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# A general two-scale criteria for logarithmic Sobolev inequalities

T. Lelièvre[1]

[1] *CERMICS, Université Paris-Est, ENPC, 6-8 avenue Blaise Pascal, Champs sur Marne, 77455 Marne La Vallée Cedex 2, France*

*CERMICS — Université Paris-Est*
*ENPC, 6 et 8 avenue Blaise Pascal*
*Cité Descartes - Champs sur Marne*
*77455 Marne la Vallée Cedex 2*

*http://cermics.enpc.fr*

# A general two-scale criteria for logarithmic Sobolev inequalities

Tony Lelièvre [a,b],

[a] *CERMICS, Ecole Nationale des Ponts et Chaussées, 77455 Marne-La-Vallée Cedex 2, FRANCE*

[b] *INRIA Rocquencourt, MICMAC project, Domaine de Voluceau, B.P. 105, 78153 Le Chesnay Cedex, FRANCE*

**Abstract**

We present a general criteria to prove that a probability measure satisfies a logarithmic Sobolev inequality, knowing that some of its marginals and associated conditional laws satisfy a logarithmic Sobolev inequality. This is a generalization of a result by N. Grunewald et al. [5].

## 1 Motivation and notation

The motivation behind this work is molecular dynamics (in the canonical statistical ensemble), and more precisely, $(i)$ the analysis of numerical methods for the computation of *free energy differences* [7] and $(ii)$ the derivation of *effective dynamics on coarse-grained variables* [6]. In both cases, it appears that estimates based on entropies for measures related to the Boltzmann-Gibbs measure is a useful tool. One important question is the following: what is the link between the logarithmic Sobolev inequality (LSI) constant of the Boltzmann-Gibbs measure for the original variables (microscopic level) and the LSI constant of the Boltzmann-Gibbs measure for some coarse-grained variables (macroscopic level). The aim of this work is to give an answer, which is a generalization to non-linear coarse-graining operators of results in [8,5].

Let $\mathcal{D}$ be a domain of $\mathbb{R}^n$ representing the configuration space of the system under consideration, and $V : \mathcal{D} \to \mathbb{R}$ a potential, associating to each configuration an energy. Let us consider a function (representing the coarse-grained

*Email address:* `lelievre@cermics.enpc.fr` (Tony Lelièvre).

variables, also called the *reaction coordinates*)

$$\xi : \mathcal{D} \to \mathcal{M},$$

with $\mathcal{M} \subset \mathbb{R}^p$ (and $1 \leq p < n$). Let us introduce the Gram matrix $G : \mathcal{D} \to \mathbb{R}^{p\times p}$ of the derivative $\nabla\xi : \mathcal{D} \to \mathbb{R}^{p\times n}$: $G = \nabla\xi\nabla\xi^T$, *i.e.*, componentwise, $\forall \alpha, \beta \in \{1,\ldots,p\}$,

$$G_{\alpha,\beta} = \nabla\xi_\alpha \cdot \nabla\xi_\beta. \tag{1}$$

We suppose that $\xi$ is such that

**[H1]** $\xi$ is a smooth function such that $\det G \neq 0$ on $\mathcal{D}$.

The submanifolds

$$\Sigma_z = \{x \in \mathcal{D}, \xi(x) = z\}$$

are then smooth submanifolds of $\mathcal{D}$ of codimension $p$. We denote by $\sigma_{\Sigma_z}$ the surface measure on $\Sigma_z$, *i.e.* the Lebesgue measure on $\Sigma_z$ induced by the Lebesgue measure in the ambient Euclidean space $\mathcal{D}$. The submanifold $\Sigma_z$ naturally has a (complete and locally compact) Riemannian structure induced by the Euclidean structure of the ambient space $\mathcal{D}$.

Let us define the density $\psi_0$ (with respect to the Lebesgue measure on $\mathcal{D}$) of the Boltzmann-Gibbs probability measure $d\mu_0(x) = \psi_0(x)\,dx$ associated to the potential $V$:

$$\psi_0 = Z^{-1}\exp(-V),$$

where $Z = \int_{\mathcal{D}} \exp(-V)$. We denote by $\psi_0^\xi$ the density (with respect to the Lebesgue measure on $\mathcal{M}$) of the image $d\mu_0^\xi(z) = \psi_0^\xi(z)\,dz$ of the measure $\mu_0$ by $\xi$:

$$\psi_0^\xi(z) = Z^{-1}\int_{\Sigma_z} \exp(-V)(\det G)^{-1/2} d\sigma_{\Sigma_z}.$$

Let us introduce then the conditional measure $\mu_{0,z}$ of $\mu_0$ at a fixed value of $\xi$:

$$d\mu_{0,z} = \frac{Z^{-1}\exp(-V)(\det G)^{-1/2} d\sigma_{\Sigma_z}}{\psi_0^\xi(z)}.$$

Let us introduce the effective potential $A_0$ associated to $\xi$ (also called *free energy*), defined by

$$A_0(z) = -\ln\psi_0^\xi(z). \tag{2}$$

The following expression for the derivative of $A_0$ (also called *the mean force*) is obtained:

$$\nabla A_0(z) = \int_{\Sigma_z} F d\mu_{0,z}, \tag{3}$$

where $F$ is defined by: $\forall \alpha \in \{1,\ldots,p\}$,

$$F_\alpha = \sum_{\beta=1}^{p} G^{-1}_{\alpha,\beta}\nabla\xi_\beta\cdot\nabla V - \operatorname{div}\left(\sum_{\beta=1}^{p} G^{-1}_{\alpha,\beta}\nabla\xi_\beta\right), \tag{4}$$

where $G^{-1}_{\alpha,\beta}$ denotes the $(\alpha,\beta)$-component of the inverse of the matrix $G$. All these results can be derived using the co-area formula (see Lemma 2.2 below), using similar computations as in Lemma 2.3 below.

Let us also introduce the following projection operators: For any $x \in \mathcal{D}$, we denote by

$$P(x) = \mathrm{Id} - \sum_{\alpha,\beta=1}^{p} G^{-1}_{\alpha,\beta} \nabla\xi_\alpha \otimes \nabla\xi_\beta(x) \tag{5}$$

the orthogonal projection operator onto the tangent space $T_x\Sigma_{\xi(x)}$ to $\Sigma_{\xi(x)}$ at point $x$, and by

$$Q(x) = \mathrm{Id} - P(x) = \sum_{\alpha,\beta=1}^{p} G^{-1}_{\alpha,\beta} \nabla\xi_\alpha \otimes \nabla\xi_\beta(x) \tag{6}$$

the orthogonal projection operator onto the normal space $N_x\Sigma_{\xi(x)}$ to $\Sigma_{\xi(x)}$ at point $x$. We denote by $\otimes$ the tensor product: for two vectors $u, v \in \mathbb{R}^n$, $u \otimes v$ is a $n \times n$ matrix with components $(u \otimes v)_{i,j} = u_i v_j$.

For any two probability measures $\mu$ and $\nu$ such that $\mu$ is absolutely continuous with respect to $\nu$ (this property being denoted $\mu \ll \nu$ in the following), we introduce the relative entropy

$$H(\mu|\nu) = \int \ln\left(\frac{d\mu}{d\nu}\right) d\mu.$$

Let us also introduce the Fisher information: For any two probability measures $\mu$ and $\nu$ such that $\mu \ll \nu$,

$$I(\mu|\nu) = \int \left|\nabla \ln\left(\frac{d\mu}{d\nu}\right)\right|^2 d\mu. \tag{7}$$

In (7) and in the following, $|\cdot|$ denotes the Euclidean norm (in $\mathbb{R}^n$ or in $\mathbb{R}^p$). In the case $\nu$ is a probability measure on the (Riemannian) submanifold $\Sigma_z$, $\nabla$ actually denotes the gradient on $\Sigma_z$ in (7), namely

$$\nabla_{\Sigma_z} = P\nabla. \tag{8}$$

We recall the definition of the Logarithmic Sobolev Inequality (LSI):

**Definition** 1.1 *The probability measure $\nu$ satisfies a logarithmic Sobolev inequality with constant $\rho > 0$ (in short: LSI($\rho$)) if for all probability measures $\mu$ such that $\mu \ll \nu$,*

$$H(\mu|\nu) \leq \frac{1}{2\rho} I(\mu|\nu).$$

The main result of this paper states under which condition a LSI holds for $\mu_0$, assuming that a LSI holds for the conditional probability measure $\mu_{0,z}$ (this is [H2]) and for for the marginal $\mu_0^\xi$ (this is [H3]).

**Theorem 1.2** *In addition to [H1], let us assume (recall that the local mean force $F$ is defined by (4)):*

$$\textbf{[H2]} \quad \begin{cases} V \text{ and } \xi \text{ are such that } \exists \rho > 0, \text{ for all } z \in \mathcal{M}, \\ \text{the conditional measure } \mu_{0,z} \text{ satisfies LSI}(\rho). \end{cases}$$

**[H3]** *$V$ and $\xi$ are such that $\exists r > 0$, the measure $d\mu_0^\xi = \psi_0^\xi(z)\,dz$ satisfies LSI($r$).*

$$\textbf{[H4]} \quad \begin{cases} V \text{ and } \xi \text{ are sufficiently differentiable functions such that: } \exists m > 0, G \geq m\,\mathrm{Id} \text{ and} \\ \text{(a) } \|\nabla_{\Sigma_z} F\|_{L^\infty} \leq M < \infty \text{ or (b) } \|F\|_{L^\infty} \leq \frac{M}{\sqrt{\rho}} < \infty. \end{cases}$$

*Then $\mu_0$ satisfies LSI(R) for some constant $R$ which satisfies:*

$$R \geq \frac{1}{2}\left( rm + \frac{M^2 m}{\rho} + \rho - \sqrt{\left( rm + \frac{M^2 m}{\rho} + \rho \right)^2 - 4rm\rho} \right). \qquad (9)$$

In [H4], $G \geq m\,\mathrm{Id}$ should be understood in the following sense: for any vector $u \in \mathbb{R}^p$, $u^T G u \geq m|u|^2$. In [H4-a] or [H4-b], the $L^\infty$ norm is with respect to $x \in \mathcal{D}$: $\|F\|_{L^\infty} = \sup_{x \in \mathcal{D}} |F|$ and $\|\nabla_{\Sigma_z} F\|_{L^\infty} = \sup_{x \in \mathcal{D}} |\nabla_{\Sigma_z} F|$, where $|\cdot|$ here denotes the operator norm on the matrix $\nabla_{\Sigma_z} F$ associated to the Euclidean norm on the vectors: $|\nabla_{\Sigma_z} F(x)| = \sup_{u \in T_x \Sigma_z} \frac{|\nabla F(x) u|}{|u|}$.

Assumption [H4-a] is an assumption on the coupling in the following sense. Assume that $V(x) = \frac{1}{2} x^T H x$ for some symmetric positive matrix $H \in \mathbb{R}^{n \times n}$ (so that $\mu_0$ is a Gaussian law), and that $\xi(x_1, \ldots, x_n) = (x_1, \ldots, x_p)$. In this case, $G = \mathrm{Id}$, and $\nabla_{\Sigma_z} F = 0$ is equivalent to the fact that the covariance $\mathrm{Cov}\left((X_1, \ldots, X_p), (X_{p+1}, \ldots, X_n)\right) = 0$, where $(X_1, \ldots, X_n)$ is a random variable with law $\mu_0$. In this case of Gaussian laws and a linear function $\xi$, it can be checked that (9) is optimal (see [8]).

## 2 Proof

To prove the result, we need to introduce a few other notation. Let $\psi$ be a probability density functional on $\mathcal{D}$. We denote the *total entropy* by

$$E = H(\psi|\psi_0),$$

and the *macroscopic entropy* by

$$E_M = H(\psi^\xi|\psi_0^\xi),$$

where

$$\psi^\xi(z) = \int_{\Sigma_z} \psi (\det G)^{-1/2} d\sigma_{\Sigma_z}.$$

We denote the conditioned probability measures of $\psi$ at a fixed value $z$ of the reaction coordinate by

$$d\mu_z = \frac{\psi (\det G)^{-1/2} d\sigma_{\Sigma_z}}{\psi^\xi(z)},$$

the "local entropy" by

$$e_m(z) = H(\mu_z|\mu_{0,z}) = \int_{\Sigma_z} \ln \left( \frac{\psi}{\psi^\xi(z)} \Big/ \frac{\psi_0}{\psi_0^\xi(z)} \right) d\mu_z,$$

and finally the *microscopic entropy* by

$$E_m = \int_{\mathcal{M}} e_m(z) \psi^\xi(z)\, dz.$$

It is straightforward to obtain the following result which can be seen as a property of extensivity of the entropy:

**Lemma 2.1** *It holds*

$$E = E_M + E_m.$$

We will need the co-area formula (see [1,4]):

**Lemma 2.2** *For any smooth function $\phi : \mathbb{R}^n \to \mathbb{R}$,*

$$\int_{\mathbb{R}^n} \phi(x) (\det G(x))^{1/2} dx = \int_{\mathbb{R}^p} \int_{\Sigma_z} \phi \, d\sigma_{\Sigma_z}\, dz, \tag{10}$$

*where $G$ is defined by* (1).

*Remark 1 The co-area formula shows that if the random variable $X$ has law $\psi(x)\, dx$ in $\mathbb{R}^n$, then $\xi(X)$ has law*

$$\int_{\Sigma_z} \psi \, (\det G)^{-1/2} \, d\sigma_{\Sigma_z}\, dz,$$

*and the law of $X$ conditioned to a fixed value $z$ of $\xi(X)$ is*

$$d\mu_z = \frac{\psi \, (\det G)^{-1/2} \, d\sigma_{\Sigma_z}}{\int_{\Sigma_z} \psi \, (\det G)^{-1/2} \, d\sigma_{\Sigma_z}}.$$

*Indeed, for any bounded functions $f$ and $g$,*

$$\begin{aligned}
\mathbb{E}(f(\xi(X))g(X)) &= \int_{\mathbb{R}^n} f(\xi(x))g(x)\psi(x)\,dx,\\
&= \int_{\mathbb{R}^p}\int_{\Sigma_z} f\circ\xi\, g\,\psi\,(\det G)^{-1/2} d\sigma_{\Sigma_z}\,dz,\\
&= \int_{\mathbb{R}^p} f(z)\frac{\int_{\Sigma_z} g\,\psi\,(\det G)^{-1/2} d\sigma_{\Sigma_z}}{\int_{\Sigma_z}\psi\,(\det G)^{-1/2} d\sigma_{\Sigma_z}}\int_{\Sigma_z}\psi\,(\det G)^{-1/2} d\sigma_{\Sigma_z}\,dz.
\end{aligned}$$

*The measure $(\det G)^{-1/2} d\sigma_{\Sigma_z}$ is sometimes denoted by $\delta_{\xi(x)-z}$ in the literature.*

From the co-area formula, we get:

**Lemma 2.3** *The derivative of $\psi^\xi$ reads: $\forall \alpha \in \{1,\dots,p\}$,*

$$\partial_{z_\alpha}\psi^\xi(z) = \int_{\Sigma_z}\sum_{\beta=1}^{p}\left(G^{-1}_{\alpha,\beta}\nabla\xi_\beta\cdot\nabla\psi + \operatorname{div}\left(G^{-1}_{\alpha,\beta}\nabla\xi_\beta\right)\psi\right)(\det G)^{-1/2} d\sigma_{\Sigma_z}.$$

*Proof :* For any smooth test function $g : \mathcal{M} \to \mathbb{R}^p$, we obtain (using the co-area formula (10) and an integration by parts) [1] :

$$\begin{aligned}
\int_{\mathcal{M}} \psi^\xi \operatorname{div} g &= \int_{\mathcal{D}} \psi\,(\operatorname{div} g)\circ\xi,\\
&= \int_{\mathcal{D}} \psi\, G^{-1}_{\alpha,\beta}\nabla\xi_\beta\cdot\nabla(g_\alpha\circ\xi),\\
&= -\int_{\mathcal{D}} \operatorname{div}\left(\psi\, G^{-1}_{\alpha,\beta}\nabla\xi_\beta\right) g_\alpha\circ\xi,\\
&= -\int_{\mathcal{M}} g_\alpha(z)\int_{\Sigma_z}\left(G^{-1}_{\alpha,\beta}\nabla\xi_\beta\cdot\nabla\psi + \operatorname{div}\left(G^{-1}_{\alpha,\beta}\nabla\xi_\beta\right)\psi\right)(\det G)^{-1/2} d\sigma_{\Sigma_z}\,dz,
\end{aligned}$$

which yields the result. ♢

A corollary of Lemma 2.3 applied with $\psi = \psi_0$ is Equation (3). Let us now introduce the mean force associated with $\psi$ (compare with (3)):

$$D(z) = \int_{\Sigma_z} F d\mu_z.$$

Notice that, in general, $D \neq -\nabla \ln \psi^\xi$, and $\operatorname{curl} D \neq 0$. We need a measure of the difference between $D$ and $\nabla A_0$, in terms of the difference between $\psi$ and $\psi_0$:

**Lemma 2.4** *The difference between $D$ and $\nabla A_0$ can be expressed in terms*

[1] In all the following proofs, we use the summation convention on repeated Greek indices going from 1 to $p$.

*of $\psi$ and $\psi_0$ as: for $\alpha \in \{1,\dots,p\}$, for all $z \in \mathcal{M}$,*

$$(D_\alpha - \partial_{z_\alpha} A_0)(z) = \int_{\Sigma_z} \sum_{\beta=1}^{p} G^{-1}_{\alpha,\beta} \nabla \xi_\beta \cdot \nabla \ln\left(\frac{\psi}{\psi_0}\right) \frac{\psi (\det G)^{-1/2}\, d\sigma_{\Sigma_z}}{\psi^\xi} - \partial_{z_\alpha} \ln\left(\frac{\psi^\xi}{\psi_0^\xi}\right).$$

*Proof :* Using Lemma 2.3 and the definition of $D$, it holds:

$$\begin{aligned}
&\int_{\Sigma_z} G^{-1}_{\alpha,\beta} \nabla \xi_\beta \cdot \nabla \ln\left(\frac{\psi}{\psi_0}\right) \frac{\psi (\det G)^{-1/2}\, d\sigma_{\Sigma_z}}{\psi^\xi} - \partial_{z_\alpha} \ln\left(\frac{\psi^\xi}{\psi_0^\xi}\right) \\
&\quad = \frac{1}{\psi^\xi} \int_{\Sigma_z} G^{-1}_{\alpha,\beta} \nabla \xi_\beta \cdot \nabla \psi (\det G)^{-1/2}\, d\sigma_{\Sigma_z} + \int_{\Sigma_z} G^{-1}_{\alpha,\beta} \nabla \xi_\beta \cdot \nabla V \frac{\psi (\det G)^{-1/2}\, d\sigma_{\Sigma_z}}{\psi^\xi} \\
&\qquad - \partial_{z_\alpha} \ln \psi^\xi + \partial_{z_\alpha} \ln \psi_0^\xi \\
&\quad = - \int_{\Sigma_z} \mathrm{div}\,(G^{-1}_{\alpha,\beta} \nabla \xi_\beta) \frac{\psi (\det G)^{-1/2}\, d\sigma_{\Sigma_z}}{\psi^\xi} + \int_{\Sigma_z} G^{-1}_{\alpha,\beta} \nabla \xi_\beta \cdot \nabla V \frac{\psi (\det G)^{-1/2}\, d\sigma_{\Sigma_z}}{\psi^\xi} \\
&\qquad - \partial_{z_\alpha} A_0 \\
&\quad = D_\alpha - \partial_{z_\alpha} A_0.
\end{aligned}$$

◊

From Lemma 2.4, the following estimates are obtained:

**Lemma 2.5** *Let us assume [H2] and [H4]. Then for all $z \in \mathcal{M}$,*

$$|D(z) - \nabla A_0(z)| \le M \sqrt{\frac{2}{\rho} e_m(z)}.$$

*Proof :* If we suppose [H4-b], then we have:

$$\begin{aligned}
|D(z) - \nabla A_0(z)| &= \left| \int F d\mu_z - \int F d\mu_{0,z} \right| \\
&\le \|F\|_{L^\infty} \|\mu_z - \mu_{0,z}\|_{VT}, \\
&\le \frac{M}{\sqrt{\rho}} \|\mu_z - \mu_{0,z}\|_{VT},
\end{aligned}$$

where $\|\mu_z - \mu_{0,z}\|_{VT}$ denotes the total variation norm of the signed measure $(\mu_z - \mu_{0,z})$. The result then follows from the Csiszar-Kullback inequality (see for example [2]):

$$\|\mu_z - \mu_{0,z}\|_{VT} \le \sqrt{2H(\mu_z|\mu_{0,z})}.$$

Let us now assume [H4-a]. For any coupling measure $\pi \in \Pi(\mu_z, \mu_{0,z})$ defined on $\Sigma_z \times \Sigma_z$ (namely any probability measure on $\Sigma \times \Sigma$ such that its marginals are $\mu_z$ and $\mu_{0,z}$), it holds:

$$
\begin{aligned}
|D(z) - \nabla A_0(z)| &= \left| \int_{\Sigma_z \times \Sigma_z} (F(x) - F(x'))\pi(dx, dx') \right|, \\
&\leq \|\nabla_{\Sigma_z} F\|_{L^\infty} \int_{\Sigma_z \times \Sigma_z} d_{\Sigma_z}(x, x')\pi(dx, dx'), \\
&\leq M \int_{\Sigma_z \times \Sigma_z} d_{\Sigma_z}(x, x')\pi(dx, dx'),
\end{aligned}
$$

where $d_{\Sigma_z}$ denotes the geodesic distance on $\Sigma_z$: $\forall x, y \in \Sigma_z$,

$$
d_{\Sigma_z}(x, y) = \inf \left\{ \sqrt{\int_0^1 |\dot{w}(t)|^2 \, dt} \;\middle|\; w \in \mathcal{C}^1([0,1], \Sigma_z), \; w(0) = x, \; w(1) = y \right\}.
$$

Taking now the infimum over all $\pi \in \Pi(\mu_z, \mu_{0,z})$, we obtain

$$
|D(z) - \nabla A_0(z)| \leq M W(\mu_z, \mu_{0,z})
$$

where $W(\mu_z, \mu_{0,z})$ denotes the Wasserstein distance with linear cost (see for example [2]). It is known that if $\mu_{0,z}$ satisfies a LSI (which is [H2]), then we have the following Talagrand inequality (see [3,9]):

$$
W(\mu_z, \mu_{0,z}) \leq \sqrt{\frac{2}{\rho} H(\mu_z|\mu_{0,z})}.
$$

This implies the result. ♢

**Lemma 2.6** *Let us assume [H2]. Then it holds*

$$
E_m \leq \frac{1}{2\rho} \int_{\mathcal{D}} \left| \nabla_{\Sigma_z} \ln \left( \frac{\psi}{\psi_0} \right) \right|^2 \psi.
$$

*Proof :* Notice that the Fisher information of $\mu_z$ with respect to $\mu_{0,z}$ writes

$$
I(\mu_z|\mu_{0,z}) = \int_{\Sigma_z} \left| \nabla_{\Sigma_z} \ln \left( \frac{\psi}{\psi_0} \right) \right|^2 \frac{\psi (\det G)^{-1/2} d\sigma_{\Sigma_z}}{\psi^\xi(z)}.
$$

Therefore, using [H2], it follows:

$$
\begin{aligned}
E_m &= \int_{\mathcal{M}} e_m \psi^\xi \, dz, \\
&\leq \int_{\mathcal{M}} \frac{1}{2\rho} \int_{\Sigma_z} \left| \nabla_{\Sigma_z} \ln \left( \frac{\psi}{\psi_0} \right) \right|^2 \frac{\psi (\det G)^{-1/2} d\sigma_{\Sigma_z}}{\psi^\xi(z)} \psi^\xi \, dz,
\end{aligned}
$$

which yields the result, using the co-area formula (10). ♢

We are now in position to prove Theorem 1.2. We have (using [H2], [H3], Lemma 2.1, Lemma 2.4, and the inequality $(a+b)^2 \leq (1+\varepsilon)a^2 + (1+\varepsilon^{-1})b^2$, for a positive $\varepsilon$ to be fixed later on):

$$
\begin{aligned}
E = E_m + E_M &\leq \frac{1}{2\rho} \int_{\mathcal{D}} \left| \nabla_{\Sigma_z} \ln \left( \frac{\psi}{\psi_0} \right) \right|^2 \psi + \frac{1}{2r} \int_{\mathcal{M}} \left| \nabla \ln \left( \frac{\psi^\xi}{\psi_0^\xi} \right) \right|^2 \psi^\xi, \\
&\leq \frac{1}{2\rho} \int_{\mathcal{D}} \left| \nabla_{\Sigma_z} \ln \left( \frac{\psi}{\psi_0} \right) \right|^2 \psi + \frac{1+\varepsilon}{2r} \int_{\mathcal{M}} |D - \nabla A_0|^2 \psi^\xi \\
&\quad + \frac{1+\varepsilon^{-1}}{2r} \int_{\mathcal{M}} \sum_{\alpha=1}^{p} \left| \int_{\Sigma_z} G^{-1}_{\alpha,\beta} \nabla \xi_\beta \cdot \nabla \ln \left( \frac{\psi}{\psi_0} \right) \frac{\psi (\det G)^{-1/2} \, d\sigma_{\Sigma_z}}{\psi^\xi} \right|^2 \psi^\xi.
\end{aligned}
$$

Using the Cauchy-Schwarz inequality:

$$
\begin{aligned}
&\left| \int_{\Sigma_z} G^{-1}_{\alpha,\beta} \nabla \xi_\beta \cdot \nabla \ln \left( \frac{\psi}{\psi_0} \right) \frac{\psi (\det G)^{-1/2} \, d\sigma_{\Sigma_z}}{\psi^\xi} \right|^2 \\
&\leq \int_{\Sigma_z} \left| G^{-1}_{\alpha,\beta} \nabla \xi_\beta \cdot \nabla \ln \left( \frac{\psi}{\psi_0} \right) \right|^2 \frac{\psi (\det G)^{-1/2} \, d\sigma_{\Sigma_z}}{\psi^\xi}
\end{aligned}
$$

and Lemma 2.5, we thus obtain

$$
\begin{aligned}
E &\leq \frac{1}{2\rho} \int_{\mathcal{D}} \left| \nabla_{\Sigma_z} \ln \left( \frac{\psi}{\psi_0} \right) \right|^2 \psi + \frac{(1+\varepsilon) M^2}{r\rho} \int_{\mathcal{M}} e_m \psi^\xi \\
&\quad + \frac{1+\varepsilon^{-1}}{2r} \int_{\mathcal{M}} \int_{\Sigma_z} \sum_{\alpha=1}^{p} \left| G^{-1}_{\alpha,\beta} \nabla \xi_\beta \cdot \nabla \ln \left( \frac{\psi}{\psi_0} \right) \right|^2 \psi (\det G)^{-1/2} \, d\sigma_{\Sigma_z}.
\end{aligned}
$$

For any vector $u \in \mathbb{R}^n$, notice that $|Qu|^2 = G^{-1}_{\alpha,\beta} \nabla \xi_\alpha \cdot u \nabla \xi_\beta \cdot u$, and that $|u|^2 = |Pu|^2 + |Qu|^2$ (where $P$ and $Q$ are the projection operators defined by (5) and (6)). Using [H4], we thus have:

$$
\begin{aligned}
\sum_{\alpha=1}^{p} |G^{-1}_{\alpha,\beta} \nabla \xi_\beta \cdot u|^2 &= G^{-1}_{\alpha,\beta} \nabla \xi_\beta \cdot u \, G^{-1}_{\alpha,\gamma} \nabla \xi_\gamma \cdot u, \\
&\leq \frac{1}{m} G^{-1}_{\beta,\gamma} \nabla \xi_\beta \cdot u \nabla \xi_\gamma \cdot u, \\
&= \frac{1}{m} |Qu|^2.
\end{aligned}
$$

Applying this inequality with $u = \nabla \ln\left(\frac{\psi}{\psi_0}\right)$ and using Lemma 2.6, we get:

$$\begin{aligned}
E &\leq \frac{1}{2\rho}\int_{\mathcal{D}}\left|P\nabla \ln\left(\frac{\psi}{\psi_0}\right)\right|^2 \psi + \frac{(1+\varepsilon)M^2}{r\rho}E_m \\
&\quad + \frac{1+\varepsilon^{-1}}{2rm}\int_{\mathcal{M}}\int_{\Sigma_z}\left|Q\nabla \ln\left(\frac{\psi}{\psi_0}\right)\right|^2 \psi(\det G)^{-1/2}\, d\sigma_{\Sigma_z}, \\
&\leq \left(\frac{1}{2\rho} + \frac{(1+\varepsilon)M^2}{2r\rho^2}\right)\int_{\mathcal{D}}\left|P\nabla \ln\left(\frac{\psi}{\psi_0}\right)\right|^2 \psi \\
&\quad + \frac{1+\varepsilon^{-1}}{2rm}\int_{\mathcal{D}}\left|Q\nabla \ln\left(\frac{\psi}{\psi_0}\right)\right|^2 \psi.
\end{aligned}$$

This shows that $\psi$ satisfies a LSI with constant $R$ satisfying

$$\begin{aligned}
R &\geq \frac{1}{2}\left(\max\left(\frac{1}{2\rho} + \frac{(1+\varepsilon)M^2}{2r\rho^2}, \frac{1+\varepsilon^{-1}}{2rm}\right)\right)^{-1}, \\
&= \min\left(\frac{\rho^2}{\rho + (1+\varepsilon)M^2/r}, \frac{rm}{1+\varepsilon^{-1}}\right).
\end{aligned}$$

Optimizing in $\varepsilon$, namely solving $\frac{\rho^2}{\rho+(1+\varepsilon)M^2/r} = \frac{rm}{1+\varepsilon^{-1}}$ concludes the proof.


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