilon}\cdot(\tau\nu_T)\,ds \tag{6.1}$$

$$\int_\Omega (2\mu\varepsilon_h - p_h\mathbf{I}) : D_{\mathcal{T}} v\,dx = \int_\Omega g\cdot v\,dx + \sum_{T\in\mathcal{T}} \int_{\partial T} (2\mu\hat{\varepsilon}_T - \hat{p}_T I) : v\otimes\nu_T\,ds, \tag{6.2}$$

$$\int_\Omega \frac{1}{\lambda} p_h q\,dx = \int_\Omega u_h\cdot\nabla_{\mathcal{T}} q\,dx - \sum_{T\in\mathcal{T}} \int_{\partial T} (\hat{u}_{T,p}\cdot\nu_T) q\,ds. \tag{6.3}$$

Proposition 6.1 *Given u_h, ε_h, and p_h with (6.1) and (6.2), define $\sigma_h := 2\mu\varepsilon_h - p_h\mathbf{I}$. Then,*

$$L_1 = -r(\underline{[u_h]}) + c_1\ell(\underline{[u_h]}\cdot\beta) \quad \textit{and} \quad L_2 = r^*([u_h]) + c_2\ell^*(\mathsf{d}\cdot[u_h] + \kappa_p\,[p_h])$$

satisfy

$$\mathbb{C}^{-1}\sigma_h = \varepsilon(u_h) + L_1 - \frac{\lambda}{n\lambda + 2\mu}(L_2 + \operatorname{tr}(L_1))\mathbf{I}.$$

Proof A direct calculation reveals

$$\varepsilon_h = \varepsilon(u_h) + r([\hat{u}_\varepsilon - u_h]) + \ell(\{\hat{u}_\varepsilon - u_h\}),$$
$$\lambda^{-1}\,p_h = -\operatorname{div} u_h + r^*([u_h - \hat{u}_p]) + \ell^*(\{u_h - \hat{u}_p\}).$$

This leads to the assertion

$$\mathbb{C}^{-1}\sigma_h = 2\mu\mathbb{C}^{-1}\varepsilon_h - p_h\mathbb{C}^{-1}\mathbf{I} = \varepsilon(u_h) + L_1 - \frac{\lambda}{n\lambda + 2\mu}(L_2 + \mathrm{tr}(L_1))\mathbf{I}. \tag{6.4}$$

□

6.3 Unified a posteriori error control

The unified a posteriori analysis extends [15,19] and, in the exceptional fully incompressible case for the LDG [21, Proposition 2.1] $\kappa_p > 0$, and jumps of the pressures arise. In all other cases, $\kappa_p = 0$ and $\|\sigma - \sigma_h\|_L \lesssim \eta + \zeta$.

Theorem 6.1 *Suppose u and σ satisfies the linear elasticity problem stated in Subsection 6.1 while $(u_h, \varepsilon_h, p_h)$ solves (6.1)–(6.3). Then, $\sigma_h := 2\mu\varepsilon_h - p_h\mathbf{I}$ satisfies*

$$\|\sigma - \sigma_h\|_L \lesssim \eta + \zeta + \left(\sum_{E\in\mathcal{E}_\Omega} \kappa_p^2 \frac{k^2}{h_E} \|\underline{[p_h]}\|^2_{L^2(E)}\right)^{1/2}.$$

The last sum always vanishes for the IP method (set $\kappa_p = 0$ for IP) and for the LDG method if $\kappa_p = 0$.

Proof Given the unique discrete solution $\sigma_h := 2\mu\varepsilon_h - p_h\mathbf{I}$, define the residual functional $\mathcal{R}es_V \in V^*$ by

$$\mathcal{R}es_V(v) = \int_\Omega \sigma_h : Dv - \int_\Omega g \cdot v\, dx \quad \text{for } v \in V. \tag{6.5}$$

Define the global numerical fluxes $\hat{u}_\varepsilon$, $\hat{u}_p$, $\hat{\varepsilon}$ and $\hat{p}$ elementwise by means of the respective local flux functions. Then on each $E \in \mathcal{E}_\Omega$, the jump $[2\mu\hat{\varepsilon} - \hat{p}]$ vanishes for all $\hat{\varepsilon}$ and $\hat{p}$ listed in the Table 3. Therefore, identity (6.2) together with (6.5) implies that $\mathcal{R}es_V(v_h) = 0$ for all $v_h \in V_h^c$ and verifies the assumption (A2).

In the spirit of [15], for any $\tilde{u}_h \in V$, the residual functional $\mathcal{R}es_L$, defined by

$$\mathcal{R}es_L(\tau) := -a(\sigma_h, \tau) - b(\tau, \tilde{u}_h),$$

has the norm

$$\|\mathcal{R}es_L\|_{L^*} = \|\mathbb{C}^{-1}\sigma_h - \varepsilon(\tilde{u}_h)\|_L.$$

For any $\tilde{u}_h \in V$, with $\sigma = \mathbb{C}\varepsilon(u)$, (1.6) and (6.4) imply

$$\|\sigma - \sigma_h\|_L + \|u - \tilde{u}_h\|_V \lesssim \|\varepsilon(u_h) - \varepsilon(\tilde{u}_h)\|_{L^2(\Omega)} + \|L_1\| + \|L_2\| + \|\mathcal{R}es_V\|_{V^*}.$$

A triangle inequality and Theorem 3.1 complete the proof. □

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References

1. Arnold, D.N., Brezzi, F., Cockburn, B., Marini, L.D.: Unified analysis of discontinuous Galerkin methods for elliptic problems. SIAM J. Numer. Anal. **39**, 1749–1779 (2002)
2. Ainsworth, M.: A posteriori error estimation for discontinuous Galerkin finite element approximation. SIAM J. Numer. Anal. **45**, 1777–1798 (2007)
3. Braess, D., Carstensen, C., Reddy, B.D.: Uniform convergence and a posteriori error estimators for the enhanced strain finite element method. Numer. Math. **96**, 461–479 (2004)
4. Brezzi, F., Fortin, M.: Mixed and Hybrid Finite Element Methods. Springer, Berlin (1991)
5. Bustinza, R., Gatica, G.N., Cockburn, B.: An a posteriori error estimate for the local discontinuous Galerkin method applied to linear and nonlinear diffusion problems. J. Sci. Comput. **22–23**, 147–185 (2005)
6. Becker, R., Hansbo, P., Larson, M.G.: Energy norm a posteriori error estimation for discontinuous Galerkin methods. Comput. Methods Appl. Mech. Eng. **192**, 723–733 (2003)
7. Brezzi, F., Manzini, G., Marini, D., Pietra, P., Russo, A.: Discontinuous finite elements for diffusion problems. In Atti Convegno in onore di F. Brioschi (Milan, 1991), Istituto Lambardo, Accademia di Scienze e Lettere, Milan, Italy (1991), pp. 197–217
8. Brezzi, F., Manzini, G., Marini, D., Pietra, P., Russo, A.: Discontinuous finite elements for elliptic problems. Numer. Methods Partial Differ. Equ. **16**, 365–378 (2000)
9. Baumann, C.E., Oden, J.T.: A discontinuous *hp* finite element method for convection–diffusion problems. Comput. Methods Appl. Mech. Eng. **175**, 311–341 (1999)
10. Bassi, F., Rebay, S.: A higher order accurate discontinuous Galerkin finite element method for the numerical solution of the compressible Navier–Stokes equations. J. Comput. Phys. **131**, 267–279 (1997)
11. Braess, D.: Finite Elements. Cambridge University Press, Cambridge (1997)
12. Bassi, F., Rebay, S., Mariotti, G., Pedinotti, S., Savini, M.: A higher-order accurate discontinuous finite element method for inviscid and viscous turbomachinery flows. In: Decuypere, R., Dilbelius, G. (eds.) Proceedings of 2nd European Conference on Turbomachinery, Fluid Dynamics and Thermodynamics, pp. 99–108. Technologisch Instituut, Antwerpen, Belgium (1997)
13. Babuska, I., Zlamal, M.: Nonconforming elements in the finite element method with penalty. SIAM J. Numer. Anal. **10**, 863–875 (1973)
14. Carstensen, C.L.: Numerical analysis of the primal problem of elastoplasticity with hardening. Numer. Math **82**(4), 577–597 (1999) MR 2000d:74070
15. Carstensen, C.L.: A unifying theory of a posteriori finite element error control. Numer. Math. **100**, 617–637 (2005)
16. Carstensen, C., Dolzmann, G.: A posteriori error estimates for mixed FEM in elasticity. Numer. Math. **81**, 187–209 (1998)
17. Carstensen, C., Funken, S.A.: Constants in Clément-interpolation error and residual-based a posteriori error estimates in finite element methods. East-West J. Numer. Math. 8(3):153–175 (2000) MR 2002a:65173
18. Cockburn, B., Schötzau, D., Kanschat, G., Schwab, C.: Local discontinuous Galerkin methods for the Stokes systems. SIAM J. Numer. Anal. **40**, 319–343 (2002)
19. Carstensen, C., Hu, J.: A unifying theory of a posteriori error control for nonconforming finite element methods. J. Numer. Math. **107**(3), 473–502 (2007)
20. Carstensen, C., Jensen, M.: Averaging techniques for reliable and efficient a posteriori finite element error control: analysis and applications. Contemp. Math. **383**, 15–34 (2006)
21. Cockburn, B., Kanschat, G., Schötzau, D., Schwab, C.: Discontinuous Galerkin methods for incompressible elastic materials. Comput. Methods Appl. Mech. Eng. **195**, 3184–3204 (2006)
22. Clément, P.: Approximation by finite element functions using local regularization. RAIRO Sér. Rouge Anal. Numér. **2**, 77–84 (1975)
23. Cockburn, B., Shu, C.-W.: The local discontinuous Galerkin method for time-dependent convection-diffusion systems. SIAM J. Numer. Anal. **35**, 2440–2463 (1998)

24. Douglas, J. Jr., Dupont, T.: Interior penalty procedures for elliptic and parabolic Galerkin methods. Lecture Notes in Physics, vol. 58, pp. 207–216. Springer, Berlin (1976)
25. Hansbo, P., Larson, M.G.: Discontinuous Galerkin methods for incompressible and nearly incompressible elasticity by Nitsche's method. Comput. Methods Appl. Mech. Eng. **191**, 1895–1908 (2002)
26. Houston, P., Schötzau, D., Wihler, T.P.: Energy norm a posteriori error estimation for mixed discontinuous Galerkin approximations of the Stokes problem. J. Sci. Comput. **22–23**, 347–370 (2005)
27. Houston, P., Schötzau, D., Wihler, T.P.: An hp-adaptive mixed discontinuous Galerkin FEM for nearly incompressible linear elasticity. Comput. Methods Appl. Mech. Eng. **195**, 3224–3246 (2006)
28. Karakashian, O.A., Pascal, F.: A posteriori error estimation for a discontinuous Galerkin approximation of second order elliptic problems. SIAM J. Numer. Anal. **41**, 2374–2399 (2003)
29. Melenk, J.M., Wohlmuth, B.: On residual-based aposteriori error estimation in *hp*-FEM. Adv. Comput. Math. **15**, 311–331 (2001)
30. Riviere, B., Wheeler, M.F., Girault, V.: Improved energy estimates for interior penalty, constrained and discontinuous Galerkin methods for elliptic problems I. Comput. Geosci. **3**, 337–360 (1999)
31. Schötzau, D., Schwab, C., Toselli, A.: Mixed *hp*-dGFEM for incompressible flows. SIAM J. Numer. Anal. **40**, 2171–2194 (2003)
32. Toselli, A.: *hp*-Discontinuous Galerkin approximations for Stokes problems. Math. Models Methods Appl. Sci. **12**, 1565–1616 (2002)
33. Wihler, T.P.: Locking-free adaptive discontinuous Galerkin FEM for linear elasticity problems. Math. Comput. **75**, 1087–1102 (2006)