ERGODICITY OF THE 3D STOCHASTIC NAVIER-STOKES EQUATIONS DRIVEN BY MILDLY DEGENERATE NOISE

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Abstract. We prove that the any Markov solution to the 3D stochastic Navier-Stokes equations driven by a mildly degenerate noise (i. e. all but finitely many Fourier modes are forced) is uniquely ergodic. This follows by proving strong Feller regularity and irreducibility.

1. Introduction

The well-posedness of three dimensional Navier-Stokes equations is still an open problem, in both the deterministic and stochastic cases (see [9] for a general introduction to the deterministic problem and [14] for the stochastic one). Although the existence of global weak solutions have been proven in both cases ([18], [10]), the uniqueness is still unknown. Inspired by the Hadamard definition of well-posedness for Cauchy problems, it is natural to ask if there are ways to find a good selection among the weak solutions to obtain additional properties, such as Markovianity or continuity with respect to the initial data.

Da Prato and Debussche proved in [3] that there exists a continuous selection (i. e. the selection is strong Feller) with unique invariant measure by studying the Kolmogorov equation associated to the stochastic Navier-Stokes equations (SNSE). Later Debussche and Odasso [6] proved that this selection is also Markovian. However, their approach essentially depends on the non-degeneracy of the driving noise. A different and slightly more general approach to Markov solutions, which includes the cases of degenerate noise and even deterministic equations, was introduced in [14]. Under the assumption of non-degeneracy and regularity of the covariance, the authors also proved that every Markov solution is strong Feller. Under the same assumptions every such dynamics is uniquely ergodic and exponentially mixing ([22]). However, both approaches rely on the non-degeneracy of the driving noise to obtain the strong Feller property, and consequently

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ergodicity.

The strong Feller property and ergodicity of SPDEs driven by *degenerate* noise have been intensively studied in recent years (see for instance [8],[16], [7], [17], [21]). For the two dimensional case there are several results on ergodicity, among which the most remarkable one is by Hairer and Mattingly [16]. They prove that the 2D stochastic dynamics has a unique invariant measure as long as the noise forces at least two linearly independent Fourier modes. In this respect the three dimensional case is still open (only partial results are known, see the aforementioned [3], [14], [22], see also [21], [20]) and this paper tries to partly fill this gap. More precisely, we will study the three dimensional Navier-Stokes equations

$$\begin{cases} \dot{u} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = \dot{\eta}, \\ \operatorname{div} u = 0, \\ u(0) = x, \end{cases} \tag{1.1}$$

on the torus $[0, 2\pi]^3$ with periodic boundary conditions and forced by a Gaussian noise $\dot{\eta}$. We assume that *all except finitely many Fourier modes* are driven by the noise, and prove that any Markov solution to the problem is strong Feller and ergodic.

Essentially, our approach combines the Malliavin calculus developed in [8] and the *weak-strong uniqueness* principle of [14]. Comparing with well-posed problems, the dynamics here exists only in the weak martingale sense and the standard tools of stochastic analysis are not available. Hence, the computations are made on an approximate cutoff dynamics (see Section 2.3), which equals any dynamics up to a small time. On the other hand, due to the degeneracy of the noise, the Bismut-Elworthy-Li formula cannot directly be applied to prove the strong Feller property. To fix this problem, we divide the dynamics into high and low frequencies, applying the formula only to the dynamics of high modes (thanks to the essential non-degeneracy of the noise).

Finally, we remark that, at least with the approach presented here, general results such as the truly hypoelliptic case in [16] seem to be hardly achievable. Here (as well as in [14]) the strong Feller property is essential to propagate smoothness from small times (where trajectories are regular with high probability) to all times. To overcome this difficulty and understand how to study the general case, the second author (with one of his collaborator) is proving in a work in progress ([1]) some results similar to those in this paper, via the Kolmogorov equation approach originally used in [3].

The paper is organized as follows. Section 2 gives a detailed description of the problem, the assumptions on the noise and the main results (Theorems 2.4 and 2.5). Section 3 contains the proof of strong Feller regularity, while Section 4 applies Malliavin calculus to prove the crucial Lemma 3.3. Section 5 shows the irreducibility of the dynamics, the appendix contains additional details and the proofs of some technical results.

2. Description of the problem and main results

Before stating the main results of the paper, we recast the problem in an abstract form, give the assumption on the noise and recall a few known results.

2.1. **Settings and notations.**

Let us start by writing (1.1) in an abstract form, using the standard formalism for the equations (see Temam [26] for details). Let $\mathbb{T}^3 = [0, 2\pi]^3$ be the three-dimensional torus, let H be the subspace of $L^2(\mathbb{T}^3; \mathbf{R}^3)$ of mean-zero *divergence-free* vector fields and let $\mathcal{P}$ be the projection from $L^2(\mathbb{T}^3, \mathbf{R}^3)$ onto H. Denote by A the Stokes operator (that is, $A = -\mathcal{P}\Delta$ is the projection on H of the Laplace operator) and by $B(u, v) = \mathcal{P}(u \cdot \nabla)v$ the projection of the nonlinearity. Following Temam [26], we consider the spaces $V_\alpha = D(A^{\alpha/2})$ and in particular we set $V = V_1$.

Problem (1.1) is recast in the following form,

$$\begin{cases} du + [\nu A u + B(u, u)]\, dt = Q\, dW_t, \\ u(0) = x. \end{cases} \tag{2.1}$$

where Q is a bounded operator on H satisfying suitable assumptions (see below) and W is a cylindrical Brownian motion on H. In the rest of the paper we shall assume $\nu = 1$, as its exact value will play no essential role.

Consider on H the Fourier basis $(e_k)_{k\in\mathbf{Z}^3_*}$ defined in (A.1) and, given $N \geq 1$, let $\pi_N : H \to H$ be the projection onto the subspace of H generated by all modes k such that $|k|_\infty := \max |k_i| \leq N$.

Assumption 2.1 (Assumptions on Q)**.** *The operator $Q : H \to H$ is linear bounded and there are $\alpha_0 > \frac{1}{2}$ and an integer $N_0 \geq 1$ such that*

- [A1] *(diagonality) Q is diagonal on the Fourier basis $(e_k)_{k\in\mathbf{Z}^3_*}$,*
- [A2] *(finite degeneracy) $\pi_{N_0} Q = 0$ and $\ker((Id - \pi_{N_0})Q) = \{0\}$,*
- [A3] *(regularity) $(Id - \pi_{N_0})A^{\alpha_0+3/4}Q$ is bounded invertible (with bounded inverse) on $(Id - \pi_{N_0})H$.*

Further details can be found in Subsection A.1. We only remark that [A3] is essentially the same as in [14] (we restrict here to $\alpha_0 > \frac{1}{2}$ for simplicity), while [A2] is the main assumption. The restriction $\pi_{N_0} Q = 0$ in [A2] (as well as property [A1]) has been taken to simplify the exposition.

2.2. **Markov solutions.**

Following the framework introduced in [14] (to which we refer for further details), we define the weak martingale solutions to problem (2.1) (cfr. Definition 3.3, [14]).

Definition 2.2 (Weak martingale solutions)**.** Given a probability measure μ on H, a solution P to problem (2.1) with initial condition μ is a probability measure on $\Omega = C([0, \infty); D(A)')$ such that

1. the marginal at time $t = 0$ of P is equal to μ,
2. $P[L^\infty_{loc}([0, \infty); H) \cap L^2_{loc}([0, \infty); V)] = 1$,

(3) For every $\phi \in D(A)$, the process

$$M_t^\phi = \langle \xi_t - \xi_0, \phi \rangle_H + \int_0^t \langle \xi_s, A\phi \rangle_H \, ds - \int_0^t \langle B(\xi_s, \phi), \xi_s \rangle_H \, ds$$

is square integrable and $(M_t^\phi, \mathcal{B}_t, P)_{t \geq 0}$ is a continuous martingale with quadratic variation $t|Q\phi|_H^2$,

where $(\xi_t)_{t\geq 0}$ is the canonical process on Ω and $\mathcal{B}_t$ is the Borel σ-field of $C([0,t]; D(A)')$.

A Markov solution $(P_x)_{x\in H}$ to problem (2.1) is a family of weak martingale solutions such that P_x has initial condition δ_x and the *almost sure Markov property* holds: for every $x \in H$ there is a Lebesgue null-set $T_x \subset (0, \infty)$ such that for every $t \geq 0$ and all $s \notin T_x$,

$$\mathbb{E}^{P_x}[\phi(\xi_{t+s})|\mathcal{B}_s] = \mathbb{E}^{P_{\xi_s}}[\phi(\xi_t)], \qquad P_x - \text{a. s.} \tag{2.2}$$

Existence of at least a Markov solution is ensured by Theorem 3.7 of [14] (see also [12], [15]), for weak martingale solutions that satisfy either a super-martingale type energy inequality ([14]) or an almost sure energy balance ([24]). More details on the martingale problem associated to these equations can be found in [23]. Given a Markov solution $(P_x)_{x\in H}$, define the *a. s. transition semigroup* $P_t : \mathcal{B}_b(H) \to \mathcal{B}_b(H)$ as

$$P_t\phi(x) = \mathbb{E}^{P_x}[\phi(\xi_t)].$$

Thanks to (2.2), for every $x \in H$, there is a Lebesgue null-set $T_x \subset (0, \infty)$ such that $P_{t+s}\phi(x) = P_s P_t \phi(x)$ for all $t \geq 0$ and all $s \notin T_x$.

2.3. **A regularized cut-off problem.** The dynamics (1.1) is dissipative, hence it is possible to prove existence of a unique local solution up to a small random time. Within this time, the solution to the following equation (2.3) coincides with any Markov solution. Let us make this rough observation more precise.

Let $\chi : [0, \infty) \to [0, 1]$ be a smooth function such that $\chi(r) \equiv 1$ for $r \leq 1$ and $\chi(r) \equiv 0$ for $r \geq 2$. Set

$$\mathcal{W} = V_{2\alpha_0+\frac{1}{2}}, \qquad \mathcal{W}' = V_{-(2\alpha_0+\frac{1}{2})}, \qquad \widetilde{\mathcal{W}} = V_{2\alpha_0+\frac{3}{4}},$$

(where α_0 is the constant in the Assumption 2.1). Given $\rho > 0$, and $x \in \mathcal{W}$, consider

$$\begin{cases} du^\rho + [Au^\rho + B(u^\rho, u^\rho)\chi(\frac{|u^\rho|_{\mathcal{W}}}{3\rho})] \, dt = Q(u^\rho) \, dW_t \\ u^\rho(0) = x, \end{cases} \tag{2.3}$$

where

$$Q(u) = Q + \big(1 - \chi(\frac{|u|_{\mathcal{W}}}{\rho})\big)\overline{Q}$$

and $\overline{Q}$ is a non-degenerate operator on $\pi_{N_0} H$ (see (A.2) for a detailed definition). It is easy to see that $Q(u)$ is non-degenerate as $|u|_{\mathcal{W}} \leq \rho$.

Theorem 2.3 (Weak-strong uniqueness). *For every $x \in \mathcal{W}$, there exists a unique weak solution to (2.3) so that the associated distribution P_x^ρ satisfies $P_x^\rho[C([0,\infty); \mathcal{W})] = 1$. Moreover, given $\rho \geq 1$, define $\tau_\rho : \Omega \to [0, \infty]$ by*

$$\tau_\rho(\omega) = \inf\{t \geq 0 : |\omega(t)|_{\mathcal{W}} \geq \rho\},$$

(and $\tau_\rho(\omega) = \infty$ if the set is empty). If $x \in \mathcal{W}$ and $|x|_{\mathcal{W}} < \rho$, then on $[0, \tau_\rho]$, P_x^ρ coincides with any Markov solution $(P_x)_{x\in\mathcal{W}}$ of (2.1), i. e., for all $t > 0$ and $\phi \in \mathcal{B}_b(H)$,

$$\mathbb{E}^{P_x^\rho}[\phi(\xi_t)1_{\{\tau_\rho \geq t\}}] = \mathbb{E}^{P_x}[\phi(\xi_t)1_{\{\tau_\rho \geq t\}}]. \tag{2.4}$$

Finally, if $|x|_{\mathcal{W}} < \rho$, then

$$\lim_{\epsilon\to 0} P_{x+h}^\rho[\tau_\rho \geq \epsilon] = 1, \tag{2.5}$$

uniformly for h in any closed subset of $\{h \in \mathcal{W} : |x + h|_{\mathcal{W}} < \rho\}$.

Proof. Existence and uniqueness for problem (2.3) are standard, since the nonlinearity and the operator $Q(u^\rho)$ are Lipschitz. Let $\widetilde{u}^\rho$ be the solution to problem (2.3) with $Q(u^\rho)$ replaced by Q, then $\tau_\rho(u^\rho) = \tau_\rho(\widetilde{u}^\rho)$. By pathwise uniqueness, $u^\rho(t) = \widetilde{u}^\rho(t)$ on $[0, \tau_\rho]$. This immediately implies (2.4) and (2.5) by Theorem 5.12 of [14]. □

2.4. **Main results.** The strong Feller and ergodicity results of [14], [13], [22] are obtained under a strong non-degeneracy assumption on the covariance. This paper relaxes this assumption, as shown by the following results.

Theorem 2.4. *Assume Assumption 2.1. Let $(P_x)_{x\in H}$ be a Markov solution to (2.1), and let $(P_t)_{t\geq 0}$ be the associated transition semigroup. Then $(P_t)_{t\geq 0}$ is strong Feller in $\mathcal{W}$.*

Proof. The theorem is a straightforward application of Theorem 5.4 of [14], once Theorems 2.3 and 3.1 are taken into account. □

Theorem 2.5. *Under the same assumptions of the previous theorem, every Markov solution $(P_x)_{x\in H}$ to (2.1) is uniquely ergodic and strongly mixing. Moreover, the (unique) invariant measure μ corresponding to a given Markov solution is* fully supported *on $\mathcal{W}$, i. e. $\mu(\mathcal{W}) = 1$ and $\mu(U) > 0$ for every open set U of $\mathcal{W}$.*

Proof. Given a Markov solution $(P_x)_{x\in H}$, there exists at least one invariant measure (Theorem 3.1, [22]). Uniqueness follows from Doob's theorem (Theorem 4.2.1 of [4]), since by Theorem 2.4 and Proposition 5.1 the system is both strong Feller and irreducible. The claim on the support follows again from Proposition 5.1. □

Remark 2.6. The strong Feller estimate on the transition semigroup can be made more quantitative with the same method used in [13], but unfortunately this only gives a Lipschitz estimate for the semigroup up to a logarithmic correction (compare with [3]).

Moreover, by Theorem 3.3 of [22], the convergence to the invariant measure is exponentially fast, if the Markov solutions satisfy an almost sure version of the energy inequality (see [22], [24]). The theorem in [22] is proved under an assumption of non-degeneracy of the noise, but the only arguments really used are that the dynamics is strong Feller and irreducible.

3. Strong Feller property of cutoff dynamics

This section will mainly prove the following theorem:

Theorem 3.1. *There is $\rho_0 > 0$ (depending only on N_0 and Q) such that for $\rho \geq \rho_0$ the transition semigroup P_t^ρ associated to equation (2.3) is strong Feller.*

Fix $N \geq N_0$ (whose value will be suitably chosen later in Proposition 4.5). In this and the following section we shall denote with the superscript L the quantities projected onto the modes smaller than N and with the superscript H those projected onto the modes larger than N. We divide the equation (2.3) into the low and high frequency parts (*dropping the* ρ *in* u^ρ for simplicity),

$$\begin{cases} du^L + [Au^L + B_L(u,u)\chi(\frac{|u|_{\mathcal{W}}}{3\rho})]\,dt = Q_L(u)dW_t^L \\ du^H + [Au^H + B_H(u,u)\chi(\frac{|u|_{\mathcal{W}}}{3\rho})]\,dt = Q_H dW_t^H \end{cases} \tag{3.1}$$

where $u^L = \pi_N u$, $u^H = (Id - \pi_N)u$, $W^L = \pi_N W$, $W^H = (Id - \pi_N)W$, $B_L = \pi_N B$, $B_H = (Id - \pi_N)B$, $Q_L(u) = Q(u)\pi_N$ and $Q_H = Q(u)(Id - \pi_N)$. In particular, Q_H is independent of u.

With the above separation for the dynamics, it is natural to define the *Frechet derivatives* for their low and high frequency parts. More precisely, for any stochastic process $X(t,x)$ on H with $X(0,x) = x$, the Frechet derivative $D_h X(t,x)$ is defined by

$$D_h X(t,x) := \lim_{\epsilon \to 0} \frac{X(t, x+\epsilon h) - X(t,x)}{\epsilon}, \qquad h \in H,$$

provided the limit exists. Moreover, it is natural to define the linear map $DX(t,x) : H \to H$ by

$$DX(t,x)h = D_h X(t,x), \qquad h \in H.$$

One can easily define $D_L X(t,x)$, $D_H X(t,x)$, $D_L X^H(t,x)$, $D_H X^L(t,x)$ and so on in a similar way, for instance, $D_H X^L(t,x) : H^H \to H^L$ is defined by

$$D_H X^L(t,x)h = D_h X^L(t,x), \qquad h \in H^H$$

with $D_h X^L(t,x) = \frac{1}{\epsilon}\lim_{\epsilon\to 0}[X^L(t,x+\epsilon h) - X^L(t,x)]$.

Let $C_b^k(\mathcal{W})$ be the set of functions on $\mathcal{W}$ with bounded 0-th, ..., k-th order derivatives. Given a $\psi \in C_b^1(\mathcal{W})$, for any $h \in \mathcal{W}$, the derivative of $\psi(x)$ along h, denoted by $D_h\psi(x)$, is defined by

$$D_h\psi(x) = \lim_{\epsilon\to 0} \frac{\psi(x+\epsilon h) - \psi(x)}{\epsilon}.$$

Clearly, the map $D\psi(x) : \mathcal{W} \to \mathbf{R}$, defined by $D\psi(x)h = D_h\psi(x)$ for all $h \in \mathcal{W}$, is linear bounded. Hence $D\psi(x) \in \mathcal{W}'$. Similarly, $D_L\psi(x)$ and $D_H\psi(x)$ can be defined (e.g. $D_L\psi(x)h = \lim_{\epsilon\to 0}[\psi(x+\epsilon h) - \psi(x)]/\epsilon$, $h \in \mathcal{W}^L$).

To prove Theorem 3.1, we need to approximate (3.1) by the following more regular dynamics:

$$\begin{cases} du^{\delta,\rho} + [Au^{\delta,\rho} + \mathrm{e}^{-A_H\delta}B(u^{\delta,\rho}, u^{\delta,\rho})\chi(\frac{|u^{\delta,\rho}|_{\mathcal{W}}}{3\rho})]\,dt = Q(u^{\delta,\rho})dW_t \\ u^{\delta,\rho}(0) = x \end{cases} \tag{3.2}$$

where $\delta > 0$ and $A_H = (Id - \pi_N)A$ (the existence and uniqueness of weak solution to equation (3.2) is standard). The reason for introducing this approximation, roughly speaking, is that one cannot prove $B(u,v) \in Ran(Q)$ but easily has $\mathrm{e}^{-A_H\delta}B(u,v) \in$

$Ran(Q)$, which is the key point for finding a suitable direction for the Malliavin derivatives (see Section 4).

Define *two maps* $\Phi_t(\cdot)$ and $\Phi_t^\delta(\cdot)$ from H to H by

$$\Phi_t(x) := u^\rho(t) \qquad \text{and} \qquad \Phi_t^\delta(x) := u^{\delta,\rho}(t),$$

where $u^\rho(t), u^{\delta,\rho}(t)$ are the solutions to (2.3) and (3.2) respectively. The following proposition shows that Φ_t is the limit of Φ_t^δ as $\delta \to 0^+$ in the some sense, and will be proven in the appendix.

Proposition 3.2. *For every $T > 0$ and $p \geq 2$, there exist some $C_i = C_i(p, \rho, \alpha_0) > 0$, $i = 1, 2$ such that*

$$\mathbb{E}[\sup_{0\leq t\leq T} |\Phi_t - \Phi_t^\delta|_{\mathcal{W}}^p] \leq C_1 \mathrm{e}^{C_1 T} |\mathrm{e}^{-A\delta} - Id|_{\mathcal{W}}^p, \tag{3.3}$$

$$\mathbb{E}[\sup_{0\leq t\leq T} |D\Phi_t - D\Phi_t^\delta|_{\mathcal{L}(\mathcal{W})}^p] \leq C_2 \mathrm{e}^{C_2 T} |\mathrm{e}^{-A\delta} - Id|_{\mathcal{W}}^p. \tag{3.4}$$

For any $\psi \in C_b^1(\mathcal{W})$, $h \in \mathcal{W}$ and $t > 0$,

$$\lim_{\delta\to 0+} |D_h \mathbb{E}[\psi(\Phi_t^\delta)] - D_h \mathbb{E}[\psi(\Phi_t)]| = 0. \tag{3.5}$$

The main ingredients of the proof of Theorem 3.1 are the following two lemmas, i.e. Lemmas 3.3 (proved in Section 4) and 3.4 (proved in the appendix, see page 23).

Lemma 3.3. *For any $t_0 > 0$, there exists $C = C(t_0, \alpha_0, \rho) > 0$ such that for every $x \in \widetilde{\mathcal{W}}$, $h \in \mathcal{W}^L$, $\psi \in C_b^1(H)$ and $t \geq t_0$,*

$$|\mathbb{E}[(D_L\psi)(\Phi_t^\delta(x)) D_h \Phi_t^{\delta,L}(x)]| \leq C\mathrm{e}^{Ct} \|\psi\|_\infty (1 + |x|_{\widetilde{\mathcal{W}}}) |h|_{\mathcal{W}}.$$

Lemma 3.4. *For any $T > 0$, $p \geq 2$ and $\delta \geq 0$, there exist some $C_i = C_i(p, \alpha_0, \rho)$, $i = 1, \ldots, 7$, such that*

$$\mathbb{E}(\sup_{0\leq t\leq T} |\Phi_t^\delta(x)|_{\mathcal{W}}^p) \leq C_1 \mathrm{e}^{C_1 T} |x|_{\mathcal{W}}^p, \tag{3.6}$$

$$\mathbb{E}[\sup_{0\leq t\leq T} |\Phi_t^\delta(x)|_{\widetilde{\mathcal{W}}}^p] \leq C_2 \mathrm{e}^{C_2 T} |x|_{\widetilde{\mathcal{W}}}^p, \tag{3.7}$$

$$\mathbb{E}[\sup_{0\leq t\leq T} |t^{1/8} \Phi_t^\delta(x)|_{\widetilde{\mathcal{W}}}^p] \leq C_3 \mathrm{e}^{C_3 T} |x|_{\mathcal{W}}^p, \tag{3.8}$$

$$\mathbb{E}[\sup_{0\leq t\leq T} |D_h \Phi_t^\delta(x)|_{\mathcal{W}}^p] \leq C_4 \mathrm{e}^{C_4 T} |h|_{\mathcal{W}}^p, \qquad h \in \mathcal{W}, \tag{3.9}$$

$$\mathbb{E}[\int_0^t |A^{1/2} D_h \Phi_s^\delta(x)|_{\mathcal{W}}^2 \, ds] \leq C_5 \mathrm{e}^{C_5 t} |h|_{\mathcal{W}}^2, \qquad h \in \mathcal{W}, \tag{3.10}$$

$$\mathbb{E}[\sup_{0\leq t\leq T} |D_{h^L} \Phi_t^{\delta,H}(x)|_{\mathcal{W}}^p] \leq (T^{p/2} \vee T^{p/8}) C_6 \mathrm{e}^{C_6 T} |h^L|_{\mathcal{W}}^p, \qquad h^L \in \mathcal{W}^L, \tag{3.11}$$

$$\mathbb{E}[\sup_{0\leq t\leq T} |D_{h^H} \Phi_t^{\delta,L}(x)|_{\mathcal{W}}^p] \leq (T^{p/2} \vee T^{p/8}) C_7 \mathrm{e}^{C_7 T} |h^H|_{\mathcal{W}}^p, \qquad h^H \in \mathcal{W}^H. \tag{3.12}$$

Proof of Theorem 3.1. Here we follow the idea in the proof of Proposition 5.2 of [8]. Set $S_t\psi(x) = \mathbb{E}[\psi(\Phi_t^\delta)]$ for any $\psi \in C_b^2(\mathcal{W})$, we prove the theorem in the following two steps.
Step 1. Estimate $DS_t\psi(x)$ for all $x \in \widetilde{\mathcal{W}}$: Since by Assumption 2.1 the operator $A_H^{3/4+\alpha_0}$ is bounded invertible on H, we know by (3.10) that $y_t^H = Q_H^{-1} D_{h^H}\Phi_t^{\delta,H} \in H^H$ $dt \times P-a.s.$, hence we can proceed as in the proof of Proposition 5.2 of [8] (more precisely, formula (5.8)) to get

$$D_{h^H} S_t\psi(x) = \frac{2}{t}\mathbb{E}\Big[\psi(\Phi_t^\delta)\int_{\frac{t}{4}}^{\frac{3t}{4}} \langle y_s^H, dW_s^H\rangle_H\Big] + \frac{2}{t}\int_{\frac{t}{4}}^{\frac{3t}{4}} \mathbb{E}[D_L S_{t-s}\psi(\Phi_s^\delta) D_{h^H}\Phi_s^{\delta,L}]\, ds$$

Hence, by Burkholder-Davis-Gundy's inequality,

(3.13)
$$\begin{aligned}\big|D_{h^H} S_t\psi(x)\big| &\le \frac{2}{t}\|\psi\|_\infty\Big(\int_{\frac{t}{4}}^{\frac{3t}{4}} \mathbb{E}|y_s^H|_H^2\, ds\Big)^{\frac{1}{2}} + \frac{2}{t}\int_{\frac{t}{4}}^{\frac{3t}{4}} \mathbb{E}[|D_L S_{t-s}\psi(\Phi_s^\delta)|_{\mathcal{W}'}|D_{h^H}\Phi_s^{\delta,L}|_{\mathcal{W}}]\, ds\\ &\le \frac{C_1}{t}\mathrm{e}^{C_1 t}\|\psi\|_\infty |h^H|_{\mathcal{W}} + \frac{2}{t}\int_{\frac{t}{4}}^{\frac{3t}{4}} \mathbb{E}[|D_L S_{t-s}\psi(\Phi_s^\delta)|_{\mathcal{W}'}|D_{h^H}\Phi_s^{\delta,L}|_{\mathcal{W}}]\, ds\end{aligned}$$

with $C_1 = C_1(p, \alpha_0, \rho)$, since by (3.10),

$$\int_{\frac{t}{4}}^{\frac{3t}{4}} \mathbb{E}|y_s^H|_H^2\, ds = \int_{\frac{t}{4}}^{\frac{3t}{4}} \mathbb{E}|Q_H^{-1} D_{h^H}\Phi_t^{\delta,H}|_H^2\, ds \le c\int_{\frac{t}{4}}^{\frac{3t}{4}} \mathbb{E}|A^{1/2} D_{h^H}\Phi_t^{\delta,H}|_{\mathcal{W}}^2\, ds \le c\mathrm{e}^{ct}|h^H|_{\mathcal{W}}^2.$$

For the low frequency part, according to Lemma 3.3, for any $0 < t_0 < 1$, there exists $C_2 = C_2(t_0, \alpha_0, \rho)$ such that for $t \ge t_0$,

(3.14)
$$\begin{aligned}|D_{h^L} S_t\psi(x)| &= |D_{h^L} S_{t/2}(S_{t/2}\psi)(x)| = |\mathbb{E}[D_L S_{t/2}\psi(\Phi_{t/2}^\delta) D_{h^L}\Phi_{t/2}^{\delta,L}]| + |\mathbb{E}[D_H S_{t/2}\psi(\Phi_{t/2}^\delta) D_{h^L}\Phi_{t/2}^{\delta,H}]|\\ &\le C_2\mathrm{e}^{C_2 t}\|\psi\|_\infty |h^L|_{\mathcal{W}}(1+|x|_{\widetilde{\mathcal{W}}}) + \mathbb{E}[|D_H S_{t/2}\psi(\Phi_{t/2}^\delta)|_{\mathcal{W}'}|D_{h^L}\Phi_{t/2}^{\delta,H}|_{\mathcal{W}}]\end{aligned}$$

Denote for all $T > t_0$,

$$\psi_{t_0,T} = \sup_{x\in\widetilde{\mathcal{W}}, t_0\le t\le T} \frac{t|DS_t\psi(x)|_{\mathcal{W}'}}{1+|x|_{\widetilde{\mathcal{W}}}},$$

combine (3.13) and (3.14), then for every $t \in [t_0, T]$ with $t_0 < T \le 1$,

$$\begin{aligned}|D_h S_t\psi(x)| &\le \frac{C_1}{t}\mathrm{e}^{C_1 T}\|\psi\|_\infty |h|_{\mathcal{W}} + C_2\mathrm{e}^{C_2 T}\|\psi\|_\infty |h|_{\mathcal{W}}(1+|x|_{\widetilde{\mathcal{W}}})\\ &\quad + \psi_{t_0,T}\frac{2}{t}\Big[\int_{\frac{t}{4}}^{\frac{3t}{4}} \frac{1}{t-s}\mathbb{E}[(1+|\Phi_s^\delta|_{\widetilde{\mathcal{W}}})|D_{h^H}\Phi_s^{\delta,L}|_{\mathcal{W}}]\, ds + \mathbb{E}[(1+|\Phi_{t/2}^\delta|_{\widetilde{\mathcal{W}}})|D_{h^L}\Phi_{t/2}^{\delta,H}|_{\mathcal{W}}]\Big]\\ &\le \frac{C_1\mathrm{e}^{C_1 T}}{t}\|\psi\|_\infty |h|_{\mathcal{W}} + \big[C_2\mathrm{e}^{C_2 T}\|\psi\|_\infty |h|_{\mathcal{W}} + \frac{1}{t}\psi_{t_0,T} C_3\mathrm{e}^{C_3 T} T^{1/8}|h|_{\mathcal{W}}\big](1+|x|_{\widetilde{\mathcal{W}}}),\end{aligned}$$

where $C_3 = C_3(p, \alpha_0, \rho) > 0$ and the last inequality is due to

$$\big(\mathbb{E}[(1+|\Phi_s^\delta|_{\widetilde{\mathcal{W}}})|D_{h^H}\Phi_s^{\delta,L}|_{\mathcal{W}}]\big)^2 \le \mathbb{E}[\sup_{0\le s\le T}(1+|\Phi_s^\delta|_{\widetilde{\mathcal{W}}})^2]\mathbb{E}[\sup_{0\le s\le T}|D_{h^H}\Phi_s^{\delta,L}|^2_{\mathcal{W}}]$$
$$\le T^{1/4}Ce^{CT}|h|^2_{\mathcal{W}}(1+|x|_{\widetilde{\mathcal{W}}})^2,$$

which follows from (3.7) and (3.12). Hence

$$\psi_{t_0,T} \le C_1 e^{C_1 T}\|\psi\|_\infty + C_2 e^{C_2 T} T\|\psi\|_\infty + \psi_{t_0,T} C_3 e^{C_3 T} T^{1/8},$$

as t_0 and T are both sufficiently small, we have $\psi_{t_0,T} \le C_4\|\psi\|_\infty$ with $C_4 = C_4(t_0, T, \rho, \alpha_0)$, and thus for $t_0 \le t \le T$,

$$|DS_t\psi(x)|_{\mathcal{W}'} \le \frac{C_4}{t}(1+|x|_{\widetilde{\mathcal{W}}})\|\psi\|_\infty. \tag{3.15}$$

Step 2. Strong Feller property of P_t^ρ. Applying Cauchy-Schwartz inequality, (3.15) and (3.8), for any $h \in \mathcal{W}$ and any $t_0 \le t \le T$, we have

$$|D_h S_{2t}\psi(x)|^2 = |\mathbb{E}[DS_t\psi(\Phi_t^\delta)D_h\Phi_t^\delta]|^2 \le \mathbb{E}[|DS_t\psi(\Phi_t^\delta)|^2_{\mathcal{W}'}]\mathbb{E}[|D_h\Phi_t^\delta|^2_{\mathcal{W}}] \le$$
$$\le \frac{C}{t^2}\|\psi\|^2_\infty \mathbb{E}[(1+|\Phi_t^\delta|_{\widetilde{\mathcal{W}}})^2]|h|^2_{\mathcal{W}} \le \frac{C}{t^{9/4}}\|\psi\|^2_\infty|x|^2_{\mathcal{W}}|h|^2_{\mathcal{W}}$$

where $C = C(t_0, \alpha_0, \rho, T)$. Let $\delta \to 0^+$, we have by (3.5)

$$|D_h P_{2t}^\rho\psi(x)| \le \frac{C}{t^{9/8}}\|\psi\|_\infty|x|_{\mathcal{W}}|h|_{\mathcal{W}}, \qquad t_0 \le t \le T. \tag{3.16}$$

Since $t_0 > 0$ can be arbitrarily small, (3.16) implies that $(P_t^\rho)_{t\in(0,T]}$ is strong Feller ([4]). The extension of the strong Feller property to arbitrary $T > 0$ is also standard. □

4. Malliavin Calculus and Proof of Lemma 3.3

In this section, we will *only* study the equation (3.2), following the idea in [8] to prove Lemma 3.3. A very important point is that all the estimates in lemmas 4.2 and 4.3 are *independent of* δ (thanks to the cutoff and to that our Malliavin calculus is essentially on low frequency part of Φ_t^δ). We will simply write $\underline{\Phi_t = \Phi_t^\delta}$ throughout this section.

4.1. **Proof of Lemma 3.3.** Given $v \in L^2_{\mathrm{loc}}(\mathbf{R}_+, H)$, the Malliavin derivative of Φ_t in direction v, denoted by $\mathcal{D}_v\Phi_t$, is defined by

$$\mathcal{D}_v\Phi_t = \lim_{\epsilon\to 0}\frac{\Phi_t(W+\epsilon V, x) - \Phi_t(W, x)}{\epsilon}$$

where $V(t) = \int_0^t v(s)\,ds$. The direction v can be random and is adapted to the filtration generated by W. The Malliavin derivatives on the low and high frequency parts, denoted by $\mathcal{D}_v\Phi_t^L$ and $\mathcal{D}_v\Phi_t^H$, can be defined in a similar way. $\mathcal{D}_v\Phi_t^L$ and $\mathcal{D}_v\Phi_t^H$ satisfies the following two SPDEs respectively:

$$d\mathcal{D}_v\Phi^L + [A\mathcal{D}_v\Phi^L + D_L(B_L(\Phi,\Phi)\chi(\frac{|\Phi|_{\mathcal{W}}}{3\rho}))\mathcal{D}_v\Phi^L + D_H(B_L(\Phi,\Phi)\chi(\frac{|\Phi|_{\mathcal{W}}}{3\rho}))\mathcal{D}_v\Phi^H]\,dt =$$
$$= [D_L Q_L(\Phi)\mathcal{D}_v\Phi^L + D_H Q_L(\Phi)\mathcal{D}_v\Phi^H]dW_t^L + Q_L(\Phi)v^L\,dt, \tag{4.1}$$

$$(4.2)\quad d\mathcal{D}_v\Phi^H + [A\mathcal{D}_v\Phi^H + D_L(\mathrm{e}^{-A_H\delta}B_H(\Phi,\Phi)\chi(\frac{|\Phi|_{\mathcal{W}}}{3\rho}))\mathcal{D}_v\Phi^L + \\ + D_H(\mathrm{e}^{-A_H\delta}B_H(\Phi,\Phi)\chi(\frac{|\Phi|_{\mathcal{W}}}{3\rho}))\mathcal{D}_v\Phi^H]\,dt = Q_H v^H\,dt$$

with $\mathcal{D}_v\Phi_0^L = 0$ and $\mathcal{D}_v\Phi_0^H = 0$.

Define the derivative flow of $\Phi^L(x)$ between s and t by $J_{s,t}(x)$, $s \le t$, which satisfies the following equation: for all $h \in H^L$

$$dJ_{s,t}h + \Big[AJ_{s,t}h + D_L[B_L(\Phi_t,\Phi_t)\chi(\frac{|\Phi_t|_{\mathcal{W}}}{3\rho})]J_{s,t}h\Big]\,dt = D_L Q_L(\Phi_t)J_{s,t}h dW_t^L$$

with $J_{s,s}(x) = Id \in \mathcal{L}(H^L,H^L)$. The inverse $J_{s,t}^{-1}(x)$ satisfies
(4.3)

$$dJ_{s,t}^{-1}h - J_{s,t}^{-1}\Big[Ah + D_L[B_L(\Phi_t,\Phi_t)\chi(\frac{|\Phi_t|_{\mathcal{W}}}{3\rho})]h - \mathrm{Tr}((D_L Q_L(\Phi_t))^2)h\Big]\,dt = -J_{s,t}^{-1}D_L Q_L(\Phi_t)h dW_t^L$$

with $\mathrm{Tr}((D_L Q_L(\Phi_t))^2)h = \sum_{k\in Z_L(N)}\sum_{i=1}^2 D[q_k(\Phi_t)e_k^i]D[q_k(\Phi_t)e_k^i]h$ and $q_k(x) = (1 - \chi(|x|_{\mathcal{W}}/\rho))q_k$ (recall the notations in Appendix A.1). Simply writing $J_t = J_{0,t}$, clearly,

$$J_{s,t} = J_t J_s^{-1}, \qquad J_t = D_L\Phi_t^L.$$

We follow the ideas in Section 6.1 of [8] to develop a Malliavin calculus for (3.2). One of the key points for this approach is to find an adapted process $v \in L^2_{loc}(\mathbf{R}_+, H)$ so that

$$(4.4)\qquad Q_H v^H(t) = D_L(\mathrm{e}^{-A_H\delta}B_H(\Phi_t,\Phi_t)\chi(\frac{|\Phi_t|_{\mathcal{W}}}{3\rho}))\mathcal{D}_v\Phi_t^L,$$

which implies that $\mathcal{D}_v\Phi_t^H = 0$ for all $t > 0$ (hence, the Malliavin calculus is essentially restricted in *low* frequency part). More precisely,

Proposition 4.1. *There exists $v \in L^2_{loc}(\mathbf{R}_+; H)$ satisfying (4.4), and*

$$\mathcal{D}_v\Phi_t^L = J_t\int_0^t J_s^{-1}Q_L(\Phi_s)v^L(s)\,ds \qquad \text{and} \qquad \mathcal{D}_v\Phi_t^H = 0.$$

Proof. We first claim that

$$(4.5)\qquad D_L(\mathrm{e}^{-A_H\delta}B_H(\Phi_t,\Phi_t)\chi(\frac{|\Phi_t|_{\mathcal{W}}}{3\rho}))\mathcal{D}_v\Phi_t^L \in (D(A^{\alpha_0+3/4}))^H.$$

Indeed, $\Phi_t \in \widetilde{\mathcal{W}}$ from (3.8). Since $\mathcal{D}_v\Phi_t^L$ is finite dimensional, $\mathcal{D}_v\Phi_t^L \in \widetilde{\mathcal{W}}$. It is easy to see

$$D_L(\mathrm{e}^{-A_H\delta}B_H(\Phi_t,\Phi_t)\chi(\frac{|\Phi_t|_{\mathcal{W}}}{3\rho}))\mathcal{D}_v\Phi_t^L = \mathrm{e}^{-A_H\delta}B_H(\mathcal{D}_v\Phi_t^L,\Phi_t)\chi(\frac{|\Phi_t|_{\mathcal{W}}}{3\rho}) + \\ + \mathrm{e}^{-A_H\delta}B_H(\Phi_t,\mathcal{D}_v\Phi_t^L)\chi(\frac{|\Phi|_{\mathcal{W}}}{3\rho}) + \mathrm{e}^{-A_H\delta}B_H(\Phi_t,\Phi_t)\chi'(\frac{|\Phi_t|_{\mathcal{W}}}{3\rho})\frac{\langle\Phi_t,\mathcal{D}_v\Phi_t^L\rangle_{\mathcal{W}}}{3\rho|\Phi_t|_{\mathcal{W}}}.$$

The three terms on the right hand of the above equality can all be bounded in the same way, for instance, applying (A.6) with $\beta = \alpha_0 + 1/8$, the first term is bounded by

$$|\mathrm{e}^{-A_H\delta} B_H(\mathcal{D}_v\Phi_t^L, \Phi_t)\chi(\frac{|\Phi|_{\mathcal{W}}}{3\rho})|_{D(A^{\alpha_0+\frac{3}{4}})} = |A^{\frac{7}{8}}\mathrm{e}^{-A_H\delta} A^{\alpha_0-\frac{1}{8}} B_H(\mathcal{D}_v\Phi_t^L, \Phi_t)|_H \le \frac{C_1}{\delta^{\frac{7}{8}}} |\mathcal{D}_v\Phi_t^L|_{\widetilde{\mathcal{W}}} |\Phi_t|_{\widetilde{\mathcal{W}}},$$

and (4.5) follows immediately. Hence, by Assumption [A3] for Q, there exists at least one $v \in L^2_{loc}(\mathbf{R}_+; H)$ so that v^H satisfies (4.4) (we will see in (4.6) that $\mathcal{D}_v\Phi_t^L$ *does not depend on* v^H). Thus equation (4.2) is a homogeneous linear equation and has a unique solution

$$\mathcal{D}_v\Phi_t^H = 0,$$

for all $t > 0$. Hence, equation (4.1) now reads

$$d\mathcal{D}_v\Phi^L + [A\mathcal{D}_v\Phi^L + D_L(B_L(\Phi,\Phi)\chi(\frac{|\Phi|_{\mathcal{W}}}{3\rho}))\mathcal{D}_v\Phi^L]\,dt = D_L Q_L(\Phi)\mathcal{D}_v\Phi^L dW_t^L + Q_L(\Phi)v^L\,dt,$$

with $\mathcal{D}_v\Phi_0^L = 0$, which is solved by

$$\mathcal{D}_v\Phi_t^L = \int_0^t J_{s,t} Q_L(\Phi_s) v^L(s)\,ds = J_t \int_0^t J_s^{-1} Q_L(\Phi_s) v^L(s)\,ds \tag{4.6}$$

□

Let $N \ge N_0$ be the integer fixed at the beginning of Section 3 and consider $M = 2(2N+1)^3 - 2$ vectors $v_1, \ldots, v_M \in L^2_{loc}(\mathbf{R}_+; H)$, with each of them satisfying Proposition 4.1 (notice that M is the dimension of $H^L = \pi_N H$). Set

$$v = [v_1, \ldots, v_M], \tag{4.7}$$

we have

$$\mathcal{D}_v\Phi_t^H = 0, \qquad \mathcal{D}_v\Phi_t^L = J_t \int_0^t J_s^{-1} Q_L(\Phi_s) v^L(s)\,ds,$$

where Q_L is defined in (3.1). Choose

$$v^L(s) = (J_s^{-1} Q_L(\Phi_s))^*$$

and define the *Malliavin matrix*

$$\mathcal{M}_t = \int_0^t J_s^{-1} Q_L(\Phi_s)(J_s^{-1} Q_L(\Phi_s))^*\,ds.$$

Since $J_t^{-1} \in \mathcal{L}(\mathcal{W}^L, \mathcal{W}^L)$ and $Q_L \in \mathcal{L}(\mathcal{W}^L, \mathcal{W}^L)$, it follows that $\mathcal{M}_t \in \mathcal{L}(\mathcal{W}^L, \mathcal{W}^L)$. By Parseval identity (using the notation in Section A.1),

$$\langle \mathcal{M}_t\eta, \eta\rangle >_{\mathcal{W}} = \int_0^t |(J_s^{-1}Q_L(\Phi_s))^*\eta|^2_{\mathcal{W}}\,ds = \sum_{k\in Z_L(N), i=1,2} \frac{1}{|k|^{4\alpha_0+1}} \int_0^t |\langle J_s^{-1}Q_L(\Phi_s)e_k^i, \eta\rangle_{\mathcal{W}}|^2\,ds$$

$$= \sum_{k\in Z_L(N), i=1,2} \frac{1}{|k|^{4\alpha_0+1}} \int_0^t |\langle J_s^{-1}(q_k x_k^i e_k), \eta\rangle_{\mathcal{W}}|^2\,ds. \tag{4.8}$$

The following two lemmas are crucial for the proof of Lemma 3.3. The first one will be proven in the appendix (see page 26), while the other in Section 4.3.

Lemma 4.2. *For any $T > 0$ and $p \geq 2$, there exist some $C_i = C_i(p, \rho, \alpha_0) > 0$ ($i = 1, 2, 3, 4$) such that*

$$\mathbb{E}(\sup_{0\leq t\leq T} |J_t(x)h^L|^p_{\mathcal{W}}) \leq C_1 \mathrm{e}^{C_1 T}|h^L|^p_{\mathcal{W}}, \tag{4.9}$$

$$\mathbb{E}(\sup_{0\leq t\leq T} |J_t^{-1}(x)h^L|^p_{\mathcal{W}}) \leq C_2 \mathrm{e}^{C_2 T}|h^L|^p_{\mathcal{W}}, \tag{4.10}$$

$$\mathbb{E}(\sup_{0\leq t\leq T} |J_t^{-1}(x)h^L - h^L|^p_{\mathcal{W}}) \leq T^{p/2} C_3 \mathrm{e}^{C_3 T}|h^L|^p_{\mathcal{W}}, \tag{4.11}$$

$$\mathbb{E}(\sup_{0\leq t\leq T} |\Phi_t(x) - \mathrm{e}^{-At}x|^p_{\mathcal{W}}) \leq (T^{p/8} \vee T^{p/2}) C_4 \mathrm{e}^{C_4 T}. \tag{4.12}$$

Suppose that v_1, v_2 satisfy Proposition 4.1 and $p \geq 2$, then

$$\mathbb{E}(\sup_{0\leq t\leq T} |\mathcal{D}_{v_1}\Phi^L_t(x)|^p_{\mathcal{W}}) \leq C_5 \mathrm{e}^{C_5 T}\mathbb{E}[\int_0^T |v_1^L(s)|^p_{\mathcal{W}}\, ds] \tag{4.13}$$

(4.14)

$$\mathbb{E}\Big(\sup_{0\leq t\leq T} |\mathcal{D}^2_{v_1 v_2}\Phi^L_t(x)|^p_{\mathcal{W}}\Big) \leq C_6 \mathrm{e}^{C_6 T}\Big(\mathbb{E}[\int_0^T |v_1^L(s)|^{2p}_{\mathcal{W}}\, ds]\Big)^{1/2}\Big(\mathbb{E}[\int_0^T |v_2^L(s)|^{2p}_{\mathcal{W}}\, ds]\Big)^{1/2}$$

$$\mathbb{E}\Big(\sup_{0\leq t\leq T} |\mathcal{D}_{v_1}D_h\Phi^L_t(x)|^p_{\mathcal{W}}\Big) \leq C_7 \mathrm{e}^{C_7 T}|h|^p_{\mathcal{W}}\Big(\mathbb{E}[\int_0^T |v_1^L(s)|^{2p}_{\mathcal{W}}\, ds]\Big)^{1/2} \tag{4.15}$$

with $h \in \mathcal{W}$ and $C_i = C_i(p, \rho, \alpha_0) > 0$, $i = 5, 6, 7$.

Lemma 4.3. *Suppose that Φ_t is the solution to equation (3.2) with initial data $x \in \widetilde{\mathcal{W}}$. Then $\mathcal{M}_t \in \mathcal{L}(\mathcal{W}^L, \mathcal{W}^L)$ is invertible almost surely and $1/\lambda_{min}(t) \in L^p(P)$ for all $p > 0$, where $\lambda_{min}(t)$ is the smallest eigenvalue of $\mathcal{M}_t$. Moreover, for every $p > 0$ and $t_0 > 0$, there is some $C = C(p, \rho, \alpha_0, t_0)$ such that for all $t \geq t_0$*

$$P[|1/\lambda_{min}(t)| \geq 1/\epsilon] \leq C\epsilon^p(1 + |x|_{\widetilde{\mathcal{W}}})^p, \quad \text{uniformly for } \delta > 0.$$

Proof of Lemma 3.3. Under an orthonormal basis of $\mathcal{W}^L$, J_t and $\mathcal{M}_t$ can be represented respectively by an invertible $M \times M$ matrix and a symmetric $M \times M$ matrix, where M is the dimension of $\mathcal{W}^L$. Let us consider

$$\psi_{ik}(\Phi_t) = \psi(\Phi_t)(\mathcal{M}_t^{-1})_{ik}, \qquad i, k = 1, \ldots, M.$$

Given any $h \in \mathcal{W}^L$, it is easy to see that

$$D_L\psi_{ik}(\Phi_t)\mathcal{D}_v\Phi^L_t h = D_L\psi(\Phi_t)(\mathcal{D}_v\Phi^L_t h)(\mathcal{M}_t^{-1})_{ik} + \psi(\Phi_t)D_L(\mathcal{M}_t^{-1})_{ik}(\mathcal{D}_v\Phi^L_t h), \tag{4.16}$$

where $v = v(t)$ is defined by (4.7) with $v^L(t) = (J_t^{-1}Q_L(\Phi_t))^*$. Note that $\mathcal{W}^L$ is isomorphic to $\mathbf{R}^M$, given the standard orthonormal basis $\{h_i : i = 1, \ldots, M\}$ of $\mathbf{R}^M$, it can be taken as a presentation of the orthonormal basis of $\mathcal{W}^L$. Denoting $v_i = vh_i$, clearly, $v_i(t) = (J_t^{-1}Q_L(\Phi_t))^* h_i$ (note that the v_i here is not the same as in (4.7)). Setting $h = h_i$ in (4.16), summing over i and noticing the identity $\mathcal{D}_v\Phi^L_t = J_t\mathcal{M}_t$, we obtain

$$\mathbb{E}\Big[\sum_{j=1}^M D_{h_j}\psi(\Phi_t)(J_t)_{jk}\Big] = \mathbb{E}\Big[\sum_{i=1}^M D_L\psi_{ik}(\Phi_t)\mathcal{D}_{v_i}\Phi^L_t\Big] - \mathbb{E}\Big[\sum_{i=1}^M \psi(\Phi_t)\mathcal{D}_{v_i}(\mathcal{M}_t^{-1})_{ik}\Big]. \tag{4.17}$$

Let us estimate the two terms on the right hand of (4.17) as follows. By Bismut formula,

$$\begin{aligned}\Big|\mathbb{E}\Big[\sum_{i=1}^{M} D_L\psi_{ik}(\Phi_t)\mathcal{D}_{v_i}\Phi_t^L\Big]\Big| &= \Big|\mathbb{E}\Big[\sum_{i=1}^{M}\psi(\Phi_t)(\mathcal{M}_t^{-1})_{ik}\int_0^t\langle v_i^L, dW_s\rangle_H\Big]\Big| \\ &\le C\|\psi\|_\infty\mathbb{E}\Big[1/\lambda_{min}\sum_{i=1}^{M}\Big|\int_0^t\langle v_i^L, dW_s\rangle_H\Big|\Big] \\ &\le C\|\psi\|_\infty\big(\mathbb{E}[|1/\lambda_{min}|^2]\big)^{\frac12}\sum_{i=1}^{M}\Big(\int_0^t\mathbb{E}[|v_i^L|_{\mathcal{W}}^2]\,ds\Big)^{\frac12} \\ &\le C_1\mathrm{e}^{C_1t}\|\psi\|_\infty(1+|x|_{\widetilde{\mathcal{W}}})\end{aligned}$$

where $t\ge t_0$ ($t_0>0$ can be arbitrarily small), $C_1=C_1(t_0,\alpha_0,\rho)>0$ and the last inequality is also due to the following estimate (thanks to Parseval's identity and (4.10))

$$\mathbb{E}[|v_i^L(s)|_{\mathcal{W}}^2]=\mathbb{E}[|(J_s^{-1}Q^L)^*h_i|_{\mathcal{W}}^2]\le C\sum_{j,k}\mathbb{E}[|\langle h_i, J_s^{-1}Q^Le_k^j\rangle|^2]\le Ce^{Ct} \tag{4.18}$$

On the other hand,

$$\begin{aligned}&\mathbb{E}\Big(\sum_{i=1}^{M}\psi(\Phi_t)\mathcal{D}_{v_i}(\mathcal{M}_t^{-1})_{ik}\Big)= \\ &\quad=\mathbb{E}\Big(\sum_{i=1}^{M}\psi(\Phi_t)(\mathcal{M}_t^{-1}J_t^{-1}\mathcal{D}_{v_i}J_t)_{ik}\Big)+\mathbb{E}\Big(\sum_{i=1}^{M}\psi(\Phi_t)(\mathcal{M}_t^{-1}J_t^{-1}\mathcal{D}_{v_i}(\mathcal{D}_v\Phi_t^L)\mathcal{M}_t^{-1})_{ik}\Big),\end{aligned}$$

we have by the symmetricity of $\mathcal{M}_t^{-1}$, (4.15) and (4.18),

$$\begin{aligned}\Big|\mathbb{E}\Big[\sum_{i=1}^{M}\psi(\Phi_t)(\mathcal{M}_t^{-1}J_t^{-1}\mathcal{D}_{v_i}J_t)_{ik}\Big]\Big| &= \Big|\mathbb{E}\Big[\sum_{i,j=1}^{M}\psi(\Phi_t)(\mathcal{M}_t^{-1})_{ij}(J_t^{-1}\mathcal{D}_{v_i}J_t)_{jk}\Big]\Big| \\ &\le\|\psi\|_\infty\sum_{i=1}^{M}\mathbb{E}\big[|\langle\mathcal{M}_t^{-1}h_i,(J_t^{-1}\mathcal{D}_{v_i}J_t)h_k\rangle_{\mathcal{W}}|\big] \\ &\le\|\psi\|_\infty\sum_{i=1}^{M}\mathbb{E}\big[1/\lambda_{min}|(J_t^{-1}\mathcal{D}_{v_i}J_t)h_k|_{\mathcal{W}}\big] \\ &=\|\psi\|_\infty\sum_{i=1}^{M}\mathbb{E}\big[1/\lambda_{min}|J_t^{-1}\mathcal{D}_{v_i}D_{h_k}\Phi_t^L|_{\mathcal{W}}\big] \\ &\le C_2\mathrm{e}^{C_2t}\|\psi\|_\infty(1+|x|_{\widetilde{\mathcal{W}}})\end{aligned}$$

and by (4.10), (4.14) and (4.18),

$$
\begin{aligned}
\Big|\mathbb{E}\Big[\sum_{i=1}^{M}\psi(\Phi_t)(\mathcal{M}_t^{-1}J_t^{-1}\mathcal{D}_{v_i}(\mathcal{D}_v\Phi_t^L)\mathcal{M}_t^{-1})_{ik}\Big]\Big| &= \Big|\mathbb{E}\Big[\sum_{i,j=1}^{M}\psi(\Phi_t)[\mathcal{M}_t^{-1}J_t^{-1}(\mathcal{D}_{v_i}\mathcal{D}_v\Phi_t)]_{ij}(\mathcal{M}_t^{-1})_{jk}\Big]\Big| \\
&\le \|\psi\|_\infty\mathbb{E}\Big[\Big|\langle\mathcal{M}_t^{-1}h_k,\sum_{i=1}^{M}h_i^T\mathcal{M}_t^{-1}J_t^{-1}(\mathcal{D}_{v_i}\mathcal{D}_v\Phi_t^L)\rangle_{\mathcal{W}}\Big|\Big] \\
&\le \|\psi\|_\infty\mathbb{E}\Big[1/\lambda_{min}\sum_{i,j=1}^{M}|h_i^T\mathcal{M}_t^{-1}(J_t^{-1}\mathcal{D}_{v_j}\mathcal{D}_{v_i}\Phi_t^L)|\Big] \\
&\le \|\psi\|_\infty\mathbb{E}\Big[1/\lambda_{min}\sum_{i,j=1}^{M}1/\lambda_{min}|J_t^{-1}\mathcal{D}_{v_j}\mathcal{D}_{v_i}\Phi_t^L|_{\mathcal{W}}\Big] \\
&\le C_3\mathrm{e}^{C_3t}\|\psi\|_\infty(1+|x|_{\widetilde{\mathcal{W}}}),
\end{aligned}
$$

where $t\ge t_0$ and $C_i=C_i(t_0,\alpha_0,\rho)$ $(i=2,3)$. Combining the above three estimates with (4.17) and noticing $J_t=D_L\Phi_t^L$, we obtain

$$|\mathbb{E}\Big(D_L\psi(\Phi_t(x))D_L\Phi_t^L(x)h_k\Big)|\le C_4\mathrm{e}^{C_4t}\|\psi\|_\infty(1+|x|_{\widetilde{\mathcal{W}}})$$

with $t\ge t_0$ and $C_4=C_4(t_0,\alpha_0,\rho)$. Since the above argument is in the frame of $\mathcal{W}^L$ with the orthonormal base $\{h_k;1\le k\le M\}$, we have

$$|\mathbb{E}\Big(D_L\psi(\Phi_t(x))D_h\Phi_t^L(x)\Big)|\le C_4\mathrm{e}^{C_4t}\|\psi\|_\infty(1+|x|_{\widetilde{\mathcal{W}}})|h|_{\mathcal{W}},$$

for every $h\in\mathcal{W}^L$ and $t\ge t_0$. □

4.2. **Hörmander's systems.** This is an auxiliary subsection for the proof of Lemma 4.3 given in the next subsection and we use the notations detailed in Section A.1 (in particular Subsection A.1.1). Let us consider the SPDE for u^L in Stratanovich form as

(4.19)

$$du^L+[Au^L+B_L(u,u)\chi(\frac{|u|_{\mathcal{W}}}{3\rho})-\frac{1}{2}\sum_{\substack{k\in Z_L(N_0),\\ i=1,2}}D_{q_k(u)e_k^i}q_k(u)e_k^i]\,dt=\sum_{\substack{k\in Z_L(N_0),\\ i=1,2}}q_k(u)\circ dw_k(t)e_k$$

where $q_k(u)=(1-\chi(\frac{|u|_{\mathcal{W}}}{\rho}))q_k$ for $k\in Z_L(N_0)$ and $q_k(u)=q_k$ for $k\in Z_L(N)\setminus Z_L(N_0)$. For any $x\in\mathcal{W}$, it is clear that if $k\in Z_L(N_0)$ and $i=1,2$,

$$D_{q_k(x)e_k^i}q_k(x)e_k^i=-\frac{1}{\rho}\chi'(\frac{|x|_{\mathcal{W}}}{\rho})\big(1-\chi(\frac{|x|_{\mathcal{W}}}{\rho})\big)\frac{\langle x,e_k^i\rangle_{\mathcal{W}}}{|x|_{\mathcal{W}}}.$$

For any two Banach spaces E_1 and E_2, denote by $P(E_1,E_2)$ the set of all C^∞ functions $E_1\to E_2$ with all orders derivatives being polynomially bounded. If $K\in P(H,H^L)$ and $X\in P(H,H)$, define $[X,K]_L$ by

$$[X,K]_L(x)=DK(x)X(x)-D_LX^L(x)K(x),\qquad x\in H.$$

For instance, $[A, K]_L \in P(D(A), H^L)$ with $[A, K]_L(x) = DK(x)Ax - A_L K(x)$. Define

$$X^0(x) = Ax + \chi\Big(\frac{|x|_{\mathcal{W}}}{3\rho}\Big)\mathrm{e}^{-\delta A_H} B(x,x) + \frac{1}{2\rho}\sum_{k\in Z_L(N_0), i=1,2} \chi'\Big(\frac{|x|_{\mathcal{W}}}{\rho}\Big)\Big(1-\chi\Big(\frac{|x|_{\mathcal{W}}}{\rho}\Big)\Big)\frac{\langle x, e_k^i\rangle_{\mathcal{W}}}{|x|_{\mathcal{W}}} e_k^i$$

The brackets $[X^0, K]_L$ and $[A, K]_L$ will appear when *applying the Itô formula on* $J_t^{-1} q_k^i(\Phi_t)$ (see (4.26)) in the proof of Lemma 4.3.

Definition 4.4. The Hörmander's system $\mathbf{K}$ for equation (4.19) is defined as follows: given any $y \in \mathcal{W}$, define

$$\begin{aligned}
\mathbf{K}_0(y) &= \{q_k(y)e_k^i \ : \ k \in Z_L(N),\ i = 1, 2\}\\
\mathbf{K}_1(y) &= \{[X^0(y), q_k(y)e_k^i]_L \ : \ k \in Z_L(N),\ i = 1, 2\}\\
\mathbf{K}_2(y) &= \{[q_k(y)e_k^i, K(y)]_L \ : \ K \in \mathbf{K}_1(y),\ k \in Z_L(N),\ i = 1, 2\}
\end{aligned}$$

and $\mathbf{K}(y) = \mathbf{K}_0(y) \cup \mathbf{K}_1(y) \cup \mathbf{K}_2(y)$.

Proposition 4.5. *There exist* $\overline{\rho} > 0$ *and* $\overline{N} \geq N_0$ *(which depend only on* N_0 *and* Q*) such that if* $\rho \geq \overline{\rho}$ *and* $N \geq \overline{N}$*, then the following property holds: for every* $x \in \mathcal{W}$ *and* $h \in H^L$ *there exist* $\sigma > 0$ *and* $R > 0$ *such that*

$$\inf_{\delta>0} \sup_{K\in\mathbf{K}} \inf_{|y-x|_{\mathcal{W}}\leq R} |\langle K(y), h\rangle_{\mathcal{W}}| \geq \sigma |h|_{\mathcal{W}}. \tag{4.20}$$

Proof. We are going to show that there are $\sigma > 0$ and $R > 0$ (independent of δ) such that for every $x \in \mathcal{W}$ and $h \in \mathcal{W}^L$,

$$\sup_{K\in\mathbf{K}} \inf_{|x-y|_{\mathcal{W}}\leq R} |\langle K(y), h\rangle_{\mathcal{W}}| \geq \sigma |h|_{\mathcal{W}}.$$

To this end, it is sufficient to show that there is a (finite) set $\widetilde{\mathbf{K}} \subset \mathbf{K}(y)$ for every y, such that $\mathrm{span}(\widetilde{\mathbf{K}}) = H^L$. We choose $R \leq \frac{1}{4}\rho$.

Case 1: $|x|_{\mathcal{W}} \geq R + 2\rho$. Hence $|y|_{\mathcal{W}} \geq 2\rho$ for every y such that $|x - y|_{\mathcal{W}} \leq R$ and $q_k(y) = q_k$ for all k. So we can take $\widetilde{\mathbf{K}} = \mathbf{K}_0$ which spans the whole H^L thanks to (A.2).

Case 2: $|x| \leq \rho - R$. Hence $|y|_{\mathcal{W}} \leq \rho$ for every y such that $|x - y|_{\mathcal{W}} \leq R$ and $q_k(y) = 0$ for all $k \in Z_L(N_0)$. In particular, $X^0(y) = Ay + \mathrm{e}^{-\delta A_H}B(y, y)$ and so for l, $m \in Z_L(N) \setminus Z_L(N_0)$ and i, $j = 1, 2$ (cfr. Subsection A.1.2),

$$[q_l e_l^i, [X^0, q_m e_m^j]_L]_L = \pi_N B(q_l e_l^i, q_m e_m^j) + \pi_N B(q_m e_m^j, q_l e_l^i)$$

(which are independent of δ, thus providing the uniformity in δ we need). The proof that the vectors $[q_l e_l^i, [X^0, q_m e_m^j]_L]_L$, where l, m run over $Z_L(N)\setminus Z_L(N_0)$ and i, $j = 1, 2$, span H^L follows exactly as in [21] (using (A.3)-(A.4), since the only difference is that here we use the Fourier basis (A.1) rather than the complex exponentials). Hence, thanks to Lemma 4.2 of [21], it is sufficient to choose $N \geq N_0$ large enough so that for every $k \in Z_L(N_0)$ there are l, $m \in Z_L(N) \setminus Z_L(N_0)$ such that $|l| \neq |m|$, l and m are linearly independent and $k = l + m$ (or $k = l - m$). Take $\widetilde{\mathbf{K}} = \mathbf{K}_0 \cup \mathbf{K}_2$.

Case 3: $\rho - R \le |x|_{\mathcal{W}} \le 2\rho + R$, hence $|x|_{\mathcal{W}} \le 3\rho$ and $|y|_{\mathcal{W}} \ge \frac{1}{2}\rho$ for all y such that $|x-y|_{\mathcal{W}} \le R$. Write $X^0(y) = X^{01}(y) + X^{02}(y)$ where $X^{01}(y) = Ay + \mathrm{e}^{-\delta A_H} B(y,y)$ and

$$X^{02}(y) = \frac{1}{2\rho} \sum_{k \in Z_L(N_0), i=1,2} \chi'\big(\frac{|y|_{\mathcal{W}}}{\rho}\big)\big(1 - \chi\big(\frac{|y|_{\mathcal{W}}}{\rho}\big)\big) \frac{\langle y, e_k^1 \rangle_{\mathcal{W}}}{|y|_{\mathcal{W}}} e_k^1.$$

Choose l, $m \in Z_L(N) \setminus Z_L(N_0)$ and i, $j \in \{1,2\}$, then

$$[q_l e_l^i, [X^0(y), q_m e_m^j]_L]_L = [q_l e_l^i, [X^{01}(y), q_m e_m^j]_L]_L + [q_l e_l^i, [X^{02}(y), q_m e_m^j]_L]_L.$$

As in the previous case the vectors $[q_l e_l^i, [X^{01}(y), q_m e_m^j]_L]_L$ span the whole H^L, so, to conclude the proof we show that the other term is a small perturbation. Indeed, $[q_l e_l^i, [X^{02}(y), q_m e_m^j]_L]_L$ corresponds to a derivative of X^{02} in the directions $q_l e_l^i$ and $q_m e_m^j$ and it is easy to see by some straightforward computations that there is $c > 0$, depending only on N, χ and Q (but not on ρ, y, δ) such that $|[q_l e_l^i, [X^{02}(y), q_m e_m^j]_L]_L| \le \frac{c}{\rho^3}$. So, for ρ large enough, the vectors $[q_l e_l^i, [X^0(y), q_m e_m^j]_L]_L$ span H^L. Take $\widetilde{\mathbf{K}} = \mathbf{K}_0 \cup \mathbf{K}_2$. □

4.3. **Proof of Lemma 4.3.** The key points for the proof are Proposition 4.5 and the following Norris' Lemma (Lemma 4.1 of [19]).

Lemma 4.6 (Norris' Lemma)**.** *Let $a, y \in \mathbb{R}$. Let $\beta_t, \gamma_t = (\gamma_t^1, \dots \gamma_t^m)$ and $u_t = (u_t^1, \dots, u_t^m)$ be adapted processes. Let*

$$a_t = a + \int_0^t \beta_s\, ds + \int_0^t \gamma_s^i dw_s^i, \qquad Y_t = y + \int_0^t a_s\, ds + \int_0^t u_s^i dw_s^i,$$

where $(w_t^1, \dots, w_t^m)$ are i.i.d. standard Brownian motions. Suppose that $T < t_0$ is a bounded stopping time such that for some constant $C < \infty$:

$$|\beta_t|, |\gamma_t|, |a_t|, |u_t| \le C \qquad \text{for all } t \le T.$$

Then for any $r > 8$ and $\nu > \frac{r-8}{9}$ there is $C = C(T, q, \nu)$ such that

$$P\Big[\int_0^T Y_t^2\, dt < \epsilon^r, \int_0^T (|a_t|^2 + |u_t|^2)\, dt \ge \epsilon\Big] < C \mathrm{e}^{-\frac{1}{\epsilon^\nu}}.$$

Proof of Lemma 4.3. We follow the lines of the proof of Theorem 4.2 of [19]. Denote $\mathcal{S}^L = \{\eta \in \mathcal{W}^L; |\eta|_{\mathcal{W}^L} = 1\}$. It is sufficient to show that

$$P\Big(\inf_{\eta \in \mathcal{S}^L} \langle \mathcal{M}_t \eta, \eta \rangle_{\mathcal{W}} \le \epsilon\Big) = o(\epsilon^p)(1 + |x|_{\widetilde{\mathcal{W}}})^p,$$

for all $p > 0$, uniformly in $\delta > 0$ and for $t \ge t_0$ ($t_0 > 0$ possibly being arbitrarily small). From (4.8), one only needs to prove

$$P\Big[\inf_{\eta \in \mathcal{S}^L} \sum_{k \in Z_L(N), i=1,2} \frac{1}{|k|^{4\alpha_0 + 1}} \int_0^t |\langle J_s^{-1} q_k^i(\Phi_s), \eta \rangle_{\mathcal{W}}|^2\, ds \le \epsilon\Big] = o(\epsilon^p)(1 + |x|_{\widetilde{\mathcal{W}}})^p \tag{4.21}$$

for all $p>0$, where, for simplicity, we have set $q_k^i(\Phi_s)=q_k(\Phi_s)x_k^i e_k$ and $q_k(\Phi_s)=q_k(1-\chi(\frac{|\Phi_s|_{\mathcal{W}}}{\rho}))$ for $k\in Z_L(N_0)$ and $q_k(\Phi_s)=q_k$ for $k\in Z_L(N)\setminus Z_L(N_0)$.

Formula (4.21) is implied by
(4.22)

$$D_\theta \sup_j \sup_{\eta\in\mathcal{N}_j} P\Big[\int_0^t \sum_{k\in Z_L(N),i=1,2}\frac{1}{|k|^{4\alpha_0+1}}|\langle J_s^{-1}q_k^i(\Phi_s),\eta\rangle_{\mathcal{W}}|^2\,ds\le\epsilon\Big]=o(\epsilon^p)(1+|x|_{\widetilde{\mathcal{W}}})^p,$$

for all $p>0$, where $\{\mathcal{N}_j\}_j$ is a finite sequence of disks of radius θ covering $\mathcal{S}^L$, $D_\theta=\#\{\mathcal{N}_j\}$ and θ is small enough. Define a stopping time τ by

$$\tau=\inf\{s>0\ :\ |\Phi_s(x)-x|_{\mathcal{W}}>R,\ |J_s^{-1}-Id|_{\mathcal{L}(\mathcal{W})}>\frac{1}{4}\}.$$

where $R>0$ is the same as in (4.20). It is easy to see that (4.22) holds as long as one proves that there exists some k and i such that for all $p>0$

$$\sup_{\eta\in\mathcal{S}^L} P\Big(\int_0^{t\wedge\tau}|\langle J_s^{-1}q_k^i(\Phi_s),\eta\rangle_{\mathcal{W}}|^2\,ds\le\epsilon\Big)=o(\epsilon^p)(1+|x|_{\widetilde{\mathcal{W}}})^p \tag{4.23}$$

(uniformly for $\delta>0$ and for $t\ge t_0$).

The key estimate of the proof is to bound $P(\tau\le\epsilon)$. By (3.6) and since $|\mathrm{e}^{-At}x-x|_{\mathcal{W}}\le Ct^{1/8}|x|_{\widetilde{\mathcal{W}}}$, we have for any $p\ge 2$

$$\mathbb{E}[\sup_{0\le t\le T}|\Phi_t-x|_{\mathcal{W}}^p]\le C_1(1+|x|_{\widetilde{\mathcal{W}}})^p(T^{p/8}\vee T^{p/2}) \tag{4.24}$$

where $C_1=C_1(\alpha_0,p,\rho)$. Combining (4.24) and (4.11), we have $P(\tau\le\epsilon)=o(\epsilon^p)(1+|x|_{\widetilde{\mathcal{W}}})^p$ for all $p>0$, independent of δ. Let us now prove (4.23) in the following two cases.

Case 1: $|x|_{\mathcal{W}}>2\rho+R$. Recalling the *case 1* in the proof of Proposition 4.5, one has

$$\sup_{K\in\mathbf{K}_0}\inf_{|y-x|_{\mathcal{W}}\le R}|\langle K(y),h^L\rangle_{\mathcal{W}}|\ge\sigma|h^L|_{\mathcal{W}},\qquad\text{independent of }\delta.$$

As $s\le\tau\wedge t$, every $q_k(\Phi_s)=q_k$ is constant and non-degenerate on $\mathrm{span}(e_k^1,e_k^2)$, hence for any $\eta\in\mathcal{S}^L$ there exist $k\in Z_L(N)$ and $i\in\{1,2\}$ such that $|\langle q_k^i,\eta\rangle_{\mathcal{W}}|\ge\frac{\sigma}{2}$ and

$$\begin{aligned}P\Big(\int_0^{t\wedge\tau}|\langle J_s^{-1}q_k^i(\Phi_s),\eta\rangle_{\mathcal{W}}|^2\,ds\le\epsilon\Big)&=P\Big(\int_0^{t\wedge\tau}|\langle J_s^{-1}q_k^i,\eta\rangle_{\mathcal{W}}|^2\,ds\le\epsilon\Big)\\&\le P\Big(\tau\le\frac{C}{\sigma^2}\epsilon\Big)=o(\epsilon^p)(1+|x|_{\widetilde{\mathcal{W}}})^p.\end{aligned}$$

Case 2: $|x|_{\mathcal{W}}\le 2\rho+R$. *Assume* (4.23) *not true*, i.e. for any $C>0$, there exists some constants $p_0>0$ and four sequences $\{\eta_n\}_n\subset\mathcal{S}^L$, $\{\delta_n\}_n$, $\{\epsilon_n\}_n$ with $\lim_{n\to\infty}\epsilon_n=0$, and $\{t_n\}_n$ with $t_n\ge t_0$, such that for all k, i,

$$P\Big(\int_0^{t_n\wedge\tau}|\langle J_s^{-1}q_k^i(\Phi_s),\eta_n\rangle_{\mathcal{W}}|^2\,ds\le\epsilon_n\Big)\ge C\epsilon_n^{p_0}(1+|x|_{\widetilde{\mathcal{W}}})^{p_0}. \tag{4.25}$$

Applying Itô formula on $J_t^{-1}q_k^i(\Phi_t)$, we have

$$\begin{aligned} dJ_t^{-1}q_k^i(\Phi_t) = J_t^{-1}\Big(-[X^0,q_k^i]_L(\Phi_t)+\frac{1}{2}\sum_{l\in Z_L(N),j=1,2}[q_l^j,[q_l^j,q_k^i]_L]_L(\Phi_t)\Big)\,dt\,+ \\ +\,J_t^{-1}\sum_{l\in Z_L(N),j=1,2}[q_l^j,q_k^i]_L(\Phi_t)dw_l^j(t). \end{aligned} \tag{4.26}$$

Setting

$$\begin{aligned} &Y(t)=\langle J_t^{-1}q_k^i(\Phi_t),\eta_n\rangle_{\mathcal{W}}, \qquad u_l^j(t)=\langle J_t^{-1}[q_l^j,q_k^i]_L(\Phi_t),\eta_n\rangle_{\mathcal{W}}, \\ &a(t)=\langle -J_t^{-1}[X^0,q_k^i]_L(\Phi_t)+\frac{1}{2}J_t^{-1}\sum_{l\in Z_L(N)}\sum_{j=1}^2[q_l^j,[q_l^j,q_k^i]_L]_L(\Phi_t),\eta_n\rangle_{\mathcal{W}}, \end{aligned}$$

applying Norris' Lemma and noticing the assumption (4.25), one has

$$P\Big(\int_0^{t_n\wedge\tau}|Y(s)|^2\,ds\le\epsilon_n,\quad \int_0^{t_n\wedge\tau}|a(s)|^2+\sum_{j,l}|u_l^j(s)|^2\,ds\ge\epsilon_n^{1/r}\Big)=o(\epsilon_n^p)$$

for all $p>0$, and

$$P\Big(\int_0^{t_n\wedge\tau}|Y(s)|^2\,ds\le\epsilon_n\Big)\ge C\epsilon_n^{p_0}(1+|x|_{\widetilde{\mathcal{W}}})^{p_0},$$

they imply that for all j, l,

$$P\Big(\int_0^{t_n\wedge\tau}|Y(s)|^2\,ds\le\epsilon_n,\quad \int_0^{t_n\wedge\tau}|a(s)|^2d\,s\le\epsilon_n^{1/r}\Big)\ge C\epsilon_n^{p_0}(1+|x|_{\widetilde{\mathcal{W}}})^{p_0} \tag{4.27}$$

$$P\Big(\int_0^{t_n\wedge\tau}|Y(s)|^2\,ds\le\epsilon_n,\quad \int_0^{t_n\wedge\tau}|u_l^j(s)|^2\,ds\ge\epsilon_n^{1/r}\Big)=o(\epsilon_n^p), \tag{4.28}$$

$$P\Big(\int_0^{t_n\wedge\tau}|u_l^j(s)|^2\,ds\le\epsilon_n^{1/r}\Big)\ge C\epsilon_n^{p_0}(1+|x|_{\widetilde{\mathcal{W}}})^{p_0}. \tag{4.29}$$

We apply Norris' lemma on $J_t^{-1}[q_l^j,q_k^i]$ and obtain for any j,l and all $p>0$,

$$P\Big(\int_0^{t_n\wedge\tau}|u_l^j(s)|^2\,ds\le\epsilon_n^{1/r},\quad \int_0^{t_n\wedge\tau}|\langle J_s^{-1}[q_l^j,[q_l^j,q_k^i]_L]_L(\Phi_s),\eta_n\rangle_{\mathcal{W}}|^2\,ds\ge\epsilon_n^{1/r^2}\Big)=o(\epsilon_n^p),$$

which, combining with (4.28), implies

$$P\Big(\int_0^{t_n\wedge\tau}|Y(s)|^2\,ds\le\epsilon_n,\ \int_0^{t_n\wedge\tau}|\langle J_s^{-1}[q_l^j,[q_l^j,q_k^i]_L]_L(\Phi_s),\eta_n\rangle_{\mathcal{W}}|^2\,ds\ge\epsilon_n^{1/r^2}\Big)=o(\epsilon_n^p). \tag{4.30}$$

Hence, combining (4.27) and (4.30), one has

$$P\Big(\int_0^{t_n\wedge\tau} |Y(s)|^2\,ds \le \epsilon_n,\ \int_0^{t_n\wedge\tau} |a(s)|^2\,ds \le \epsilon_n^{1/r},$$
$$\int_0^{t_n\wedge\tau} |\langle J_s^{-1}[q_l^j,[q_l^j,q_k^i]_L]_L(\Phi_s),\eta_n\rangle_{\mathcal{W}}|^2\,ds \le \epsilon_n^{1/r^2}\Big) \ge C\epsilon_n^{p_0}(1+|x|_{\widetilde{\mathcal{W}}})^{p_0}.$$

So, for all $k\in Z_L(N)$ and $i=1,2$,

$$P\Big(\int_0^{t_n\wedge\tau} |\langle J_s^{-1}[X^0,q_k^i]_L(\Phi_s),\eta_n\rangle_{\mathcal{W}}|^2\,ds \le 2\epsilon_n^{1/r}\Big) \ge C\epsilon_n^{p_0}(1+|x|_{\widetilde{\mathcal{W}}})^{p_0}.$$

Apply Norris' Lemma on $\langle J_s^{-1}[X^0,q_k^i]_L(\Phi_s),\eta_n\rangle_{\mathcal{W}}$, we have an inequality similar to (4.29) as follows

$$P\Big(\int_0^{t_n\wedge\tau} |\langle J_s^{-1}[q_{k_1}^{i_1},[X^0,q_k^i]_L]_L(\Phi_s),\eta_n\rangle_{\mathcal{W}}|^2\,ds \le 2\epsilon_n^{1/r^3}\Big) \ge C\epsilon_n^{p_0}(1+|x|_{\widetilde{\mathcal{W}}})^{p_0} \tag{4.31}$$

for any $i,i_1\in\{1,2\}$ and $k,k_1\in Z_L(N)$. Using the set $\widetilde{\mathbf{K}}$ in the latter two cases of the proof of Proposition 4.5, for any $\eta_n\in\mathcal{S}^L$, there exists some $\ell\in\widetilde{\mathbf{K}}$ such that

$$\sup_{0\le t\le t_n\wedge\tau} |\langle J_t^{-1}\ell(\Phi_t),\eta_n\rangle_{\mathcal{W}}| \ge \frac{\sigma}{2}.$$

We have by the same argument as in case 1

$$P\Big(\int_0^{t_n\wedge\tau} |\langle J_s^{-1}\ell(\Phi_s),\eta_n\rangle_{\mathcal{W}}|^2\,ds \le \epsilon_n\Big) = o(\epsilon_n^p)(1+|x|_{\widetilde{\mathcal{W}}})^p \tag{4.32}$$

for all $p>0$. If $\ell=[q_{k_1}^{i_1},[X^0,q_k^i]_L]_L$, then (4.32) contradicts (4.31). If $\ell=q_k^i$, then (4.32) contradicts (4.25) □

5. Controllability and support

The following proposition describes the support of the distribution associated to a Markov solution.

Proposition 5.1. *Let $(P_x)_{x\in H}$ be a Markov solution. For every $x\in\mathcal{W}$ and $T>0$, the following properties hold,*

- $P_x[\xi_T\in\mathcal{W}]=1$,
- *for every $\mathcal{W}$-open set $U\subset\mathcal{W}$, $P_x[\xi_T\in U]>0$.*

The proof of the above proposition relies on the following control problem (see [25] for a general result on the same lines).

Lemma 5.2. *Given any $T>0$, $x,y\in\mathcal{W}$ and $\epsilon>0$, there exist $\rho_0=\rho_0(|x|_{\mathcal{W}},|y|_{\mathcal{W}},T)$, u and w such that*

- $w\in L^2([0,T];H)$ *and* $u\in C([0,T];\mathcal{W})$,
- $u(0)=x$ *and* $|u(T)-y|_{\mathcal{W}}\le\epsilon$,
- $\sup_{t\in[0,T]}|u(t)|_{\mathcal{W}}\le\rho_0$,

and u, w solve the following problem,

$$\partial_t u + Au + B(u,u) = Qw, \tag{5.1}$$

where Q is defined in Assumption 2.1.

Proof. Let $z \in D(A^{\alpha_0+7/4})$ such that $|y-z|_{\mathcal{W}} \le \frac{\epsilon}{2}$, it suffices to show that there exist u, w satisfying the conditions of the lemma and

$$|u(T) - z|_{\mathcal{W}} \le \frac{\epsilon}{2}. \tag{5.2}$$

Decompose $u = u^H + u^L$ where $u^H = (I - \pi_{N_0})u$ and $u^L = \pi_{N_0} u$ and N_0 is the number in Assumption 2.1, then equation (5.1) can be written as

$$\partial_t u^L + Au^L + B_L(u,u) = 0, \tag{5.3}$$

$$\partial_t u^H + Au^H + B_H(u,u) = Qw. \tag{5.4}$$

We split $[0,T]$ into the pieces $[0,T_1]$, $[T_1,T_2]$, $[T_2,T_3]$ and $[T_3,T]$, with the times T_1, T_2, T_3 to be chosen along the proof, and prove that (5.2) holds in the following four steps, provided ρ_0 is chosen large enough (depending on $|x|_{\mathcal{W}}$, $|y|_{\mathcal{W}}$ and T).

Step 1: regularization of the initial condition. Set $w \equiv 0$ in $[0,T_1]$, using (A.5), one obtains

$$\frac{d}{dt}|u|^2_{\mathcal{W}} + 2|A^{\frac{1}{2}}u|^2_{\mathcal{W}} \le 2|\langle A^{\frac{3}{4}+\alpha_0}u, A^{\alpha_0-\frac{1}{4}}B(u,u)\rangle_H| \le |A^{\frac{1}{2}}u|^2_{\mathcal{W}} + c|u|^4_{\mathcal{W}}. \tag{5.5}$$

It is easy to see, by solving a differential inequality, that $|u(t)|^2_{\mathcal{W}} + \int_0^t |A^{1/2}u|^2_{\mathcal{W}}\,ds \le 2|x|^2_{\mathcal{W}}$ for $t \le t_0 := (2c|x|^2_{\mathcal{W}})^{-1}$. In particular $u(t) \in D(A^{\alpha_0+3/4})$ for a. e. $t \in [0,t_0]$. An energy estimate similar to the one above, this time in $D(A^{\alpha_0+3/4})$ and with initial condition $u(t_0/2)$ (w.l.o.g. assume $u(t_0/2) \in D(A^{\alpha_0+3/4})$), implies that $u(t) \in D(A^{\alpha_0+5/4})$ a. e. for $t \in [t_0/2, t_0]$. By repeating the argument, we can finally find a time $T_1 \le \frac{T}{4} \wedge t_0$ such that $u(T_1) \in D(A^{\alpha_0+7/4})$.

Step 2: high modes led to zero. Choose a smooth function ψ on $[T_1,T_2]$ such that $0 \le \psi \le 1$, $\psi(T_1) = 1$ and $\psi(T_2) = 0$, and set $u^H(t) = \psi(t)u^H(T_1)$ for $t \in [T_1,T_2]$. An estimate similar to (5.5) yields

$$\frac{d}{dt}|u^L|^2_{\mathcal{W}} + |A^{\frac{1}{2}}u^L|^2_{\mathcal{W}} \le c(|u^L|^2_{\mathcal{W}} + |u^H|^2_{\mathcal{W}})^2,$$

and $|u(t)|^2_{\mathcal{W}} \le |u^L(t)|^2_{\mathcal{W}} + |u^H(T_1)|^2_{\mathcal{W}} \le 4|x|^2_{\mathcal{W}}$ for $T_1 \le t \le T_2 := \frac{T}{2} \wedge (T_1 + (4c|x|^2_{\mathcal{W}})^{-1})$. Plug u^L in (5.4), take

$$w(t) = \psi'(t)Q^{-1}u^H(T_1) + \psi(t)Q^{-1}Au^H(T_1) + Q^{-1}B_H(u(t),u(t)).$$

By the previous step $u(T_1) \in D(A^{\alpha_0+7/4})$, $|Q^{-1}Au^H(T_1)| < \infty$; by (A.5), $|Q^{-1}B_H(u(t),u(t))| \le c|Au(t)|^2_{\mathcal{W}} \le 2cN_0^4(|Au^H(T_1)|^2_{\mathcal{W}} + |u^L(t)|^2_{\mathcal{W}})$ for $t \in [T_1,T_2]$. Hence, $w \in L^2([T_1,T_2],H)$.

Step 3: low modes close to z. Let $u^L(t)$ be the linear interpolation between $u^L(T_2)$ and z^L for $t \in [T_2,T_3]$. Write $u(t) = \sum u_k(t)e_k$, then (5.3) in Fourier coordinates is given by

$$\dot{u}_k + |k|^2 u_k + B_k(u,u) = 0, \qquad k \in Z_L(N_0), \tag{5.6}$$

where $B_k(u,u) = B_k(u^L,u^L) + B_k(u^L,u^H) + B_k(u^H,u^L) + B_k(u^H,u^H)$. Let us choose a suitable u^H to simplify the above $B_k(u,u)$. To this end, consider the set $\{(l_k,m_k) : k \in Z_L(N_0)\}$ such that

(1) If $k \in Z_L(N_0)_+$, then $l_k, -m_k \in Z_H(N_0)_+$ and $l_k + m_k = k$.
(2) If $k \in Z_L(N_0)_-$, then $l_k, m_k \in Z_H(N_0)_+$ and $l_k - m_k = k$.
(3) $|l_k| \neq |m_k|$ and $l_k \not\parallel m_k$ for all $k \in Z_L(N_0)$.
(4) For every $k \in Z_L(N_0)$, $|l_k|, |m_k| \geq 2^{(2N_0+1)^3}$.
(5) If $k_1 \neq k_2$, then $|l_{k_1} \pm l_{k_2}|, |m_{k_1} \pm m_{k_2}|, |l_{k_1} \pm m_{k_2}|, |m_{k_1} \pm l_{k_2}| \geq 2^{(2N_0+1)^3}$.

Define

$$u^H(t) = \sum_{k\in Z_L(N_0)} u_{l_k}(t)e_{l_k} + u_{m_k}(t)e_{m_k},$$

with $u_{l_k}(t)$ and $u_{m_k}(t)$ to be determined by equation (5.7) below. Using the formulas (A.3)-(A.4) in Section A.1.2, it is easy to see that

- by (4), $B_k(u^L,u^H) = B_k(u^H,u^L) = 0$,
- by (5), $B_k(u_{l_{k_1}},u_{l_{k_2}}) = B_k(u_{l_{k_1}},u_{m_{k_2}}) = B_k(u_{m_{k_1}},u_{l_{k_2}}) = B_k(u_{m_{k_1}},u_{m_{k_2}}) = 0$.

Hence, using again the computations of Section A.1.2, equation (5.6) is simplified to the following equation

$$\begin{cases} (m_k \cdot X)\mathcal{P}_k Y \pm (l_k \cdot Y)\mathcal{P}_k X + 2G_k(t) = 0, \\ X \cdot l_k = 0, \quad Y \cdot m_k = 0, \quad l_k \pm m_k = k, \end{cases} \tag{5.7}$$

for each $k \in Z_L(N_0)_\pm$, where $G_k = \dot{u}_k + |k|^2 u_k + B_k(u^L,u^L)$ is a polynomial in t and clearly $G_k \cdot k = 0$. In order to see that the above equation has a solution, consider for instance the case $k \in Z_L(N_0)_+$. Let $\{\vec{k}, g_1, g_2\}$ be an orthonormal basis of $\mathbf{R}^3$ such that l_k, $m_k \in \mathrm{span}(\vec{k}, g_1)$, and $\vec{k} = \frac{k}{|k|}$. Let $X = x_0\vec{k} + x_1 g_1 + x_2 g_2$ and $Y = y_0\vec{k} + y_1 g_1 + y_2 g_2$. A simple computation yields

$$(X\cdot m_k)(\mathcal{P}_k Y) + (Y \cdot l_k)(\mathcal{P}_k X) = |k|(x_0 y_2 + x_2 y_0)g_2 - |k| c_k x_0 y_0 g_1,$$

where $c_k = \frac{|l_k|^2 - |m_k|^2}{\sqrt{|l_k|^2|m_k|^2 - (l_k\cdot m_k)^2}}$. One can for instance set $x_0 = 1$, $x_2 = 1$ and solve the problem in the unknown y_0, y_2 (notice that x_1, y_1 can be determined by the divergence free constraint).

In conclusion the solution $u^H(t)$ is smooth in t and by this construction the dynamics $u = u^L + u^H$ is finite dimensional. Hence $u(t)$ is smooth in space and time for $t \in [T_2,T_3]$ and $\sup |u(t)|_{\mathcal{W}}$ can be bounded only in terms of $|u^L(T_2)$, z^L and $T_3 - T_2$. We finally set $w = Q^{-1}[\dot{u}^H + Au^H + B_H(u,u)]$.

Step 4: high modes close to z. In the interval $[T_3,T]$ we choose u^H as the linear interpolation between $u^H(T_3)$ and z^H. Let u^L be the solution to equation (5.3) on $[T_3,T]$ with the choice of u^H given above. Since $u(T_3) \in D(A^{\alpha_0+7/4})$ and $u^L(T_3) = z^L$ from step 3, by the continuity of the dynamics, $\sup_{T_3\leq t\leq T} |u^L(t) - z^L|_{\mathcal{W}} \leq \frac{\epsilon}{2}$ if $T - T_3$ is small enough (recall that we can choose an arbitrary $T_3 \in (T_2,T)$ in the third step). Thus (5.2) holds and, as in the second step, we can find $w \in L^2([T_3,T],H)$ solving (5.4). It is clear from the above construction that $\sup_{T_3\le t\le T}|u(t)|_{\mathcal{W}} \le C|z|_{\mathcal{W}} + C|u(T_3)|_{\mathcal{W}}$. □

Proof of Proposition 5.1. The first property follows from Theorem 6.3 of [14] (which only uses strong Feller). For the second property, fix $x \in \mathcal{W}$ and $T > 0$, then it is sufficient to show that for every $y \in \mathcal{W}$ and $\epsilon > 0$, $P_x[|\xi_T - y|_{\mathcal{W}} \le \epsilon] > 0$. Consider $\rho > \rho_0$ (where ρ_0 is the constant provided by Lemma 5.2), then by Theorem 2.3,

$$P_x[|\xi_T - y|_{\mathcal{W}} \le \epsilon] \ge P_x[|\xi_T - y|_{\mathcal{W}} \le \epsilon,\ \tau_\rho > T] = P_x^\rho[|\xi_T - y|_{\mathcal{W}} \le \epsilon,\ \tau_\rho > T].$$

By Lemma 5.2 there exist $\overline{\eta}$ and $\overline{u}$ such that $\overline{u}$ is the solution to the control problem (5.1) connecting x at 0 with y at T corresponding to the control $\partial_t\overline{\eta}$. Choose $s \in (0,\frac{1}{2})$, $p > 1$ and $\beta > \frac{3}{4}$ such that $s - \frac{1}{p} > 0$ and $\beta + \frac{1}{p} - s < \frac{1}{2}$, then by Lemma $C.3$ of [14] (which does not rely on non-degeneracy of the covariance), there is $\delta > 0$ such that for all η in the δ-ball $B_\delta(\overline{\eta})$ centred at $\overline{\eta}$ in $W^{s,p}([0,T];D(A_H^{-\beta}))$, we have that $|u(T,\eta) - y|_{\mathcal{W}} \le \epsilon$ and $\sup_{[0,T]}|u(t,\eta)|_{\mathcal{W}} \le \rho_0$, where $u(\cdot,\eta)$ is the solution to the control problem with control $\partial_t\eta$. By proceeding as in the proof of Proposition 6.1 of [14], it follows that in conclusion the probability $P_x^\rho[|\xi_T - y|_{\mathcal{W}} \le \epsilon,\ \tau_\rho > T]$ is bounded from below by the (positive) measure of $B_\delta(\overline{\eta})$ with respect to the Wiener measure corresponding to the cylindrical Wiener process on H. □

Appendix A. Appendix

A.1. **Details on the geometry of modes.** Here we reformulate the problem in Fourier coordinates and explain in full details the conditions of Assumption 2.1.

Define $\mathbf{Z}^3_* = \mathbf{Z}^3 \setminus \{(0,0,0)\}$, $\mathbf{Z}^3_+ = \{k \in \mathbf{Z}^3 : k_1 > 0\} \cup \{k \in \mathbf{Z}^3 : k_1 = 0, k_2 > 0\} \cup \{k \in \mathbf{Z}^3; k_1 = 0, k_2 = 0, k_3 > 0\}$ and $\mathbf{Z}^3_- = -\mathbf{Z}^3_+$, and set

$$e_k(x) = \begin{cases} \cos k\cdot x & k \in \mathbf{Z}^3_+, \\ \sin k\cdot x & k \in \mathbf{Z}^3_-. \end{cases} \tag{A.1}$$

Fix for every $k \in \mathbf{Z}^3_*$ an arbitrary orthonormal basis (x_k^1, x_k^2) of the subspace $k^\perp$ of $\mathbf{R}^3$ and set $e_k^1 = x_k^1 e_k(x)$ and $e_k^2 = x_k^2 e_k(x)$, then $\{e_k^i : k \in \mathbf{Z}^3_*,\ i = 1,2\}$ is an *orthonormal basis* of H. In particular, $\pi_N H = \operatorname{span}(\{e_k^i : 0 < |k|_\infty \le N,\ i = 1,2\})$. Denote moreover, for any $N > 0$, $Z_L(N) = [-N,N]^3 \setminus (0,0,0)$ and $Z_H(N) = \mathbf{Z}^3_* \setminus Z_L(N)$.

A.1.1. *Assumptions on the covariance.* Under the Fourier basis of H, the *diagonality* assumption [A1] means that for each $k \in \mathbf{Z}^3_+$, there exists some linear operator $q_k : k^\perp \to k^\perp$ such that $Q(ye_k) = (q_k y)e_k$ for $y \in k^\perp$. The *finite degeneracy* assumption [A2] says that q_k is invertible on $k^\perp$ if $k \in Z_H(N_0)$ and $q_k = 0$ otherwise. If W is a cylindrical Wiener process on H, then $Q\,dW = \sum_{k\in Z_H(N_0)} e_k q_k\,dw_k$, where $(w_k)_{k\in Z_H(N_0)}$ is a sequence of independent 2d Brownian motions and each $w_k \in k^\perp$.

The $\overline{Q}$ in (2.3) is a non-degenerate operator on $\pi_{N_0}H$, which is defined under the Fourier basis by

$$\overline{Q} = \sum_{k\in Z_L(N_0)} e_k q_k \langle\cdot, e_k\rangle_H, \tag{A.2}$$

where, for each $k \in Z_L(N_0)$, q_k is an invertible operator on $k^\perp$.

A.1.2. *The nonlinearity.* In Fourier coordinates, equation (2.1) can be represented under the Fourier basis by

$$\begin{cases} du_k + [|k|^2 u_k + B_k(u,u)]\,dt = q_k\,dw_k(t), & k \in Z_H(N_0) \\ du_k + [|k|^2 u_k + B_k(u,u)]\,dt = 0, & k \in Z_L(N_0) \\ u_k(0) = x_k, & k \in \mathbf{Z}^3_*, \end{cases}$$

where $u = \sum u_k e_k$, $u_k \in k^\perp$ for all $k \in \mathbf{Z}^3_*$ and $B_k(u,u)$ is the Fourier coefficient of $B(u,u)$ corresponding to k. To be more precise,

$$B(u,u) = \sum_{l,m \in \mathbf{Z}^3_*} B(u_l e_l, u_m e_m)$$

and if, for instance, l, $-m$, $l+m \in \mathbf{Z}^3_+$,

$$B(u_l e_l, u_m e_m) = \mathcal{P}\big((u_l \cdot m) u_m e_l e_{-m}\big) = \frac{1}{2}\big[(u_l \cdot m)\mathcal{P}_{l+m} u_m e_{l+m} + (u_l \cdot m)\mathcal{P}_{l-m} u_m e_{l-m}\big],$$

where $\mathcal{P}_k$ is the projection of $\mathbf{R}^3$ onto $k^\perp$, given by $\mathcal{P}_k \eta = \eta - \frac{k\cdot\eta}{|k|^2}k$, then, clearly,

$$B_{l+m}(u_l e_l, u_m e_m) = \tfrac{1}{2}(u_l \cdot m)\mathcal{P}_{l+m} u_m e_{l+m}, \tag{A.3}$$

$$B_{l-m}(u_l e_l, u_m e_m) = \tfrac{1}{2}(u_l \cdot m)\mathcal{P}_{l-m} u_m e_{l-m}, \tag{A.4}$$

and $B_k(u_l e_l, u_m e_m) = 0$ otherwise. For the other cases (of l, m), similar formulas hold.

A.2. **Proofs of the auxiliary results.** The key points for the proofs of this section are the following two inequalities and Lemma A.1 below. Given $\beta > \frac{1}{2}$, there exist constants $C_1 > 0$, $C_2 > 0$ such that for every u, $v \in D(A^{\beta+1/4})$,

$$|A^{\beta-\frac{1}{4}} B(u,v)|_H \le C_1 |A^{\beta+\frac{1}{4}} u|_H |A^{\beta+\frac{1}{4}} v|_H, \tag{A.5}$$

$$|A^{\beta+\frac{1}{4}} \mathrm{e}^{-At} B(u,v)|_H \le \frac{C_2}{\sqrt{t}} |A^{\beta+\frac{1}{4}} u|_H |A^{\beta+\frac{1}{4}} v|_H. \tag{A.6}$$

The first inequality is given by Lemma D.2. in [14], the second follows from the standard estimate $|A^{1/2}\mathrm{e}^{-At}|_H \le Ct^{-1/2}$ for analytical semigroups. The other basic tool is the following Lemma which is a straightforward modification of Proposition 7.3 of [4].

Lemma A.1. *Let $Q : H \to H$ be a linear bounded operator such that $A^{\alpha_0+3/4}Q$ is also bounded, and let W be a cylindrical Wiener process on H. Then for any $0 < \beta < \frac{1}{4}$, $p > 2$ and $\epsilon \in [0, \frac{1}{4} - \beta)$, there exists $C > 0$ such that*

$$\mathbb{E}\Big[\sup_{0\le t\le T} |A^\beta \int_0^t \mathrm{e}^{-A(t-s)} Q\,dW_s|^p_{\mathcal{W}}\Big] \le C T^{(\frac{1}{4}-\epsilon-\beta)p} |A^{-\frac{3}{4}-\epsilon}|^p_{HS}.$$

Proof of Lemma 3.4. We simply write $\Phi_t = \Phi^\delta_t$ (with $\delta \ge 0$) and prove (3.10) at the end. Clearly, $\Phi_t(x)$ satisfies the following equation

$$\Phi_t = \mathrm{e}^{-At}x + \int_0^t \mathrm{e}^{-A(t-s)} \mathrm{e}^{-A_H \delta} B(\Phi_s, \Phi_s)\chi(\frac{|\Phi_s|_{\mathcal{W}}}{3\rho})\,ds + \int_0^t \mathrm{e}^{-A(t-s)} Q(\Phi_s)\,dW_s.$$

By inequality (A.6), the fact $|e^{-A_H\delta}|_{\mathcal W}\le 1$ and the inequality $\chi(\frac{|\Phi_t|_{\mathcal W}}{3\rho})|\Phi_t|_{\mathcal W}\le 3\rho$, it is easy to see that

$$\begin{aligned}|\Phi_t|_{\mathcal W} &\le |x|_{\mathcal W}+\int_0^t |e^{-A(t-s)}B(\Phi_s,\Phi_s)|_{\mathcal W}\chi(\frac{|\Phi_s|_{\mathcal W}}{3\rho})\,ds+\Big|\int_0^t e^{-A(t-s)}Q(\Phi_s)\,dW_s\Big|_{\mathcal W}\\ &\le |x|_{\mathcal W}+\int_0^t \frac{C\rho}{\sqrt{t-s}}|\Phi_s|_{\mathcal W}\cdot\chi(\frac{|\Phi_s|_{\mathcal W}}{3\rho})\,ds+\Big|\int_0^t e^{-A(t-s)}(1-\chi(\frac{|\Phi_s|_{\mathcal W}}{\rho}))Q\,dW_s\Big|_{\mathcal W}\\ &\le |x|_{\mathcal W}+C\rho t^{\frac12}\sup_{0\le s\le t}|\Phi_s|_{\mathcal W}+\Big|\int_0^t e^{-A(t-s)}(1-\chi(\frac{|\Phi_s|_{\mathcal W}}{\rho}))Q\,dW_s\Big|_{\mathcal W},\end{aligned}$$

and that for any $p\ge 2$, $T>0$,

$$\mathbb E\Big(\sup_{0\le t\le T}|\Phi_t|^p_{\mathcal W}\Big)\le |x|^p_{\mathcal W}+C_1T^{p/8}+C_1T^{p/2}\mathbb E\Big(\sup_{0\le t\le T}|\Phi_t|^p_{\mathcal W}\Big)$$

by Lemma A.1 (with $\epsilon=\frac18$, $\beta=0$) and some basic computation, with $C_1=C_1(p,\alpha_0,\rho)$. For T small, $\mathbb E(\sup_{0\le t\le T}|\Phi_t|^p_{\mathcal W})\le\frac{|x|^p_{\mathcal W}+C_1T^{p/8}}{1-C_1T^{p/2}}$. Now, by taking $T,2T,\dots$ as initial times, by applying the same procedure on $[T,2T],[2T,3T],\dots,$ respectively one can obtain similar estimates as the above on these time intervals. Inductively, the estimate (3.6) follows. The proof of (3.7) and (3.8) proceeds similarly.

For every $h\in\mathcal W$, $D_h\Phi_t$ satisfies the following equation

$$\begin{aligned}D_h\Phi_t = e^{-At}h+\int_0^t e^{-A(t-s)}(B(D_h\Phi_s,\Phi_s)+B(\Phi_s,D_h\Phi_s))\chi(\frac{|\Phi_s|_{\mathcal W}}{3\rho})+\\ +e^{-A(t-s)}B(\Phi_s,\Phi_s)\chi'(\frac{|\Phi_s|_{\mathcal W}}{3\rho})\frac{1}{3\rho}\cdot\frac{\langle D_h\Phi_s,\Phi_s\rangle_{\mathcal W}}{|\Phi_s|_{\mathcal W}}\,ds+\\ -\int_0^t e^{-A(t-s)}\chi'(\frac{|\Phi_s|_{\mathcal W}}{\rho})\frac1\rho\cdot\frac{\langle D_h\Phi_s,\Phi_s\rangle_{\mathcal W}}{|\Phi_s|_{\mathcal W}}Q_L\,dW^L_s,\end{aligned}$$

By (A.6) and $\chi(\frac{|\Phi_t|_{\mathcal W}}{3\rho})|\Phi_t|_{\mathcal W}\le 3\rho$,

$$\begin{aligned}|D_h\Phi_t|_{\mathcal W} &\le |h|_{\mathcal W}+\int_0^t\frac{C}{\sqrt{t-s}}\Big(\chi(\frac{|\Phi_s|_{\mathcal W}}{3\rho})|\Phi_s|_{\mathcal W}+\frac{1}{3\rho}|\Phi_s|^2_{\mathcal W}|\chi'(\frac{|\Phi_s|_{\mathcal W}}{3\rho})|\Big)|D_h\Phi_s|_{\mathcal W}\,ds\\ &\quad+\frac1\rho\Big|\int_0^t e^{-A(t-s)}\chi'(\frac{|\Phi_s|_{\mathcal W}}{\rho})\frac{\langle D_h\Phi_s,\Phi_s\rangle_{\mathcal W}}{|\Phi_s|_{\mathcal W}}Q_L\,dW^L_s\Big|_{\mathcal W}\\ &\le |h|_{\mathcal W}+\int_0^t\frac{C\rho}{\sqrt{t-s}}|D_h\Phi_s|_{\mathcal W}\,ds+\frac1\rho\Big|\int_0^t e^{-A(t-s)}\chi'(\frac{|\Phi_s|_{\mathcal W}}{\rho})\frac{\langle D_h\Phi_s,\Phi_s\rangle_{\mathcal W}}{|\Phi_s|_{\mathcal W}}Q_L\,dW^L_s\Big|_{\mathcal W},\end{aligned}$$

by Lemma A.1 (with $\beta=0$ and $\epsilon=\frac18$),

$$\mathbb E\Big[\sup_{0\le t\le T}|D_h\Phi_t|^p_{\mathcal W}\Big]\le |h|^p_{\mathcal W}+CT^{\frac p8}\mathbb E\Big[\sup_{0\le t\le T}|D_h\Phi_t|^p_{\mathcal W}\Big],\qquad 0\le T\le 1,$$

where $C=C(\alpha_0,p,\rho)>0$. For $T>0$ small enough, $\mathbb E[\sup_{0\le t\le T}|D_h\Phi_t|^p]\le\frac{1}{1-CT^{p/8}}|h|^p_{\mathcal W}$.

For $|D_{h^H}\Phi^L_t|_{\mathcal W}$, it is easy to see by a similar argument as in proving (3.9) that

$$|D_{h^H}\Phi^L_t|_{\mathcal W}\le\int_0^t\frac{C\rho}{\sqrt{t-s}}|D_{h^H}\Phi_s|_{\mathcal W}\,ds+\frac1\rho\Big|\int_0^t e^{-A(t-s)}\chi'(\frac{|\Phi_s|_{\mathcal W}}{\rho})\frac{\langle D_{h^H}\Phi_s,\Phi_s\rangle_{\mathcal W}}{|\Phi_s|_{\mathcal W}}Q_L\,dW^L_s\Big|_{\mathcal W},$$

so by Lemma A.1 and (3.9),

$$\mathbb{E}\Big[\sup_{0\le t\le T}|D_{h^H}\Phi^L_t|^p_{\mathcal{W}}\Big] \le T^{\frac{p}{8}}C\mathrm{e}^{CT}|h^H|^p_{\mathcal{W}}, \qquad 0\le T\le 1,$$

$$\mathbb{E}\Big[\sup_{0\le t\le T}|D_{h^H}\Phi^L_t|^p_{\mathcal{W}}\Big] \le T^{\frac{p}{2}}C\mathrm{e}^{CT}|h^H|^p_{\mathcal{W}}, \qquad T>1,$$

where $C=C(\alpha_0,p,\rho)>0$. Similarly but more simply, we have (3.11).

Let us now prove (3.10). By Itô formula,

$$\mathbb{E}|D_h\Phi_t|^2_{\mathcal{W}}+2\int_0^t \mathbb{E}|A^{\frac12}D_h\Phi_s|^2_{\mathcal{W}}\,ds \le$$
$$\le |h|^2_{\mathcal{W}}+C\rho\int_0^t \mathbb{E}\Big[|A^{\frac12}D_h\Phi_s|_{\mathcal{W}}|A^{\alpha_0-\frac14}D_h[\mathrm{e}^{-A_H\delta}B(\Phi_s,\Phi_s)\chi(\frac{|\Phi_s|_{\mathcal{W}}}{3\rho})]|_H\Big]\,ds.$$

By (A.5) and Cauchy inequality, we have

$$\mathbb{E}|D_h\Phi_t|^2_{\mathcal{W}}+\int_0^t \mathbb{E}|A^{\frac12}D_h\Phi_s|^2_{\mathcal{W}}\,ds\le |h|^2_{\mathcal{W}}+C\int_0^t\mathbb{E}|D_h\Phi_s|^2_{\mathcal{W}}\,ds$$

with $C=C(\alpha_0,\rho)>0$, which easily implies (3.10) by Gronwall's lemma. □

Proof of Proposition 3.2. Recall that the solutions to (2.3) and (3.2) are respectively denoted by $\Phi_t(x)$ and $\Phi^\delta_t(x)$. Denote $\Psi_t=\Phi_t-\Phi^\delta_t$, we have

$$\Psi_t=\int_0^t I_1\,ds+\int_0^t I_2\,dW_s \tag{A.7}$$

with

$$I_1=\mathrm{e}^{-A(t-s)}[B(\Phi_s,\Phi_s)\chi(\frac{|\Phi_s|_{\mathcal{W}}}{3\rho})-\mathrm{e}^{-A\delta}B(\Phi^\delta_s,\Phi^\delta_s)\chi(\frac{|\Phi^\delta_s|_{\mathcal{W}}}{3\rho})],$$

and $I_2=\mathrm{e}^{-A(t-s)}[Q(\Phi_s)-Q(\Phi^\delta_s))]$. By (A.6),

$$\begin{aligned}|I_1|_{\mathcal{W}}&\le |Id-\mathrm{e}^{-A\delta}|_{\mathcal{L}(\mathcal{W})}|\mathrm{e}^{-A(t-s)}B(\Phi_s,\Phi_s)|_{\mathcal{W}}\chi(\frac{|\Phi_s|_{\mathcal{W}}}{3\rho})\\&\quad+\Big|\mathrm{e}^{-A(t-s)}B(\Phi_s,\Phi_s)\chi(\frac{|\Phi_s|_{\mathcal{W}}}{3\rho})-\mathrm{e}^{-A(t-s)}B(\Phi^\delta_s,\Phi^\delta_s)\chi(\frac{|\Phi^\delta_s|_{\mathcal{W}}}{3\rho})\Big|_{\mathcal{W}}\\&\le\frac{C_1}{\sqrt{t-s}}|Id-\mathrm{e}^{-A\delta}|_{\mathcal{L}(\mathcal{W})}+\frac{C_2}{\sqrt{t-s}}|\Psi_s|_{\mathcal{W}}\end{aligned}\tag{A.8}$$

with $C_1=C_1(\rho,\alpha_0)$ and $C_2=C_2(\rho,\alpha_0)$, since

$$\begin{aligned}&\Big|\mathrm{e}^{-A(t-s)}B(\Phi_s,\Phi_s)\chi(\frac{|\Phi_s|_{\mathcal{W}}}{3\rho})-\mathrm{e}^{-A(t-s)}B(\Phi^\delta_s,\Phi^\delta_s)\chi(\frac{|\Phi^\delta_s|_{\mathcal{W}}}{3\rho})\Big|_{\mathcal{W}}\\&\quad=\Big|\int_0^1\mathrm{e}^{-A(t-s)}\frac{d}{d\lambda}[B(\lambda\Phi_s+(1-\lambda)\Phi^\delta_s,\lambda\Phi_s+(1-\lambda)\Phi^\delta_s)\chi(\frac{|\lambda\Phi_s+(1-\lambda)\Phi^\delta_s|_{\mathcal{W}}}{3\rho})]d\lambda\Big|_{\mathcal{W}}\\&\quad\le\frac{C_2}{\sqrt{t-s}}|\Psi_s|_{\mathcal{W}}\end{aligned}$$

By fundamental calculus and Lemma A.1 (with $\beta = 0$ and $\epsilon = 1/8$),

$$\begin{aligned}\mathbb{E}\Big[\sup_{0\le t\le T}|\int_0^t I_2\,dW_s|^p\Big] &\le \mathbb{E}\Big[\sup_{0\le t\le T}|\int_0^t \mathrm{e}^{-A(t-s)}(\chi(\frac{|\Phi_s|_{\mathcal{W}}}{\rho})-\chi(\frac{|\Phi_s^\delta|_{\mathcal{W}}}{\rho}))Q_L\,dW_s^L|^p\Big]\\ &\le \mathbb{E}\Big[\int_0^1 \sup_{0\le t\le T}|\int_0^t \mathrm{e}^{-A(t-s)}\frac{d}{d\lambda}\chi(\frac{|\lambda\Phi_s+(1-\lambda)\Phi_s^\delta|_{\mathcal{W}}}{\rho})Q_L\,dW_s^L|^p d\lambda\Big]\\ &\le C_3T^{p/2}\mathbb{E}\Big[\sup_{0\le t\le T}|\Psi_t|_{\mathcal{W}}^p\Big],\end{aligned}\tag{A.9}$$

with $p\ge 2$, $C_3 = C_3(p,\alpha_0,\rho)$ and $T>0$. Combining (A.7), (A.8) and (A.9), we have

$$\mathbb{E}\Big[\sup_{0\le t\le T}|\Psi_t|_{\mathcal{W}}^p\Big] \le C_1T^{\frac{p}{2}}|Id-\mathrm{e}^{-A\delta}|_{\mathcal{L}(\mathcal{W})}^p + C_4T^{\frac{p}{2}}\mathbb{E}\Big[\sup_{0\le t\le T}|\Psi_t|_{\mathcal{W}}^p\Big]\tag{A.10}$$

with $C_4 = C_4(p,\alpha_0,\rho) > 0$. With the estimate of (A.10) and by the same induction argument as in the proof of Lemma 3.4, estimate (3.3) follows.

As for the estimate (3.4), differentiating both sides of (A.7) along directions $h\in\mathcal{W}$, applying the same method as above but with a little more complicated computation, and noticing (3.9), we have

$$\mathbb{E}\Big[\sup_{0\le t\le T}|D_h\Psi_t|_{\mathcal{W}}^p\Big] \le C_5\mathrm{e}^{C_5T}|Id-\mathrm{e}^{-A\delta}|_{\mathcal{L}(\mathcal{W})}^p|h|_{\mathcal{W}}^p,$$

for all $h\in\mathcal{W}$, with $C_5 = C_5(\alpha_0,\rho,p)$. Formula (3.5) follows from the two estimates in the lemma immediately. □

Proof of Lemma 4.2. That the constants of the estimates in the lemma are *independent* of δ is due to the uniform estimates (in δ) of the nonlinear term and to the fact that the Malliavin derivatives $\mathcal{D}_v\Phi_t$ do not depend on v^H.

The proofs of (4.9), (4.11) are classical since the SDEs for J_t, J_t^{-1} are both finite dimensional and have the cutoff. The proof of (4.12) is by the same procedure as for (3.12). For the other estimates, we will apply the bootstrap argument in the proof of (3.6) but omit the trivial induction argument.

As for (4.10), we consider the integral form of equation (4.3) and obtain by applying some classical inequalities

$$\begin{aligned}3^{-p}|J_t^{-1}h^L|_{\mathcal{W}}^p \le |h^L|_{\mathcal{W}}^p + t^{p/q}\int_0^t |J_s^{-1}[A_L + D_L(B_L(\Phi_s,\Phi_s)\chi(\frac{|\Phi|_{\mathcal{W}}}{3\rho})) - \mathrm{Tr}((D_LQ_L(\Phi_t))^2)]h^L|_{\mathcal{W}}^p dt\\ + \Big|\int_0^t J_s^{-1}D_LQ_L(\Phi_s)h^L dW_s^L\Big|_{\mathcal{W}}^p.\end{aligned}$$

Since all the operators in the above inequalities are finite dimensional, by (A.6), Doob's martingale inequality and Birkhold-Davis-Gundy inequality, one has

$$\mathbb{E}\big[\sup_{0\le t\le T}|J_t^{-1}h^L|_{\mathcal{W}}^p\big] \le C_1\Big(1+T^p\mathbb{E}\big[\sup_{0\le t\le T}|J_t^{-1}|_{\mathcal{L}(\mathcal{W})}^p\big] + T^{\frac{p}{2}}\mathbb{E}\big[\sup_{0\le t\le T}|J_t^{-1}|_{\mathcal{L}(\mathcal{W})}^p\big]\Big)|h^L|_{\mathcal{W}}^p$$

where $C_1 = C_1(p,\rho,\alpha_0)$. When T is small enough, we have $\mathbb{E}[\sup_{0\le t\le T}|J_t^{-1}|_{\mathcal{L}(\mathcal{W})}^p] \le \frac{C_1}{1-C_1(T^p+T^{p/2})}$.

Clearly, $\mathcal{D}_v\Phi_t^L$ satisfies the following equation

$$\begin{aligned}\mathcal{D}_v\Phi_t^L &= \int_0^t \mathrm{e}^{-A(t-s)}[-B_L(\Phi_s,\mathcal{D}_v\Phi_s^L)-B_L(\mathcal{D}_v\Phi_s^L,\Phi_s)]\chi(\frac{|\Phi_s|_{\mathcal{W}}}{3\rho})\,ds\\ &\quad -\frac{1}{3\rho}\int_0^t \mathrm{e}^{-A(t-s)}B_L(\Phi_s,\Phi_s)\chi'(\frac{|\Phi_s|_{\mathcal{W}}}{3\rho})\frac{\langle D_v\Phi_s^L,\Phi_s\rangle_{\mathcal{W}}}{|\Phi_s|_{\mathcal{W}}}\,ds\\ &\quad +\int_0^t \mathrm{e}^{-A(t-s)}(1-\chi'(\frac{|\Phi_s|_{\mathcal{W}}}{\rho}))Q_Lv^L\,ds-\frac{1}{\rho}\int_0^t \mathrm{e}^{-A(t-s)}\chi'(\frac{|\Phi_s|_{\mathcal{W}}}{\rho})\frac{\langle D_v\Phi_s^L,\Phi_s\rangle_{\mathcal{W}}}{|\Phi_s|_{\mathcal{W}}}Q_L\,dW_s^L\\ &= J_1(t)+J_2(t)+J_3(t)+J_4(t)\end{aligned}$$

By (A.6) and Lemma A.1, one has

$$\begin{aligned}&|J_1(t)|_{\mathcal{W}} \le \int_0^t \frac{C_2}{\sqrt{t-s}}|\mathcal{D}_v\Phi_s^L|_{\mathcal{W}}\,ds\\ &|J_2(t)|_{\mathcal{W}} \le \int_0^t \frac{C_3}{\sqrt{t-s}}|\mathcal{D}_v\Phi_s^L|_{\mathcal{W}}\,ds\\ &\mathbb{E}\Big(\sup_{0\le t\le T}|J_3(t)|_{\mathcal{W}}^p\Big) \le C_4\mathbb{E}\Big(\int_0^T |v^L(s)|_{\mathcal{W}}^p\,ds\Big)\\ &\mathbb{E}\Big(\sup_{0\le t\le T}|J_4(t)|_{\mathcal{W}}^p\Big) \le C_5T^{p/8}\mathbb{E}\Big(\sup_{0\le t\le T}|\mathcal{D}_v\Phi_t^L|_{\mathcal{W}}^p\Big), \qquad 0\le T\le 1,\end{aligned}$$

with $C_i=C_i(\rho,\alpha_0)$ $(i=2,3)$ and $C_i=C_i(\rho,\alpha_0,p)$ $(i=4,5)$. Thus, for $p\ge 2$,

$$\mathbb{E}\Big(\sup_{0\le t\le T}|\mathcal{D}_v\Phi_t^L|_{\mathcal{W}}^p\Big)\le C_6T^{p/8}\mathbb{E}\Big(\sup_{0\le t\le T}|\mathcal{D}_v\Phi_t^L|_{\mathcal{W}}^p\Big)+C_6\mathbb{E}(\int_0^T |v^L(s)|_{\mathcal{W}}^p\,ds)$$

with $C_6=C_6(\alpha_0,\rho,p)$, and $\mathbb{E}\big(\sup_{0\le t\le T}|\mathcal{D}_v\Phi_t^L|_{\mathcal{W}}^p\big)\le \frac{C_6}{1-C_6T^{p/8}}\mathbb{E}\big(\int_0^T|v^L(s)|_{\mathcal{W}}^p\,ds\big)$ for T small enough.

The term $\mathcal{D}_{v_1}\mathcal{D}_{v_2}\Phi_t$ satisfies the following equation

$$\begin{aligned}\mathcal{D}_{v_1}\mathcal{D}_{v_2}\Phi_t^L &= -\int_0^t \mathrm{e}^{-A(t-s)}\mathcal{D}_{v_1}\mathcal{D}_{v_2}(B_L(\Phi_s,\Phi_s^L)\chi(\frac{|\Phi_s|_{\mathcal{W}}}{3\rho}))\,ds\\ &\quad +\int_0^t \mathrm{e}^{-A(t-s)}\mathcal{D}_{v_2}Q_L(\Phi_s)v_1^L(s)\,ds+\int_0^t \mathrm{e}^{-A(t-s)}\mathcal{D}_{v_1}\mathcal{D}_{v_2}Q_L(\Phi_s)dW_s^L\end{aligned}$$

Expanding the terms $\mathcal{D}_{v_1}\mathcal{D}_{v_2}(B_L(\Phi_s,\Phi_s^L)\chi(\frac{|\Phi_s|_{\mathcal{W}}}{3\rho}))$ and $\mathcal{D}_{v_1}\mathcal{D}_{v_2}Q_L(\Phi_s)$, we obtain two very complex expressions which we omit them but point out the key points for their estimates. Noticing the fact $\mathcal{D}_{v_2}\Phi_t=\mathcal{D}_{v_2}\Phi_t^L$, $|\Phi_t|_{\mathcal{W}}\chi(\frac{|\Phi_t|_{\mathcal{W}}}{3\rho})\le 3\rho$, and using (A.6) and Lemma A.1, one has

$$\begin{aligned}&|\mathrm{e}^{-A(t-s)}\mathcal{D}_{v_2}Q_L(\Phi_s)v_1^L(s)|_{\mathcal{W}}\le C_7|\mathcal{D}_{v_2}\Phi_t^L|_{\mathcal{W}}|v_1^L|_{\mathcal{W}},\\ &\Big|\mathrm{e}^{-A(t-s)}\mathcal{D}_{v_1}\mathcal{D}_{v_2}(B_L(\Phi_s,\Phi_s)\chi(\frac{|\Phi_s|_{\mathcal{W}}}{3\rho}))\Big|_{\mathcal{W}}\le \frac{C_8}{\sqrt{t-s}}\big(|\mathcal{D}_{v_1}\mathcal{D}_{v_2}\Phi_t^L|_{\mathcal{W}}+|\mathcal{D}_{v_1}\Phi_t^L|_{\mathcal{W}}|\mathcal{D}_{v_2}\Phi_t^L|_{\mathcal{W}}\big),\end{aligned}$$

and

$$\mathbb{E}\Big(\sup_{0\le t\le T}|\int_0^t \mathrm{e}^{-A(t-s)}\mathcal{D}_{v_1}\mathcal{D}_{v_2}Q_L(\Phi_s)dW_s^L|_{\mathcal{W}}^p\Big)\le$$
$$\le C_9T^{p/8}\mathbb{E}\Big[\sup_{0\le t\le T}(|\mathcal{D}_{v_1}\mathcal{D}_{v_2}\Phi_t^L|_{\mathcal{W}}^p+|\mathcal{D}_{v_1}\Phi_t^L|_{\mathcal{W}}^p|\mathcal{D}_{v_2}\Phi_t^L|_{\mathcal{W}}^p)\Big],$$

for $0<T\le 1$, with $C_i=C_i(\rho,\alpha_0)$ $(i=7,8)$ and $C_9=C_9(\rho,\alpha_0,p)$. Hence, when T is small

$$\mathbb{E}\Big(\sup_{0\le t\le T}|\mathcal{D}_{v_1}\mathcal{D}_{v_2}\Phi_t^L|_{\mathcal{W}}^p\Big)\le\frac{C_9}{1-C_9T^{p/8}}\mathbb{E}\Big(|\mathcal{D}_{v_1}\Phi_t^L|_{\mathcal{W}}^p|\mathcal{D}_{v_2}\Phi_t^L|_{\mathcal{W}}^p\Big)\le$$
$$\le\Big(\frac{C_{10}}{1-C_{10}T^{p/8}}\Big)^2\Big(1+\mathbb{E}[\int_0^T|v_1^L(s)|_{\mathcal{W}}^{2p}\,ds]\Big)^{\frac12}\Big(1+\mathbb{E}[\int_0^T|v_2^L(s)|_{\mathcal{W}}^{2p}\,ds]\Big)^{\frac12},$$

with $C_{10}=C_{10}(\rho,\alpha_0,p)$. The proof of (4.15) is similar. □

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