Homological algebra for affine Hecke algebras

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Abstract.
In this paper we study homological properties of modules over an affine Hecke algebra $\mathcal{H}$. In particular we prove a comparison result for higher extensions of tempered modules when passing to the Schwartz algebra $\mathcal{S}$, a certain topological completion of the affine Hecke algebra. The proof is self-contained and based on a direct construction of a bounded contraction of certain standard resolutions of $\mathcal{H}$-modules.
This construction applies for all positive parameters of the affine Hecke algebra. This is an important feature, since it is an ingredient to analyse how the irreducible discrete series representations of $\mathcal{H}$ arise in generic families over the parameter space of $\mathcal{H}$. For irreducible non-simply laced affine Hecke algebras this will enable us to give a complete classification of the discrete series characters, for all positive parameters (we will report on this application in a separate article).

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Contents

Introduction

Affine Hecke algebras are useful tools in the study of the representation theory and harmonic analysis of a reductive p-adic group G, cf. [BuKu1, BuKu2, Lus3, Mor1, Mor2]. A central theme in this context is the Morita equivalence of Bernstein blocks of the category of smooth representations of G with the module category of suitable Hecke algebras, often closely related to affine Hecke algebras. This could be thought of as an affine analogue of the role played by finite dimensional Iwahori–Hecke algebras in the representation theory of finite groups of Lie type, a theory which was developed in great detail by Howlett and Lehrer [HoLe]. An important point of Howlett–Lehrer theory is the fact that the Hecke algebras which arise are semisimple specializations of a generic algebra. The affine Hecke algebras which arise in the study of reductive p-adic groups are specializations of generic algebras as well. This time however, it is much more delicate to relate the representation theory of different specializations of the generic algebra. The theory developed in this paper gives an important handle on such problems.

Various aspects of the harmonic analysis on G can be transferred to Hecke algebras [HeOp]. In particular the Hecke algebra comes equipped with a Hilbert algebra structure defined by an anti-linear involution and a tracial state whose spectral measure (also called Plancherel measure) corresponds to the restriction of the Plancherel measure of G to the Bernstein block under the Morita equivalence. This should be compared to the role of generic degrees of representations of finite dimensional Hecke algebras in Howlett–Lehrer theory.

Let q be a positive parameter function for a (based) root datum $\mathcal{R}$, and let $\mathcal{H} = \mathcal{H}(\mathcal{R}, q)$ be the corresponding affine Hecke algebra. The Schwartz algebra completion $\mathcal{S} = \mathcal{S}(\mathcal{R}, q)$ of $\mathcal{H}$ plays a role which is similar to that of the Harish-Chandra Schwartz space $\mathcal{C}(G)$ in the representation theory of G. In particular the support of the Plancherel measure of $\mathcal{H}$ consists precisely of the irreducible representations which extend continuously to $\mathcal{S}$ (the irreducible tempered representations).

More restrictively we say that an irreducible $\mathcal{H}$-module belongs to the discrete series if it is contained in the left regular representation of $\mathcal{H}$ on its own Hilbert space completion. Every irreducible representation can be constructed from a discrete series representation, with a suitable version of parabolic induction. Therefore the discrete series is of utmost importance in the representation theory of $\mathcal{H}$ and of $\mathcal{S}$.

Although $\mathcal{S}$ is larger than $\mathcal{H}$, its representation theory is actually simpler. The spectrum of $\mathcal{S}$ (also called the tempered spectrum of $\mathcal{H}$) is much smaller than the spectrum of $\mathcal{H}$. For example, the discrete series corresponds to isolated points in the spectrum of $\mathcal{S}$, while the spectrum of $\mathcal{H}$ is connected. This observation leads to an especially nice property of $\mathcal{S}$, namely that discrete series representations are projective and injective as $\mathcal{S}$-modules. In contrast, $\mathcal{H}$ does not have finite dimensional projective modules. Yet with quite some representation theory [DeOp] one can reconstruct the entire spectrum of $\mathcal{H}$ from its tempered spectrum.

A priori there could exist higher extensions of tempered $\mathcal{H}$-modules which are themselves not tempered. But this does never happen. More precisely we prove in Corollary 3.7 that

$$\mathrm{Ext}^n_{\mathcal{H}}(U,V) \cong \mathrm{Ext}^n_{\mathcal{S}}(U,V) \tag{1}$$

for all finite dimensional tempered $\mathcal{H}$-modules U and V and all $n \geq 0$. Our belief that something like (1) might be true was inspired by the work of Vignéras, Schneider, Stuhler and Meyer [Vig, ScSt, Mey3].

To prove (1) we construct explicit resolutions of U and V by projective $\mathcal{H}$-modules. The remarkable part of the proof is that we can turn these into projective $\mathcal{S}$-module resolutions in the most naive way, simply by tensoring them with $\mathcal{S}$ over $\mathcal{H}$.

One instance of (1) is particularly important. Suppose that U is a discrete series representation and that V is an irreducible tempered $\mathcal{H}$-module. Theorem 3.8 states that

$$\mathrm{Ext}^n_{\mathcal{H}}(U,V) \cong \left\{ \begin{array}{ll} \mathbb{C} & \text{if } U \cong V \text{ and } n = 0 \\ 0 & \text{otherwise.} \end{array} \right. \tag{2}$$

We want to use (2) to count the number of inequivalent discrete series representations. This requires quite a few steps, which we discuss now. The Euler–Poincaré characteristic [ScSt] of two finite dimensional $\mathcal{H}$-modules is defined as

$$EP_{\mathcal{H}}(U,V) = \textstyle\sum_{n=0}^{\infty} (-1)^n \dim_{\mathbb{C}} \mathrm{Ext}^n_{\mathcal{H}}(U,V). \tag{3}$$

This extends to a symmetric, bilinear and positive semidefinite pairing on virtual $\mathcal{H}$-modules. By (2) the discrete series form an orthonormal set for this pairing.

Suppose that the based root datum $\mathcal{R}$ is given by the 5-tuple $(R_0, X, R_0^\vee, Y, F_0)$, where X and Y are dual lattices, $R_0 \subset X$ is a root system, $R_0^\vee \subset Y$ is its dual root system, and $F_0 \subset R_0$ is a basis of simple roots of R_0. We call $W = W_0 \ltimes X$ the (extended) affine Weyl group of $\mathcal{R}$. For the parameter function $q \equiv 1$ we have $\mathcal{H}(\mathcal{R}, 1) = \mathbb{C}[W]$, while $\mathcal{S}(\mathcal{R}, 1)$ is the Schwartz algebra $\mathcal{S}(W)$ of rapidly decreasing functions $W \to \mathbb{C}$. In particular (3) becomes

$$EP_W(U,V) = \textstyle\sum_{n=0}^{\infty} (-1)^n \dim_{\mathbb{C}} \mathrm{Ext}^n_W(U,V). \tag{4}$$

This is much simpler than (3), because everything about the Euler–Poincaré characteristic for groups like W can be made explicit. In Theorem 3.3 we find a conjugation-invariant "elliptic" measure μ_{ell} on W such that

$$EP_W(U,V) = \int_W \overline{\chi_U} \chi_V \, d\mu_{\mathrm{ell}}, \tag{5}$$

where χ denotes the character of a representation. The support of μ_{ell} consists precisely of the elements which have an isolated fixed point in the real vector space $\mathbb{R} \otimes_{\mathbb{Z}} X$, with respect to the canonical action of W. The number of conjugacy classes of such elements can easily be counted. This can be compared with Kazhdan's elliptic integrals [Kaz, ScSt, Bez].

Finally we relate $EP_{\mathcal{H}(\mathcal{R},q)}$ to EP_W, as follows. The label function q can be scaled to q^ϵ ($\epsilon \in \mathbb{R}$), which yields a continuous field of algebras $\mathcal{H}(\mathcal{R}, q^\epsilon)$. One can associate to any finite dimensional $\mathcal{H}$-module V a continuous family of modules $\tilde{\sigma}_\epsilon(V)$ such that

$$EP_{\mathcal{H}(\mathcal{R},q^\epsilon)}\big(\tilde{\sigma}_\epsilon(U), \tilde{\sigma}_\epsilon(V)\big) = EP_{\mathcal{H}(\mathcal{R},q)}(U,V) \qquad \forall \epsilon \in [-1,1]. \tag{6}$$

In particular we can evaluate this at $\epsilon = 0$, which in combination with the above yields a important upper bound on the number of discrete series representations of $\mathcal{H}$, see Proposition 3.9. In [OpSo] we will use this bound to obtain a complete classification of the discrete series of affine Hecke algebras $\mathcal{H}(\mathcal{R}, q)$ with $\mathcal{R}$ irreducible and q positive.

Now let us describe the contents of the chapters. In Chapter one we collect some notations and results that will be used subsequently. We do not prove any deep theorems in this chapter, but some of the results have not been published in research papers before.

Chapter two is the technical heart of the paper, here we prove everything needed for (1). In fact we do something better, we construct an explicit projective $\mathcal{H}$-bimodule resolution of $\mathcal{H}$. The crucial point is that this becomes a resolution of $\mathcal{S}$ if we tensor it with $\mathcal{S} \otimes \mathcal{S}^{op}$ over $\mathcal{H} \otimes \mathcal{H}^{op}$ and subsequently complete it to a complex of Fréchet spaces. As an immediate consequence we calculate that the global dimensions of $\mathcal{H}$ and $\mathcal{S}$ are equal to the rank of the underlying root datum $\mathcal{R}$.

Although the proof of (1) uses the combinatorial structure of affine Hecke algebras in an essential way, the result itself is of a more analytical nature. The inclusion $\mathcal{H} \to \mathcal{S}$ can be compared to embeddings of the type $F_1(G) \to F_2(G)$, where G is a locally compact group and the $F_i(G)$ are certain convolution algebras of functions on G. In many situations of this type there is a comparison result

$$\mathrm{Ext}^*_{F_1(G)}(U,V) = \mathrm{Ext}^*_{F_2(G)}(U,V) \tag{7}$$

for very general modules U and V [Mey3].

We choose to formulate our results in the category of bornological $\mathcal{S}$-modules. Bornologies are the best technique to cover both non-topological algebras like $\mathcal{H}$ and Fréchet algebras like $\mathcal{S}$, in a natural way. However, we would like to point out that the technical language of bornologies is inessential when dealing with finite dimensional modules of $\mathcal{H}$ or $\mathcal{S}$. In this case it suffices to work with algebraic tensor products, and all proofs can be adapted in such a way so as to avoid the use of results on bornologies. In particular the results on the discrete series do not rely on bornologies. We have put some necessary information on bornological modules in the Appendix.

In Chapter three we first study the Euler–Poincaré characteristic for crossed products of lattices with finite groups. This leads among others to (5). Clearly the results hold for affine Weyl groups, but they do not rely on root systems. In the last two sections we combine everything to derive the aforementioned properties of the Euler–Poincaré characteristic for affine Hecke algebras.

Chapter 1

Preliminaries

1.1 Root data

First we introduce some well-known objects associated to root data. For more background the reader is referred to [BrTi, Hum, IwMa].

Let R_0 be a reduced root system of rank r in an Euclidean space $E \cong \mathbb{R}^r$. Let W_0 be the Weyl group of R_0 and

$$F_0 = \{\alpha_1, \dots, \alpha_r\}$$

an ordered basis. This determines the set of positive (resp. negative) roots R_0^+ (resp. R_0^-). We suppose that R_0 is part of a based root datum

$$\mathcal{R} = (X, R_0, Y, R_0^\vee, F_0).$$

For $I \subset F_0$ we write

$$\begin{array}{lll} C_I^+ & := & \{x \in E : \langle x \,,\, \alpha_i^\vee \rangle = 0 \; \forall \alpha_i \in I, \, \langle x \,,\, \alpha_j^\vee \rangle \geq 0 \; \forall \alpha_j \in F_0 \setminus I\}, \\ C_I^{++} & := & \{x \in E : \langle x \,,\, \alpha_i^\vee \rangle = 0 \; \forall \alpha_i \in I, \, \langle x \,,\, \alpha_j^\vee \rangle > 0 \; \forall \alpha_j \in F_0 \setminus I\}. \end{array}$$

We call $C_\emptyset^{++}$ the positive chamber. Its closure $C_\emptyset^+$ is a fundamental domain for the action of W_0 on E. The isotropy group (in W_0) of any point of C_I^{++} is the standard parabolic subgroup W_I of W_0.

Recall that $Y \times \mathbb{Z}$ is the set of integral affine linear functions on X. Let R^{aff} be the affine root system $R_0^\vee \times \mathbb{Z} \subset Y \times \mathbb{Z}$. The subsets of positive and negative affine roots are

$$\begin{array}{lll} R_+^{\mathrm{aff}} & = & R_0^{\vee,+} \times \{0\} \;\; \cup \;\; R_0^\vee \times \mathbb{Z}_{>0}, \\ R_-^{\mathrm{aff}} & = & R_0^{\vee,-} \times \{0\} \;\; \cup \;\; R_0^\vee \times \mathbb{Z}_{<0}. \end{array}$$

The affine Weyl group of R^{aff} is $W^{\mathrm{aff}} = \mathbb{Z}R_0 \rtimes W_0$, usually considered as a group of affine linear transformations of X. It acts on R^{aff} by

$$w \cdot (\alpha^\vee, k)(x) = (\alpha^\vee, k)(w^{-1}x).$$

For $a = (\alpha^\vee, k) \in R^{\mathrm{aff}}$ consider the affine hyperplane

$$H_a := \{x \in E : \langle x \,,\, a \rangle = \langle x \,,\, \alpha^\vee \rangle + k = 0\}.$$

By definition s_a is the reflection in this hyperplane, given by the formula

$$s_a(x) = x - \langle x\,,\,\alpha^\vee\rangle\alpha - k\alpha.$$

Let F_M be the set of maximal elements of $R_0^\vee$ for the dominance ordering. Label its elements $\alpha_j^\vee$, $j = r+1, \ldots, r+r'$, where r' is the number of irreducible components of R_0. We write

$$a_j := \begin{cases} (\alpha_j^\vee, 0) & \text{if} \quad \alpha_j^\vee \in F_0^\vee \\ (-\alpha_j^\vee, 1) & \text{if} \quad \alpha_j^\vee \in F_M. \end{cases}$$

Then

$$F^{\text{aff}} := \{a_j : j = 1, \ldots, r'\}$$

is a basis of R^{aff} and $\big(W^{\text{aff}}, S^{\text{aff}}\big)$ is a Coxeter system, where

$$S^{\text{aff}} := \{s_a : a \in F^{\text{aff}}\}.$$

For $J \subset S^{\text{aff}}$ we put

$$A_J := \{x \in E : \langle x\,,\,a_j\rangle = 0\,\forall a_j \in J,\ \langle x\,,\,a_i\rangle > 0\,\forall a_i \in F^{\text{aff}} \setminus J\}.$$

All the A_J are facets of the fundamental alcove $A_\emptyset$. Its closure $\overline{A_\emptyset}$ is a fundamental domain for the action of W^{aff} on E. The isotropy group (in W^{aff}) of a point of A_J is the standard parabolic subgroup $\langle J\rangle$ of W^{aff}. We will also write facets as $f = A_J$, in which case the pointwise stabilizer is $W_f = \langle J\rangle$. Notice that this is consistent with the above notation, in the sense that W_0 is the isotropy group of the facet $\{0\}$.

All the hyperplanes $H_{(\alpha^\vee,k)}$ together give E the structure of a polysimplicial complex Σ. The interior of a polysimplex of maximal dimension is called an alcove.

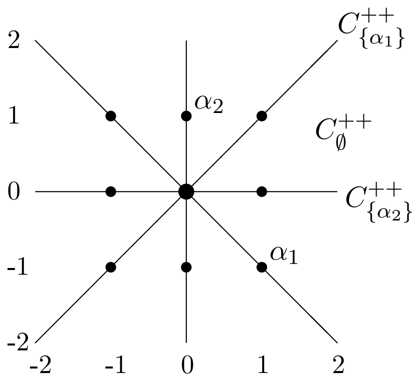
Example.
Let R_0 be the root system B_2 in $E = \mathbb{R}^2$:

$$R_0 = \{\pm(1,-1), \pm(0,1), \pm(1,0), \pm(1,1)\}.$$

The Weyl group W_0 is isomorphic to the dihedral group D_4. A basis of R_0 is

$$F_0 = \{\alpha_1 = (1,-1), \alpha_2 = (0,1)\}.$$

The positive chamber and its walls are



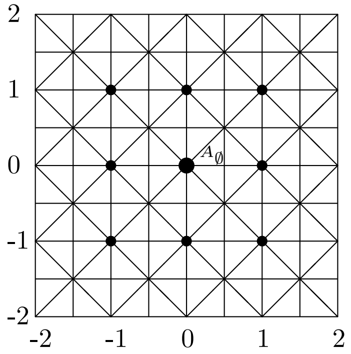
If furthermore $\alpha_3 = (1,0)$ then

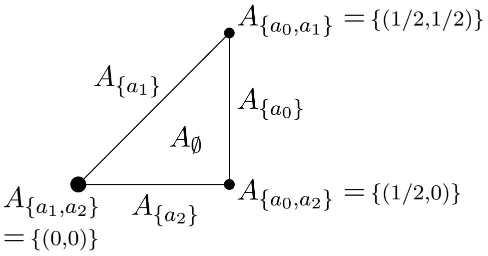
$$F^{\text{aff}} = \{(\alpha_1^\vee, 0), (\alpha_2^\vee, 0), (-\alpha_3^\vee, 1)\} = \{a_1, a_2, a_0\}.$$

The affine Weyl group W^{aff} is generated by the simple reflections

$$\begin{array}{rcrcl} s_1 & : & (x_1, x_2) & \mapsto & (x_2, x_1), \\ s_2 & : & (x_1, x_2) & \mapsto & (x_1, -x_2), \\ s_0 & : & (x_1, x_2) & \mapsto & (1 - x_1, x_2). \end{array}$$

The simplicial complex Σ and the fundamental alcove look like





In general, if A and A' are two alcoves, then a gallery of length n between A and A' is a sequence $(A_0, \ldots, A_n)$ of alcoves such that:

- $A_0 = A$,
- $A_n = A'$,
- $\overline{A_{i-1}} \cap \overline{A_i}$, is contained in exactly one hyperplane H_a, for all i.

The group W^{aff} acts simply transitively on the set of alcoves. For $w \in W^{\mathrm{aff}}$ there is a natural bijection between expressions of w in terms of the generators S^{aff} and galleries from $A_\emptyset$ to $wA_\emptyset$. This bijection is given by

$$w = s_1 \cdots s_n \qquad \longleftrightarrow \qquad (s_1 \cdots s_m A_\emptyset)_{m=0}^{n}\,. \tag{1.1}$$

Lemma 1.1. *For $w \in W^{\mathrm{aff}}$ the following numbers are equal:*

1) *the word length $\ell(w)$ in the Coxeter system $\left(W^{\mathrm{aff}}, S^{\mathrm{aff}}\right)$,*

2) $\#\left\{a \in R_+^{\mathrm{aff}} : wa \in R_-^{\mathrm{aff}}\right\}$,

3) *the number of hyperplanes H_a $(a \in R^{\mathrm{aff}})$ separating $A_\emptyset$ and $wA_\emptyset$,*

4) *the minimal length of a gallery between $A_\emptyset$ and $wA_\emptyset$.*

In particular (1.1) *restricts to a bijection between reduced expressions and galleries of minimal length.*

Proof. See [IwMa, Section 1], [BrTi, Section 2.1] or [Hum, Theorem 4.5]. □

Varying on the Bruhat order, we define a partial order $\leq_A$ on the affine Weyl group W^{aff}:

$$u \leq_A w \qquad \Longleftrightarrow \qquad \ell(u) + \ell(u^{-1}w) = \ell(w).$$

This means that $u \leq_A w$ if and only if a reduced expression for u can be extended to a reduced expression for w by writing extra terms on the right.

Let K be a subset of E, and $\alpha \in R_0$.

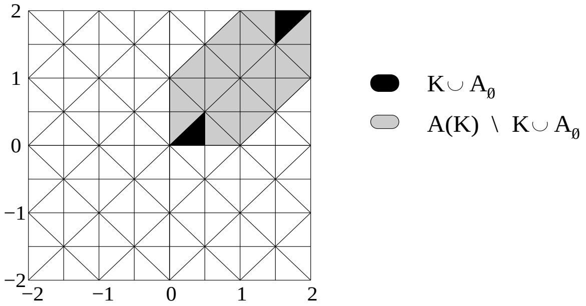
$$\begin{aligned} m(K,\alpha) &:= \inf\left\{\lfloor\langle x\,,\,\alpha^\vee\rangle\rfloor : x \in K \cup A_\emptyset\right\}, \\ M(K,\alpha) &:= \sup\left\{\lceil\langle x\,,\,\alpha^\vee\rangle\rceil : x \in K \cup A_\emptyset\right\}. \end{aligned}$$

where $\lfloor y \rfloor$ and $\lceil y \rceil$ denote respectively the floor and the ceiling of a real number y. With these numbers we define

$$\begin{aligned} A(K,\alpha) &:= \{x \in E : m(K,\alpha) \leq \langle x\,,\,\alpha^\vee\rangle \leq M(K,\alpha)\}, \\ A(K) &:= \textstyle\bigcap_{\alpha\in R_0} A(K,\alpha). \end{aligned}$$

We can interpret $A(K)$ as a kind of Σ-approximation of the convex closure of $K \cup A_\emptyset$ in E.

Example.
In the setting of our previous example $R_0 = B_2$, let K be the simplex $[(3/2,3/2),(3/2,2),(2,2)]$. Then $A(K)$ is the colored area below:



Lemma 1.2. *For any $w \in W^{\mathrm{aff}}$ we have*

$$A(wA_\emptyset) = \bigcup_{u \leq_A w} u\overline{A_\emptyset}\,.$$

Proof. "$\supset$" By Lemma 1.1 every alcove $uA_\emptyset$ with $u \leq_A w$ is part of a gallery of minimal length between $A_\emptyset$ and $wA_\emptyset$. Such a gallery cannot cross any hyperplane H_a ($a \in R^{\mathrm{aff}}$) that does not separate $A_\emptyset$ and $wA_\emptyset$. So for every $\alpha \in R_0$ we must have

$$\langle uA_\emptyset\,,\,\alpha^\vee\rangle \subset [m(wA_\emptyset,\alpha), M(wA_\emptyset,\alpha)].$$

"$\subset$" Since it is bounded by hyperplanes H_a with $a \in R^{\mathrm{aff}}$, $A(wA_\emptyset)$ is a union of closures of alcoves. If $B \subset A(wA_\emptyset)$ is an alcove, then there are no hyperplanes H_a separating B from $A_\emptyset \cup wA_\emptyset$. Hence B is part of at least one gallery of minimal length between $A_\emptyset$ and $wA_\emptyset$. So $B = uA_\emptyset$ for some $u \leq_A w$. □

We note the consequence

$$wA(\sigma) \subset A(w\sigma) \qquad \forall \sigma \subset C_\emptyset^+, w \in W_0. \tag{1.2}$$

1.2 Affine Hecke algebras

We recall a few important results on affine Hecke algebras, meanwhile fixing some notations. Reconsider the based root datum $\mathcal{R} = (X, R_0, Y, R_0^\vee, F_0)$. The extended affine Weyl group of $\mathcal{R}$ is

$$W(\mathcal{R}) = W = X \rtimes W_0.$$

It acts naturally on X, and to avoid confusion we will often denote the element of W corresponding to $x \in X$ by t_x. For any $x \in X$ and $w \in W_0$ we have $w(x) - x \in \mathbb{Z}R_0$, so $W(\mathcal{R})$ contains W^{aff} as a normal subgroup. We write

$$\begin{aligned} X^+ &:= \{x \in X : \langle x \, , \, \alpha^\vee \rangle \geq 0 \; \forall \alpha \in F_0\}, \\ X^- &:= \{x \in X : \langle x \, , \, \alpha^\vee \rangle \leq 0 \; \forall \alpha \in F_0\} = -X^+. \end{aligned}$$

It is easily seen that the center of W is the lattice

$$Z(W) = X^+ \cap X^-.$$

We also want to make W act on E. Since

$$X \otimes \mathbb{R} = E \oplus \big(Z(W) \otimes \mathbb{R} \big),$$

there is a canonical projection

$$p_E : X \otimes \mathbb{R} \to E.$$

This induces a group homomorphism

$$p_E : W \to E \rtimes W_0,$$

and the latter group acts naturally on E. The resulting action of W on E consists of automorphisms of Σ, because

$$\langle p_E(x) \, , \, \alpha^\vee \rangle = \langle x \, , \, \alpha^\vee \rangle \in \mathbb{Z} \qquad \forall x \in X, \alpha^\vee \in R_0^\vee .$$

Hence 2), 3) and 4) of Lemma 1.1 define a natural extension of the length function ℓ from W^{aff} to W. The subgroup $\Omega := \{\omega \in W : \ell(\omega) = 0\}$ of W is complementary to W^{aff}:

$$W = W^{\mathrm{aff}} \rtimes \Omega.$$

We say that $\mathcal{R}$ is semisimple if $R_0^\perp = 0 \subset Y$, or equivalently if $X \otimes \mathbb{R} = E$. If $\mathcal{R}$ is not semisimple then we can make it so by enlarging R_0 and $R_0^\vee$. Namely, pick a basis $\{\alpha_{r+1}, \ldots, \alpha_{\mathrm{rk}(X)}\}$ of $X \cap (R_0^\vee)^\perp$. Then

$$\widetilde{F}_0 = \{\alpha_1, \ldots, \alpha_{\mathrm{rk}(X)}\}$$

is a basis of a root system

$$\widetilde{R}_0 \cong R_0 \times (A_1)^{\mathrm{rk}(X) - r}.$$

Furthermore, pick $\alpha_j^\vee \in Y$ such that

$$\langle \alpha_i \, , \, \alpha_j^\vee \rangle = 2\delta_{ij} \qquad i = 1, \ldots, \mathrm{rk}(X), \; j > r.$$

This yields a semisimple based root datum

$$\widetilde{\mathcal{R}} := (X, \widetilde{R}_0, Y, \widetilde{R}_0^\vee, \widetilde{F}_0). \tag{1.3}$$

Denoting the Weyl group of $(A_1)^{\mathrm{rk}(X)-r}$ by $\tilde{G}$, we observe that

$$W(\widetilde{\mathcal{R}}) = W(\mathcal{R}) \rtimes \tilde{G} = X \rtimes \big(W_0(\mathcal{R}) \times \tilde{G} \big) = X \rtimes W_0(\widetilde{\mathcal{R}}). \tag{1.4}$$

With $\mathcal{R}$ we also associate some other root systems. There is the non-reduced root system

$$R_{nr} := R_0 \cup \{ 2\alpha : \alpha^\vee \in 2Y \}.$$

Obviously we put $(2\alpha)^\vee = \alpha^\vee / 2$. Let R_1 be the reduced root system of long roots in R_{nr}:

$$R_1 := \{ \alpha \in R_{nr} : \alpha^\vee \notin 2Y \}.$$

Let q be a positive labeling of $R_{nr}^\vee$, that is, a W_0-invariant map $R_{nr}^\vee \to (0, \infty)$. This uniquely determines a parameter function $q : W \to (0, \infty)$ with the properties

$$\begin{array}{lcll}
q(s_{\alpha^\vee}) & = & q_{\alpha^\vee} & \alpha \in R_0 \cap R_1, \\
q(s_{1+\beta^\vee}) & = & q_{\beta^\vee} & \beta \in R_0 \setminus R_1, \\
q(s_{\beta^\vee}) & = & q_{\beta^\vee/2} q_{\beta^\vee} & \beta \in R_0 \setminus R_1, \\
q(\omega) & = & 1 & \ell(\omega) = 0, \\
q(wv) & = & q(w)q(v) & w, v \in W \quad \text{with} \quad \ell(wv) = \ell(w) + \ell(v).
\end{array} \tag{1.5}$$

Conversely every function on W with the last two properties defines a labeling of $R_{nr}^\vee$. We speak of equal parameters if $q(s) = q(s') \; \forall s, s' \in S^{\mathrm{aff}}$.

The affine Hecke algebra $\mathcal{H} = \mathcal{H}(\mathcal{R}, q)$ is the unique complex associative algebra with basis $\{ T_w : w \in W \}$ and relations

$$\begin{array}{lcll}
T_w T_v & = & T_{wv} & \text{if} \quad \ell(wv) = \ell(w) + \ell(v), \\
T_s T_s & = & (q(s) - 1) T_s + q(s) T_e & \text{if} \quad s \in S^{\mathrm{aff}}.
\end{array}$$

We can extend q to a parameter function $\tilde{q}$ on $W(\widetilde{\mathcal{R}})$ by putting

$$\tilde{q}(s_{\alpha_j^\vee}) = 1 \quad \forall j > r. \tag{1.6}$$

Then $\tilde{G}$ acts on $\mathcal{H}(\mathcal{R}, q)$ and its group algebra is naturally embedded in $\mathcal{H}(\widetilde{\mathcal{R}}, \tilde{q})$, so the latter can be regarded as a crossed product algebra:

$$\mathcal{H}(\widetilde{\mathcal{R}}, \tilde{q}) \cong \tilde{G} \ltimes \mathcal{H}(\mathcal{R}, q).$$

Now we describe the Bernstein presentation of $\mathcal{H}$. For $x \in X^+$ we put

$$\theta_x := q(x)^{-1/2} T_x \, .$$

The corresponding semigroup morphism $X^+ \to \mathcal{H}(\mathcal{R}, q)^\times$ extends to a group homomorphism

$$X \to \mathcal{H}(\mathcal{R}, q)^\times : x \mapsto \theta_x .$$

Theorem 1.3. [Bernstein presentation]

a) *The sets* $\{T_w \theta_x : w \in W_0, x \in X\}$ *and* $\{\theta_x T_w : w \in W_0, x \in X\}$ *are bases of* $\mathcal{H}$.

b) *The subalgebra* $\mathcal{A} := \mathrm{span}\{\theta_x : x \in X\}$ *is isomorphic to* $\mathbb{C}[X]$.

c) *The center of* $Z(\mathcal{H}(\mathcal{R}, q))$ *of* $\mathcal{H}(\mathcal{R}, q)$ *is* $\mathcal{A}^{W_0}$, *where we define the action of* W_0 *on* $\mathcal{A}$ *by* $w \cdot \theta_x = \theta_{wx}$.

Proof. These results are due to Bernstein, see [Lus2, §3]. $\square$

Let T be the complex algebraic torus $\mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{C}^\times)$, so that $\mathcal{A} \cong \mathcal{O}(T)$ and $Z(\mathcal{H}) = \mathcal{A}^{W_0} \cong \mathcal{O}(T/W_0)$. From Theorem 1.3 we see that $\mathcal{H}$ is of finite rank over its center, and hence Noetherian.

For a set of simple roots $I \subset F_0$ we introduce the notations

$$\begin{array}{ll} R_I = \mathbb{Q}I \cap R_0 & R_I^\vee = \mathbb{Q}R_I^\vee \cap R_0^\vee, \\ X_I = X / \big(X \cap (I^\vee)^\perp \big) & X^I = X / (X \cap \mathbb{Q}I), \\ Y_I = Y \cap \mathbb{Q}I^\vee & Y^I = Y \cap I^\perp, \\ T_I = \mathrm{Hom}_{\mathbb{Z}}(X_I, \mathbb{C}^\times) & T^I = \mathrm{Hom}_{\mathbb{Z}}(X^I, \mathbb{C}^\times), \\ \mathcal{R}_I = (X_I, R_I, Y_I, R_I^\vee, I) & \mathcal{R}^I = (X, R_I, Y, R_I^\vee, I). \end{array} \tag{1.7}$$

We can define parameter functions q_I and q^I on the root data $\mathcal{R}_I$ and $\mathcal{R}^I$, as follows. Restrict q to a labeling of $(R_I)^\vee_{nr}$ and use (1.5) to extend it to $W(\mathcal{R}_I)$ and $W(\mathcal{R}^I)$. Then $\mathcal{H}(\mathcal{R}^I, q^I)$ is isomorphic to the subalgebra of $\mathcal{H}(\mathcal{R}, q)$ generated by $\mathcal{A}$ and $\mathcal{H}(W_I, q)$. With this identification in mind we call $\mathcal{H}(\mathcal{R}^I, q^I)$ a parabolic subalgebra of $\mathcal{H}(\mathcal{R}, q)$.

For any $t \in T^I$ there is a surjective algebra homomorphism

$$\begin{aligned} \phi_t &: \mathcal{H}(\mathcal{R}^I, q^I) \to \mathcal{H}(\mathcal{R}_I, q_I), \\ \phi_t(\theta_x T_w) &= t(x) \theta_{x_I} T_w \,. \end{aligned} \tag{1.8}$$

where x_I is the image of $x \in X$ in X_I. So given any representation σ of $\mathcal{H}(\mathcal{R}_I, q_I)$, we can construct the $\mathcal{H}$-representation

$$\pi(I, \sigma, t) := \mathrm{Ind}_{\mathcal{H}(\mathcal{R}^I, q^I)}^{\mathcal{H}(\mathcal{R}, q)} (\sigma \circ \phi_t).$$

Representations of this form are said to be parabolically induced.

Since $\mathcal{H}$ is of finite rank over $Z(\mathcal{H})$ every irreducible $\mathcal{H}$-representation has finite dimension. In particular an $\mathcal{H}$-module is of finite length if and only if it has finite dimension. Let $\mathrm{Mod}(\mathcal{H})$ be the category of all $\mathcal{H}$-modules and $\mathrm{Mod}_{\mathrm{fin}}(\mathcal{H})$ the subcategory of finite length $\mathcal{H}$-modules. We denote the Grothendieck group of $\mathrm{Mod}_{\mathrm{fin}}(\mathcal{H})$ by $G(\mathcal{H})$ and we write

$$G_{\mathbb{C}}(\mathcal{H}) := G(\mathcal{H}) \otimes_{\mathbb{Z}} \mathbb{C}.$$

Similarly we can define $\mathrm{Mod}(A)$, $\mathrm{Mod}_{\mathrm{fin}}(A)$, $G(A)$ and $G_{\mathbb{C}}(A)$ for any algebra or group A. For bornological algebras A we will also consider the category $\mathrm{Mod}_{\mathrm{bor}}(A)$ of bornological A-modules, see the Appendix.

The center of $\mathcal{H}(\mathcal{R},q)$ contains the group algebra of $Z(W)$, so every irreducible $\mathcal{H}$-representation admits a unique $Z(W)$-character χ. Such representations factor through the algebra

$$\mathcal{H}(\mathcal{R},q)_\chi = \mathcal{H} \otimes_{Z(W)} \mathbb{C}_\chi .$$

The algebra $\mathcal{H}$ is endowed with a trace

$$\tau\big(\textstyle\sum_{w\in W} h_w T_w\big) = h_e$$

and an involution

$$\big(\textstyle\sum_{w\in W} h_w T_w\big)^* = \textstyle\sum_{w\in W} \overline{h_w} T_{w^{-1}}$$

Because q takes only positive values, * is conjugate-linear and antimultiplicative, while τ is positive.

Our affine Hecke algebra is canonically isomorphic to the crossed product of the Iwahori-Hecke algebra corresponding to W^{aff}, and the group Ω:

$$\mathcal{H}(\mathcal{R},q) \cong \mathcal{H}(W^{\text{aff}},q) \rtimes \Omega .$$

Let f be a facet of the fundamental alcove $A_\emptyset$ and write

$$\Omega_f := \{\omega \in \Omega : p_E \omega(f) = f\}.$$

Whether or not ω changes the orientation of f is measured by the character $\epsilon_f : \Omega_f \to \{\pm 1\}$. Furthermore Ω_f acts on W_f, so we can define

$$\mathcal{H}(\mathcal{R},f,q) := \mathcal{H}(W_f,q) \rtimes \Omega_f .$$

We note that for any $\mathcal{H}(\mathcal{R},f,q)$-module (π,V) there is a well-defined $\mathcal{H}(\mathcal{R},f,q)$-module $V \otimes \epsilon_f$, where

$$(\pi \otimes \epsilon_f)(hT_\omega)(v) := \epsilon_f(\omega)\pi(hT_\omega)(v).$$

By definition $Z(W) \subset \Omega_f$, so

$$\mathbb{C}[Z(W)] \subset Z(\mathcal{H}(\mathcal{R},f,q)).$$

Lemma 1.4. *Let $\mathbb{C}_\chi$ be a onedimensional $Z(W)$-representation with character χ.*

$$\mathcal{H}(\mathcal{R},f,q)_\chi := \mathcal{H}(\mathcal{R},f,q) \otimes_{Z(W)} \mathbb{C}_\chi$$

is a finite dimensional semisimple algebra.

Proof. As vector spaces we may identify

$$\mathcal{H}(\mathcal{R},f,q)_\chi = \mathrm{Ind}_{\mathbb{C}[Z(W)]}^{\mathcal{H}(\mathcal{R},f,q)} \mathbb{C}_\chi = \mathcal{H}(W_f,q) \otimes_{\mathbb{C}} \mathbb{C}[\Omega_f / Z(W)].$$

We can extend $|\chi|$ canonically to $X \otimes \mathbb{R}$, making it 1 on E. Using this extension we define an involution $*_\chi$ on $\mathcal{H}(\mathcal{R},f,q)$ by

$$(h_w T_w)^{*_\chi} = \overline{h_w}\, |\chi|(2w(0)) T_{w^{-1}} .$$

The associated bilinear form is

$$\langle h \,,\, h' \rangle_\chi = \tau(h^{*_\chi} \cdot h').$$

By construction $\mathrm{Ind}_{\mathbb{C}[Z(W)]}^{\mathcal{H}(\mathcal{R},f,q)} \mathbb{C}_\chi$ is now a unitary representation. This makes $\mathcal{H}(\mathcal{R},f,q)_\chi$ into a finite dimensional Hilbert algebra, so in particular it is semisimple. $\Box$

1.3 The Schwartz completion

We show how to complete an affine Hecke algebra to a C^*-algebra and to a Schwartz algebra. The involution and the trace on $\mathcal{H}(\mathcal{R}, q)$ give rise to a Hermitian inner product

$$\langle h\, ,\, h' \rangle = \tau(h^* \cdot h') \qquad h, h' \in \mathcal{H}(\mathcal{R}, q)$$

and a norm

$$\|h\|_\tau = \sqrt{\langle h\, ,\, h \rangle} = \sqrt{\tau(h^* \cdot h)}.$$

With a basic calculation one can check that

$$\{N_w = q(w)^{-1/2} T_w : w \in W\} \tag{1.9}$$

is an orthonormal basis of $\mathcal{H}(\mathcal{R}, q)$ for this inner product. All this gives $\mathcal{H}(\mathcal{R}, q)$ the structure of a Hilbert algebra, in the sense of [Dix, A 54]. Let $L^2(\mathcal{R}, q)$ be its Hilbert space completion, for which (1.9) is by definition a basis. Consider the multiplication map

$$\begin{aligned} \lambda(h) : \mathcal{H}(\mathcal{R}, q) &\to \mathcal{H}(\mathcal{R}, q), \\ \lambda(h)\, h' \;&=\; h \cdot h'. \end{aligned}$$

By [Opd1, Lemma 2.3] this maps extends to a bounded operator on $L^2(\mathcal{R}, q)$, whose norm we denote by

$$\|h\|_o = \|\lambda(h)\|_{B(L^2(\mathcal{R},q))}\, .$$

Thus, $\mathcal{H}(\mathcal{R}, q)$ being a *-subalgebra of the C^*-algebra $B(L^2(\mathcal{R}, q))$ of bounded operators on $L^2(\mathcal{R}, q)$, we can consider its closure $C^*(\mathcal{R}, q)$ with respect to the operator norm topology. By definition this is a separable unital C^*-algebra, called the (reduced) C^*-algebra of $\mathcal{H}$ or of $(\mathcal{R}, q)$.

Let (π, V) be an irreducible $\mathcal{H}$-representation. We say that it belongs to the discrete series if the following equivalent conditions hold:

- (π, V) is a subrepresentation of the left regular representation $(\lambda, L^2(\mathcal{R}, q))$,
- all matrix coefficients of (π, V) are in $L^2(\mathcal{R}, q)$.

By definition a discrete series representation is unitary, and it extends continuously to $C^*(\mathcal{R}, q)$. Because this is a Hilbert algebra, a suitable version of [Dix, Proposition 18.4.2] shows that π is an isolated point in its spectrum. Moreover, since $C^*(\mathcal{R}, q)$ is unital its spectrum is compact [Dix, Proposition 3.18], so there can be only finitely many inequivalent discrete series representations.

It is also possible to complete $\mathcal{H}(\mathcal{R}, q)$ to a Schwartz algebra $\mathcal{S} = \mathcal{S}(\mathcal{R}, q)$. As a topological vector space $\mathcal{S}$ will consist of rapidly decreasing functions on W, with respect to some length function. For this purpose it is unsatisfactory that ℓ is 0 on the subgroup $Z(W)$, as this can be a large part of W. To overcome this inconvenience, let $L : X \otimes \mathbb{R} \to [0, \infty)$ be a function such that

- $L(X) \subset \mathbb{Z}$,
- $L(x + y) = L(x) \quad \forall x \in X \otimes \mathbb{R}, y \in E$,

- L induces a norm on $X \otimes \mathbb{R}/E \cong Z(W) \otimes \mathbb{R}$.

Now we define for $w \in W$

$$\mathcal{N}(w) := \ell(w) + L(w(0)),$$

so that

$$\begin{array}{lllll} \mathcal{N}(u\omega) & = & \mathcal{N}(\omega u) \; = \; \ell(u) + L(\omega(0)) & & u \in W^{\mathrm{aff}}, \, \omega \in \Omega, \\ \mathcal{N}(wv) & \leq & \mathcal{N}(w) + \mathcal{N}(v) & & w, v \in W. \end{array}$$

Since $Z(W) \oplus \mathbb{Z}R_0$ is of finite index in X, the set $\{w \in W : \mathcal{N}(w) = 0\}$ is finite. Moreover, because W is the semidirect product of a finite group and an abelian group, it is of polynomial growth and different choices of L lead to equivalent length functions $\mathcal{N}$. For $n \in \mathbb{N}$ we define the norm

$$p_n\big(\textstyle\sum_{w \in W} h_w N_w\big) := \sup_{w \in W} |h_w| (\mathcal{N}(w) + 1)^n .$$

The completion $\mathcal{S} = \mathcal{S}(\mathcal{R}, q)$ of $\mathcal{H}(\mathcal{R}, q)$ with respect to the family of norms $\{p_n\}_{n \in \mathbb{N}}$ is a nuclear Fréchet space. It consists of all possible infinite sums $h = \sum_{w \in W} h_w N_w$ such that $p_n(h) < \infty \; \forall n \in \mathbb{N}$.

Lemma 1.5. [Sol, p. 135]
Let $b = \mathrm{rk}(X) + 1$. *The sum*

$$\textstyle\sum_{w \in W} \big(\mathcal{N}(w) + 1\big)^{-b}$$

converges to a limit C_b. *If* $h \in \mathcal{S}$ *and* $n \in \mathbb{N}$ *then*

$$\textstyle\sum_{w \in W} |h_w| (\mathcal{N}(w) + 1)^n \leq C_b \, p_{n+b}(h).$$

The norms p_n behave reasonably with respect to multiplication:

Theorem 1.6. [Opd1, Section 6.2]
There exist $C_q > 0$, $d \in \mathbb{N}$ *such that* $\forall h, h' \in \mathcal{S}(\mathcal{R}, q), n \in \mathbb{N}$

$$\begin{array}{c} \|h\|_o \leq C_q p_d(h), \\ p_n(h \cdot h') \leq C_q p_{n+d}(h) p_{n+d}(h'). \end{array}$$

In particular $\mathcal{S}(\mathcal{R}, q)$ *is a unital locally convex *-algebra, and it is contained in* $C^*(\mathcal{R}, q)$.

A finite dimensional $\mathcal{H}$-module is called tempered if the $\mathcal{H}$-action extends continuously to $\mathcal{S}$. There are various ways to define infinite dimensional tempered modules, depending on which category of vector spaces one wishes to consider. In the Appendix we discuss tempered bornological modules.

From the work of Casselman [Cas, §4.4] one can deduce concrete criteria for representations to be tempered or discrete series, see [Opd1, Section 2.7]. It follows from these criteria that an $\mathcal{H}$-module can only be tempered if all its $Z(W)$-weights are unitary.

The reader is referred to [DeOp] for a study of the algebra $\mathcal{S}$ and its Fourier transform. Notice that as a Fréchet space $\mathcal{S}(\mathcal{R}, q)$ does not depend on q. The basis $\{N_w : w \in W\}$ gives rise to a canonical isomorphism between $\mathcal{S}(\mathcal{R}, q)$ and $\mathcal{S}(W)$.

For $\epsilon \in \mathbb{R}$ let q^ϵ be the parameter function $q^\epsilon(w) = q(w)^\epsilon$. For every ϵ we have the affine Hecke algebra $\mathcal{H}(\mathcal{R}, q^\epsilon)$ and its Schwartz completion $\mathcal{S}(\mathcal{R}, q^\epsilon)$. We note that $\mathcal{H}(\mathcal{R}, q^0) = \mathbb{C}[W]$ is the group algebra of W and that $\mathcal{S}(\mathcal{R}, q^0) = \mathcal{S}(W)$ is the Schwartz algebra of rapidly decreasing functions on W.

The intuitive idea is that these algebras depend continuously on ϵ. We will use this in the form of the following rather technical result.

Theorem 1.7. *For $\epsilon \in [-1, 1]$ there exists a family of additive functors*

$$\begin{aligned} \tilde{\sigma}_\epsilon &: \mathrm{Mod}_{\mathrm{fin}}(\mathcal{H}(\mathcal{R}, q)) \to \mathrm{Mod}_{\mathrm{fin}}(\mathcal{H}(\mathcal{R}, q^\epsilon)), \\ \tilde{\sigma}_\epsilon(\pi, V) &= (\pi_\epsilon, V). \end{aligned}$$

with the properties

1) *the map*

$$[-1, 1] \to \operatorname{End} V : \epsilon \mapsto \pi_\epsilon(N_w)$$

is analytic for any $w \in W$,

2) *$\tilde{\sigma}_\epsilon$ is a bijection if $\epsilon \neq 0$,*

3) *$\tilde{\sigma}_\epsilon$ preserves unitarity,*

4) *$\tilde{\sigma}_\epsilon$ preserves temperedness if $\epsilon \geq 0$,*

5) *$\tilde{\sigma}_\epsilon$ preserves the discrete series if $\epsilon > 0$.*

Proof. See [Sol, Theorem 5.16 and Lemma 5.17]. □

Chapter 2

Projective resolutions

In this chapter we will contruct projective resolutions for modules of an affine Hecke algebra $\mathcal{H}$. We do this in a functorial way, starting from an explicit projective $\mathcal{H}$-bimodule resolution of $\mathcal{H}$. This allows us to show that the global dimension of $\mathcal{H}$ equals the rank of the lattice X.

It turns out that the same contructions also work over $\mathcal{S}$. However this is by no means automatic. Namely, it is not enough to have a projective $\mathcal{H}$-bimodule resolution, to show that it can be induced to $\mathcal{S}$ we also need a contraction which is bounded in a suitable sense. The essential part of the proof takes place within the polysimplicial complex Σ associated to the root system R_0. Taking advantage of the abundant symmetry of root systems we construct a bounded contraction of the corresponding differential complex. With this contraction we establish a projective bimodule resolution of $\mathcal{S}$. As a consequence we can show that the cohomological dimension of bornological $\mathcal{S}$-modules also equals the rank of X.

Actually more is true, as Ralf Meyer kindly pointed out to us. The inclusion of complete, unital, bornological algebras $\mathcal{H} \to \mathcal{S}$ is isocohomological (in the sense discussed in the Appendix).

2.1 The bounded contraction of the polysimplicial complex

From the polysimplicial complex Σ (cf. page 7) we construct a differential complex $(C_*(\Sigma), \partial_*)$. The vector space in degree n is

$$C_n(\Sigma) := \mathbb{C}\{\sigma \in \Sigma : \dim \sigma = n\}. \tag{2.1}$$

For every σ there is a unique facet f of the fundamental alcove $A_\emptyset$ such that σ is W^{aff}-conjugate to the closure $\bar{f}$ of f in E. We fix an orientation on all the facets of $A_\emptyset$ and we decree that the map $w : f \to wf$ preserves orientation. This determines a unique orientation on every simplex of Σ. With these conventions we can identify

$$C_n(\Sigma) = \bigoplus_{f : \dim f = n} \mathbb{C}\big[W^{\text{aff}}/W_f\big]. \tag{2.2}$$

Clearly Σ is the direct product of a number (say r') simplicial complexes corresponding to the irreducible components of R_0. Let

$$\sigma = \sigma^{(1)} \times \cdots \times \sigma^{(r')}$$

be a polysimplex of Σ. Denote the vertices of $\sigma^{(j)}$ by $x_i^{(j)}$, so that we can write

$$\sigma^{(j)} = \left[x_0^{(j)}, x_1^{(j)}, \ldots, x_{d_j}^{(j)} \right].$$

The order of the vertices defines an orientation on $\sigma^{(j)}$. For a permutation $\lambda \in S_{d_j}$ with sign $\epsilon(\lambda)$ we identify

$$\left[x_{\lambda(0)}^{(j)}, x_{\lambda(1)}^{(j)}, \ldots, x_{\lambda(d_j)}^{(j)} \right] = \epsilon(\lambda) \left[x_0^{(j)}, x_1^{(j)}, \ldots, x_{d_j}^{(j)} \right].$$

The boundary of $\sigma^{(j)}$ is defined as

$$\begin{aligned} \partial\sigma^{(j)} = \partial \left[x_0^{(j)}, x_1^{(j)}, \ldots, x_{d_j}^{(j)} \right] &:= \sum_{i=0}^{d_j} (-1)^i \left[x_0^{(j)}, \ldots, x_{i-1}^{(j)}, x_{i+1}^{(j)}, \ldots, x_{d_j}^{(j)} \right], \\ \partial \big[x_0^{(j)} \big] &:= 0. \end{aligned}$$

Furthermore we define

$$\partial_n \sigma = \sum_{j=1}^{r'} (-1)^{d_1 + \cdots + d_{j-1}} \sigma^{(1)} \times \cdots \times \sigma^{(j-1)} \times \partial\sigma^{(j)} \times \sigma^{(j+1)} \times \cdots \times \sigma^{(r')}$$

if $\dim \sigma = n > 0$. It is easily verified that this operation satisfies the usual property $\partial \circ \partial = 0$. We augment this differential complex by

$$C_{-1}(\Sigma) = \mathbb{C}$$

and $\partial_0[x] = 1$ if x is a vertex of Σ. The augmented complex $(C_*(\Sigma), \partial_*)$ computes the reduced singular homology of the space E underlying Σ. This space is contractible, so by the Poincaré lemma

$$H_n(C_*(\Sigma), \partial_*) = 0 \qquad \forall n \in \mathbb{Z}. \tag{2.3}$$

The support of a chain $c = \sum_{\sigma \in \Sigma} c_\sigma \sigma \in C_*(\Sigma)$ is

$$\operatorname{supp} c = \bigcup_{\sigma : c_\sigma \neq 0} \sigma.$$

A contraction γ of $(C_*(\Sigma), \partial_*)$ is a collection of linear maps

$$\gamma_n : C_n(\Sigma) \to C_{n+1}(\Sigma) \qquad n \geq -1,$$

such that

$$\gamma_{n-1}\partial_n + \partial_{n+1}\gamma_n = \mathrm{id}_{C_n(\Sigma)} \qquad \forall n \in \mathbb{Z}.$$

The periodic nature of Σ allows us to construct a contraction with good bounds on the coefficients:

Proposition 2.1. *There exists a contraction γ with the properties*

1) $\gamma\partial + \partial\gamma = id$,

2) γ *is* W_0*-equivariant,*

3) *supp* $\gamma(\sigma) \subset A(\sigma)$ *for every* $\sigma \in \Sigma$,

4) $\gamma(\sigma) = \sum_{\tau\in\Sigma} \gamma_{\sigma\tau}\tau$ *with* $|\gamma_{\sigma\tau}| < M_\gamma$ *for some constant* M_γ *depending only on* γ.

Proof. Our construction will be rather similar to that of V. Lafforgue in [Ska, §4]. First we impose some extra conditions. 2) and 3) force

5) if $\sigma \subset C_I^+$ then supp $\gamma(\sigma) \subset C_I^+$.

In view of (1.2) and since ∂ is W_0-equivariant, it suffices to construct γ on $C_\emptyset^+$. We will use that the translations t_x with $x \in \mathbb{Z}R_0$ are orientation preserving automorphisms of Σ. For $\alpha_i \in F_0$ let β_i be the minimal element of $C^{++}_{F_0\setminus\{\alpha_i\}} \cap \mathbb{Z}R_0$. Note that β_i is an integral multiple of a vertex of $A_\emptyset$. We could also pick a fundamental weight instead of β_i, but in that case we would have keep track of the orientations. Consider the halfopen parallelogram

$$P_\emptyset = \Big\{ \sum_{i=1}^{r} y_i\beta_i : y_i \in [0,1) \Big\}.$$

Let τ be any polysimplex whose interior is contained in $P_\emptyset$. Our contraction will also satisfy

6) $\gamma(t_{(m+1)\beta_i}(\tau)) = \gamma(t_{m\beta_i}(\tau)) + t_{m\beta_i}\gamma(t_{\beta_i}(\tau) - \tau)$

for $m \geq 0$. Suppose that $\beta = \sum_{i=1}^{k} n_i\beta_i$ with $n_i \in \mathbb{N}$. Then we decree

7) $\gamma(t_\beta(\tau)) = \gamma(t_{n_k\beta_k}(\tau)) + t_{n_k\beta_k}\gamma(t_{\beta - n_k\beta_k}(\tau) - \tau)$.

Here we use the ordering on the set F_0 of simple roots. The idea underlying 6) and 7) is that we want to make γ equivariant with respect to certain translations.

Now we really start constructing γ. In degree -1 we put

$$\gamma_{-1}(1) = [0].$$

Suppose that γ_m has already been defined for $m < n$, satisfying conditions 1) - 7). Let σ be any n-dimensional polysimplex whose interior is contained in

$$P_1 := P_\emptyset \cup t_{\beta_1}P_\emptyset \cup \cdots \cup t_{\beta_r}P_\emptyset.$$

By 1) we have

$$\partial(\sigma - \gamma\partial(\sigma)) = (\mathrm{id} - \partial\gamma)(\partial\sigma) = \gamma\partial(\partial\sigma) = 0.$$

Together with (2.3) this implies that the equation

$$\partial\gamma(\sigma) = \sigma - \gamma\partial(\sigma)$$

has a solution $\gamma(\sigma) \in C_{n+1}(\Sigma)$. By 3) and 5) we have

$$\mathrm{supp}\,(\sigma - \gamma\partial(\sigma)) \subset A(\sigma) \cap C_I^+ \quad \text{if} \quad \sigma \subset C_I^+.$$

Since $A(\sigma) \cap C_I^+$ is convex, we can pick $\gamma(\sigma)$ with support in this set. We do this for any n-dimensional $\sigma \in \Sigma$ whose interior is contained in P_1. Now 6) and 7) determine γ_n uniquely on $C_\emptyset^+$.

We will show that the other required properties follow from this construction. Write $\beta' = \sum_{i=1}^{k-1} n_i\beta_i$ and $\beta'' = \sum_{i=1}^{k-1} n_i'\beta_i$ for some $n_i' \in \mathbb{N}$. By 7) we have

$$\gamma t_{n_k\beta_k}\big(t_{\beta'}(\tau) - t_{\beta''}(\tau)\big) = t_{n_k\beta_k}\gamma\big(t_{\beta'}(\tau) - t_{\beta''}(\tau)\big). \tag{2.4}$$

We claim that the following stronger version of 7) holds

7') $\gamma t_{n_k\beta_k}\big(t_{\beta - n_k\beta_k}(\sigma) - \sigma\big) = t_{n_k\beta_k}\gamma\big(t_{\beta - n_k\beta_k}(\sigma) - \sigma\big) \qquad \forall \sigma \subset C_\emptyset^+$.

Indeed, write $\sigma = t_x\tau$ with τ as in 7) and $x = \sum_{j=1}^r m_j\beta_j$. Then by a repeated application of (2.4) the left hand side of 7') becomes

$$\begin{aligned} \gamma t_{n_k\beta_k}\big(t_{\beta'}(t_x\tau) - t_x\tau\big) &= t_{(n_k+m_k)\beta_k + m_{k+1}\beta_{k+1} + \cdots + m_r\beta_r}\gamma(t_{\beta'} - \mathrm{id})t_{m_1\beta_1 + \cdots + m_{k-1}\beta_{k-1}}(\tau) \\ &= t_{n_k\beta_k}\gamma t_x(t_{\beta'}(\tau) - \tau) \\ &= t_{n_k\beta_k}\gamma(t_{\beta'}(\sigma) - \sigma). \end{aligned}$$

It follows easily from 6) that

$$\gamma t_{m\beta_i}\big(t_{m'\beta_i}(\tau) - t_{m''\beta_i}(\tau)\big) = t_{m\beta_i}\gamma\big(t_{m'\beta_i}(\tau) - t_{m''\beta_i}(\tau)\big) \qquad \forall m, m', m'' \in \mathbb{N}. \tag{2.5}$$

There also is a stronger version of 6) :

6') $\gamma t_{m\beta_i}\big(t_{\beta_i}(\sigma) - \sigma\big) = t_{m\beta_i}\gamma\big(t_{\beta_i}(\sigma) - \sigma\big) \qquad \forall \sigma \subset C_\emptyset^+$.

Indeed, in the above notation and by 7') and (2.5) the left hand side equals

$$\begin{aligned} &\gamma t_{m\beta_i}\big(t_{\beta_i + x}(\tau) - t_x(\tau)\big) &&= \\ &t_{m_{i+1}\beta_{i+1} + \cdots + m_r\beta_r}\gamma t_{(m+m_i)\beta_i}(t_{\beta_i} - \mathrm{id})t_{m_1\beta_1 + \cdots + m_{i-1}\beta_{i-1}}(\tau) &&= \\ &t_{m_{i+1}\beta_{i+1} + \cdots + m_r\beta_r}\Big(\gamma t_{(m+m_i)\beta_i}\big(t_{\beta_i}(\tau) - \tau\big) \\ &\quad + t_{(m+m_i)\beta_i}(t_{\beta_i} - \mathrm{id})\gamma\big(t_{m_1\beta_1 + \cdots + m_{i-1}\beta_{i-1}}(\tau) - \tau\big)\Big) &&= \\ &t_{m\beta_i + m_{i+1}\beta_{i+1} + \cdots + m_r\beta_r}\Big(\gamma t_{m_i\beta_i}\big(t_{\beta_i}(\tau) - \tau\big) \\ &\quad + t_{m_i\beta_i}(t_{\beta_i} - \mathrm{id})\gamma\big(t_{m_1\beta_1 + \cdots + m_{i-1}\beta_{i-1}}(\tau) - \tau\big)\Big) &&= \\ &t_{m\beta_i + m_{i+1}\beta_{i+1} + \cdots + m_r\beta_r}\gamma(t_{\beta_i} - \mathrm{id})t_{m_1\beta_1 + \cdots + m_i\beta_i}(\tau) &&= \\ &t_{m\beta_i}\gamma(t_{\beta_i} - \mathrm{id})t_x(\tau) &&= t_{m\beta_i}\gamma\big(t_{\beta_i}(\sigma) - \sigma\big). \end{aligned}$$

Now we can see that the relations 6) and 7) are compatible with 1). Assume that 1) holds for $t_{m\beta_i}(\tau)$. Then by 6')

$$\begin{aligned} &(\partial_{n+1}\gamma_n + \gamma_{n-1}\partial_n)\big(t_{(m+1)\beta_i}(\tau)\big) &&= \\ &\partial_{n+1}\gamma_n(t_{m\beta_i}\tau) + \partial_{n+1}t_{m\beta_i}\gamma_n\big(t_{\beta_i}(\tau) - \tau\big) + \gamma_{n-1}t_{(m+1)\beta_i}\partial_n(\tau) &&= \\ &\partial_{n+1}\gamma_n(t_{m\beta_i}\tau) + t_{m\beta_i}\partial_{n+1}\gamma_n\big(t_{\beta_i}(\tau) - \tau\big) \\ &\quad + \gamma_{n-1}t_{m\beta_i}\partial_n(\tau) + t_{m\beta_i}\gamma_{n-1}\big(t_{\beta_i}(\partial_n\tau) - \partial_n\tau\big) &&= \\ &t_{m\beta_i}(\tau) + t_{m\beta_i}\big(t_{\beta_i}(\tau) - \tau\big) &&= t_{(m+1)\beta_i}(\tau). \end{aligned}$$

Similarly, suppose that $t_{n_k\beta_k}(\sigma)$ and $t_{\beta-n_k\beta_k}(\sigma)$ both satisfy 1). It follows from 7') that

$$\begin{array}{ll}
(\partial_{n+1}\gamma_n+\gamma_{n-1}\partial_n)\big(t_\beta(\sigma)\big) & = \\
\partial_{n+1}\gamma_n(t_{n_k\beta_k}(\sigma))+\partial_{n+1}(t_{n_k\beta_k}\gamma_n(t_{\beta-n_k\beta_k}(\sigma)-\sigma)))+\gamma_{n-1}(t_\beta\partial_n(\sigma)) & = \\
\partial_{n+1}\gamma_n(t_{n_k\beta_k}(\sigma))+t_{n_k\beta_k}\partial_{n+1}\gamma_n(t_{\beta-n_k\beta_k}(\sigma)-\sigma)) & \\
\quad +\gamma_{n-1}(t_{n_k\beta_k}\partial_n(\sigma))+t_{n_k\beta_k}\gamma_{n-1}(t_{\beta-n_k\beta_k}(\partial_n\sigma)-\partial_n(\sigma))) & = \\
t_{n_k\beta_k}(\sigma)+t_{n_k\beta_k}(t_{\beta-n_k\beta_k}(\sigma)-\sigma)) & = \ t_\beta(\sigma).
\end{array}$$

Thus we can construct γ respecting all conditions, except possibly 3) and 4). The parallelogram $P_2 = 2\overline{P}_\emptyset$ consists of finitely many polysimplices, so there is a real number M such that

$$\gamma(\tau)=\sum_\sigma \gamma_{\tau\sigma}\sigma \quad \text{with} \quad |\gamma_{\tau\sigma}|<M$$

for all polysimplices $\tau \subset P_2$. Let us examine the size of the coefficients of $\gamma(t_{(m+1)\beta_i}(\sigma))$ for τ with interior in $P_\emptyset$. By induction to m we may suppose that

$$\gamma(t_{m\beta_i}(\tau))=\sum_\sigma \lambda^m_\sigma\sigma \quad \text{with} \quad |\lambda^m_\sigma|\begin{cases}
= & 0 & \text{if} & \sigma & \not\subset & A(t_{m\beta_i}(\tau)) \\
< & M & \text{if} & \sigma & \subset & P_2 \\
< & M & \text{if} & \sigma & \not\subset & A(t_{(m-1)\beta_i}(\tau)) \\
< & 3M & \text{if} & \sigma & \subset & A(t_{(m-1)\beta_i}(\tau)).
\end{cases} \tag{2.6}$$

By construction we have

$$t_{m\beta_i}\gamma(t_{\beta_i}(\tau)-\tau)=\sum_\sigma \lambda'_\sigma\sigma \quad \text{with} \quad |\lambda'_\sigma|\begin{cases}
= & 0 & \text{if} & \sigma & \not\subset & A(t_{(m+1)\beta_i}(\tau)) \\
= & 0 & \text{if} & \sigma & \not\subset & t_{m\beta_i}C^+_\emptyset \\
< & M & \text{if} & \sigma & \not\subset & A(t_{m\beta_i}(\tau)) \\
< & 2M & \text{if} & \sigma & \subset & A(t_{(m+1)\beta_i}(\tau)).
\end{cases}$$

With 6) this implies that (2.6) also holds with $m+1$ instead of m.

Let β be as above. By induction to k we may assume that

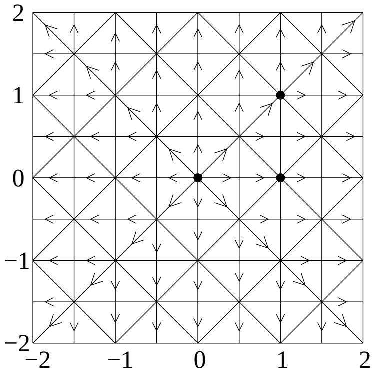
$$t_{n_k\beta_k}\gamma(t_{\beta-n_k\beta_k}(\tau)-\tau)=\sum_\sigma \mu^k_\sigma\sigma \quad \text{with} \quad |\mu^k_\sigma|\begin{cases}
= & 0 & \text{if} & \sigma & \not\subset & A(t_\beta(\sigma)) \\
= & 0 & \text{if} & \sigma & \not\subset & t_{n_k\beta_k}C^+_\emptyset \\
< & M & \text{if} & \sigma & \not\subset & t_{\beta'}(\sigma) \\
< & 2M & \text{if} & \sigma & \subset & A(t_{n_k\beta_k}(\sigma)) \\
< & 3M & \text{if} & \sigma & \subset & A(t_\beta(\sigma)).
\end{cases} \tag{2.7}$$

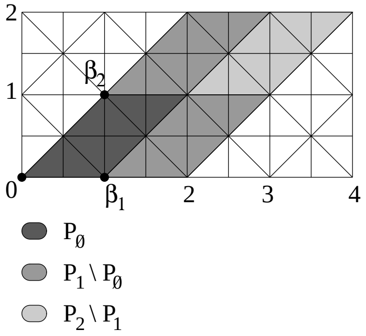
where $\beta' = \beta - \beta_i$ with i minimal for $n_i > 0$. In view of 7) the above implies that

$$\gamma(t_\beta(\tau))=\sum_\sigma \mu'_\sigma\sigma \quad \text{with} \quad |\mu'_\sigma|\begin{cases}
= & 0 & \text{if} & \sigma & \not\subset & A(t_\beta(\sigma)) \\
< & M & \text{if} & \sigma & \subset & P_2 \\
< & M & \text{if} & \sigma & \not\subset & A(t_{\beta'}(\sigma)) \\
< & 3M & \text{if} & \sigma & \subset & A(t_\beta(\sigma)).
\end{cases}$$

This in turn implies (2.7) with $k+1$ instead of k. Hence condition 4) is fulfilled, with $M_\gamma = 3M$. $\square$

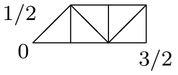
Example.
In the case $R_0 = B_2$ we have $\beta_1 = (1,0)$ and $\beta_2 = (1,1)$. We drew the sets $P_\emptyset$, P_1 and P_2 below. If x is a vertex of Σ then $\gamma[x]$ is a path from 0 to x, along the following lines:





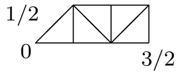
We define

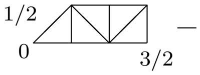
$$\begin{array}{lcccl} \gamma\left[(1/2,0),(1/2,1/2)\right] & = & A_\emptyset & = & [(0,0),(1/2,0),(1/2,1/2)] \\ \gamma\left[(1,1/2),(1,1)\right] & = & t_{(1/2,1/2)}A_\emptyset & = & [(1/2,1/2),(1,1/2),(1,1)] \\ \gamma\left[(3/2,1),(3/2,3/2)\right] & = & t_{(1,1)}A_\emptyset & = & [(1,1),(3/2,1),(3/2,3/2)] \end{array}$$

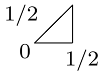
$$\gamma\left[(3/2,0),(3/2,1/2)\right] \quad =$$ 

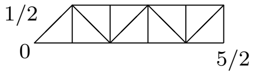
According to 6)

$$\gamma\left[(5/2,0),(5/2,1/2)\right] =$$
$$\gamma\left[(3/2,0),(3/2,1/2)\right] + t_{(1,0)}\gamma\Big(\left[(3/2,0),(3/2,1/2)\right] - \left[(1/2,0),(1/2,1/2)\right]\Big) =$$



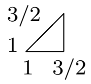
$+\ t_{(1,0)}\ \Big($ 

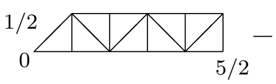
$-$ 

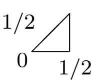
$\Big) =$ 

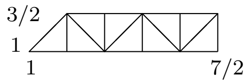
Condition 7) says that

$$\gamma\left[(7/2,1),(7/2,3/2)\right] =$$
$$\gamma\left[(3/2,1),(3/2,3/2)\right] + t_{(1,1)}\gamma\Big(\left[(5/2,0),(5/2,1/2)\right] - \left[(1/2,0),(1/2,1/2)\right]\Big) =$$



$+\ t_{(1,1)}\ \Big($ 

$-$ 

$\Big) =$ 

2.2 Projective resolutions for affine Hecke algebras

For $(\pi, V) \in \mathrm{Mod}(\mathcal{H})$ and $n \in \mathbb{N}$ we consider the $\mathcal{H}$-module

$$P_n(V) := \bigoplus_{f:\dim f = n} \mathcal{H} \otimes_{\mathcal{H}(W_f,q)\otimes\mathbb{C}[Z(W)]} V \otimes_{\mathbb{C}} \mathbb{C}\{f\} = \bigoplus_{f:\dim f = n} \mathcal{H} \otimes_{\mathcal{H}(W_f,q)\otimes\mathbb{C}[Z(W)]} V.$$

where the sum runs over facets of $A_\emptyset$. Recall that we already fixed an (arbitrary) orientation of all these facets. Hence we can express the boundary of a polysimplex f as

$$\partial(f) = \textstyle\sum_{f'} [f : f']\, f',$$

with suitable numbers $[f : f'] \in \{-1, 0, 1\}$. We define $\mathcal{H}$-module homomorphisms

$$\begin{aligned} &d_n : P_n(V) \to P_{n-1}(V), \\ &d_n\big(h \otimes_{\mathcal{H}(W_f,q)\otimes\mathbb{C}[Z(W)]} v \otimes_{\mathbb{C}} f\big) = \sum_{f' : \dim f' = n-1} h \otimes_{\mathcal{H}(W_{f'},q)\otimes Z(W)} v \otimes_{\mathbb{C}} [f : f'] f'. \end{aligned} \tag{2.8}$$

Furthermore we define

$$\begin{aligned} &d_0 : P_0(V) \to V, \\ &d_0\big(h \otimes_{\mathcal{H}(W_x,q)\otimes\mathbb{C}[Z(W)]} v \otimes_{\mathbb{C}} x\big) = \pi(h)v, \end{aligned} \tag{2.9}$$

if x is a vertex of $A_\emptyset$. Now $\big(P_*(V), d_*\big)$ is an augmented differential complex because $\partial \circ \partial = 0$. The group Ω acts naturally on this complex by

$$\omega(h \otimes_{\mathcal{H}(W_f,q)\otimes\mathbb{C}[Z(W)]} v \otimes f) = hT_\omega^{-1} \otimes_{\mathcal{H}(W_{\omega(f)},q)\otimes\mathbb{C}[Z(W)]} \pi(T_\omega)v \otimes \omega(f),$$

where we consider $\omega(f)$ with orientation. This action commutes with the $\mathcal{H}$-action and with the differentials d_n, so $\big(P_*(V)^\Omega, d_*\big)$ is again an augmented differential complex. Note that $P_n(V)$ and $P_n(V)^\Omega$ are finitely generated $\mathcal{H}$-modules if V has finite dimension.

Theorem 2.2. *Consider $\mathcal{H}$ as a $\mathcal{H}$-bimodule.*

$$0 \longleftarrow \mathcal{H} \xleftarrow{d_0} P_0(\mathcal{H})^\Omega \xleftarrow{d_1} P_1(\mathcal{H})^\Omega \longleftarrow \cdots \xleftarrow{d_r} P_r(\mathcal{H})^\Omega \longleftarrow 0 \tag{2.10}$$

is a resolution of $\mathcal{H}$ by $\mathcal{H} \otimes \mathcal{H}^{op}$-modules. Every $P_n(\mathcal{H})^\Omega$ is projective as a left and as a right $\mathcal{H}$-module. Moreover if $\mathcal{R}$ is semisimple then $P_n(\mathcal{H})^\Omega$ is projective as a $\mathcal{H} \otimes \mathcal{H}^{op}$-module.

Proof. This result stems from joint work of Mark Reeder and the first author, see [Opd2, Proposition 8.1]. The proof is based on constructions of Kato [Kat1].

First we consider the case $\Omega = Z(W) = \{e\}$, $W = W^{\mathrm{aff}}$. There is a linear bijection

$$\begin{aligned} &\phi : \mathbb{C}[W] \otimes_{\mathbb{C}} \mathcal{H} \to \mathcal{H} \otimes_{\mathbb{C}} \mathcal{H}, \\ &\phi(w \otimes h') = T_w \otimes T_w^{-1} h'. \end{aligned} \tag{2.11}$$

For $s_i \in S_{\mathrm{aff}}$ we write $q_i = q(s_i)$ and

$$\begin{array}{rcll} L_i & := & \mathrm{span}\{hT_{s_i} \otimes T_{s_i}^{-1}h' - h \otimes h' : h, h' \in \mathcal{H}\} & \subset \quad \mathcal{H} \otimes_{\mathbb{C}} \mathcal{H}, \\ \mathbb{C}[W]_i & := & \big\{ \sum_{w\in W} x_w w : x_{ws_i} = -x_w \;\forall w \in W \big\} & \subset \quad \mathbb{C}[W]. \end{array} \tag{2.12}$$

This L_i is interesting because

$$\mathcal{H} \otimes_{\mathcal{H}(W_f,q)} \mathcal{H} = \big(\mathcal{H} \otimes_{\mathbb{C}} \mathcal{H}\big) / \textstyle\sum_{s_i \in W_f} L_i.$$

Let $w \in W$ be such that $\ell(ws_i) > \ell(w)$. For any $h' \in \mathcal{H}$ we have

$$\phi((ws_i - w) \otimes h') = T_{ws_i} \otimes T_{ws_i}^{-1} h' - T_w \otimes T_w^{-1} h' = T_w T_{s_i} \otimes T_{s_i}^{-1} T_w^{-1} h' - T_w \otimes T_w^{-1} h' \in L_i,$$

so $\phi(\mathbb{C}[W]_i \otimes \mathcal{H}) \subset L_i$. On the other hand, L_i is spanned by elements as in (2.12) with $h = T_w$ or $h = T_{ws_i}$.

$$\begin{array}{lr}
\phi^{-1}(T_{ws_i} T_{s_i} \otimes T_{s_i}^{-1} h' - T_{ws_i} \otimes h') & = \\
\phi^{-1}(q_i T_w + (q_i - 1) T_{ws_i} \otimes T_{s_i}^{-1} h') - ws_i \otimes T_{ws_i} h' & = \\
q_i w \otimes T_w T_{s_i}^{-1} h' + (q_i - 1) ws_i \otimes T_{ws_i} T_{s_i}^{-1} h' - ws_i \otimes T_{ws_i} h' & = \\
q_i (w - ws_i) \otimes T_w T_{s_i}^{-1} h' + ws_i \otimes \big(q_i T_w T_{s_i}^{-1} + (q_i - 1) T_{ws_i} T_{s_i}^{-1} - T_{ws_i} \big) h' & = \\
(w - ws_i) \otimes T_w q_i T_{s_i}^{-1} h' + ws_i \otimes \big(T_w (T_{s_i} + 1 - q_i) + (q_i - 1) T_w - T_w T_{s_i} \big) h' & = \\
(w - ws_i) \otimes T_w (T_{s_i} + 1 - q_i) h' \qquad \in \qquad \mathbb{C}[W]_i \otimes \mathcal{H}. &
\end{array}$$

We conclude that $\phi^{-1}(L_i) = \mathbb{C}[W]_i \otimes \mathcal{H}$. Now we bring the linear bijections

$$\mathbb{C}[W] / \textstyle\sum_{s_i \in W_f} \mathbb{C}[W]_i \to \mathbb{C}[W/W_f] : w \mapsto w W_f. \tag{2.13}$$

into play. Under these identifications our differential complex becomes

$$0 \leftarrow \mathcal{H} \leftarrow \cdots \leftarrow \bigoplus_{f : \dim f = n} \mathbb{C}[W/W_f] \otimes \mathcal{H} \otimes \mathbb{C}\{f\} \leftarrow \cdots \leftarrow \mathbb{C}[W] \otimes \mathcal{H} \otimes \mathbb{C}\{A_\emptyset\} \leftarrow 0.$$

But this is just the complex $\big(C_*(\Sigma), \partial_*\big)$ tensored with $\mathcal{H}$, so by (2.3) its homology vanishes. This shows that indeed we have a resolution in the special case $\Omega = \{e\}$.

Now the general case. Since the action of Ω on $A_\emptyset$ factors through the finite group $\Omega / Z(W)$ we can construct a Reynolds operator

$$R_\Omega := [\Omega : Z(W)]^{-1} \sum_{\omega \in \Omega / Z(W)} \omega \quad \in \quad \mathrm{End}_{\mathcal{H} \otimes \mathcal{H}^{op}} \big(P_n(\mathcal{H}) \big).$$

Since this is an idempotent,

$$P_n(\mathcal{H})^\Omega = R_\Omega \cdot P_n(\mathcal{H}) \tag{2.14}$$

is a direct summand of $P_n(\mathcal{H})$. We generalize (2.11) to a bijection

$$\begin{aligned}
&\phi : \mathbb{C}[W/Z(W)] \otimes_{\mathbb{C}} \mathcal{H} \to \mathcal{H} \otimes_{\mathbb{C}[Z(W)]} \mathcal{H}, \\
&\phi(w \otimes h') = T_w \otimes T_w^{-1} h'.
\end{aligned} \tag{2.15}$$

Just as above this leads to bijections

$$\textstyle\bigoplus_{f : \dim f = n} \mathbb{C}[W/(W_f \times Z(W))] \otimes \mathcal{H} \otimes \mathbb{C}\{f\} \to P_n(\mathcal{H}). \tag{2.16}$$

The group $\Omega / Z(W)$ acts on the left hand side by

$$\omega \cdot (w \otimes h' \otimes f) = w \omega^{-1} \otimes h' \otimes \omega(f).$$

Since both sides of (2.16) are free $\Omega / Z(W)$-modules, we also get a linear bijection

$$\begin{aligned}
&\textstyle\bigoplus_{f : \dim f = n} \mathbb{C}\big[W^{\mathrm{aff}} / W_f\big] \otimes \mathcal{H} \otimes \mathbb{C}\{f\} \to P_n(\mathcal{H})^\Omega, \\
&w \otimes h' \otimes f \mapsto R_\Omega \big(T_w \otimes_{\mathcal{H}(W_f, q) \otimes \mathbb{C}[Z(W)]} T_w^{-1} h' \otimes f \big).
\end{aligned} \tag{2.17}$$

Now the same argument as in the special case shows that the modules $P_n(\mathcal{H})^\Omega$ form a resolution of $\mathcal{H}$.

For any facet f, $\mathcal{H}$ is a free $\mathcal{H}(W_f,q)\otimes\mathbb{C}[Z(W)]$-module, both from the left and from the right. Therefore every $P_n(\mathcal{H})$ is a projective $\mathcal{H}$-module, from the left and from the right.

For $\mathcal{R}$ semisimple $P_n(\mathcal{H})$ is a direct sum of $\mathcal{H}\otimes\mathcal{H}^{op}$-modules of the form $\mathcal{H}\otimes_{\mathcal{H}(W_f,q)}\mathcal{H}$. For every irreducible representation V_i of $\mathcal{H}(W_f,q)$ we pick an idempotent $e_i\in\mathcal{H}(W_f,q)$ which acts as a rank one projection on V_i and as 0 on all other irreducible representations. Consider the element $e_f=\sum_i e_i\otimes e_i\in\mathcal{H}\otimes\mathcal{H}^{op}$. From

$$\mathcal{H}\otimes_{\mathcal{H}(W_f,q)}\mathcal{H}\cong\big(\mathcal{H}\otimes_{\mathbb{C}}\mathcal{H}^{op}\big)e_f \tag{2.18}$$

we see that $P_n(\mathcal{H})$ is a projective $\mathcal{H}$-bimodule. By (2.14) $P_n(\mathcal{H})^\Omega$ is projective in the same senses as $P_n(\mathcal{H})$. $\Box$

Corollary 2.3. a) *Let V be any $\mathcal{H}$-module.*

$$0\longleftarrow V\xleftarrow{d_0}P_0(V)^\Omega\xleftarrow{d_1}P_1(V)^\Omega\longleftarrow\cdots\xleftarrow{d_r}P_r(V)^\Omega\longleftarrow 0$$

is a resolution of V. It is bornological if V is.

b) *If V admits a $Z(W)$-character χ then every $P_n(V)^\Omega$ is a projective $\mathcal{H}(\mathcal{R},q)_\chi$-module.*

c) *The cohomological dimensions of* $\mathrm{Mod}\big(\mathcal{H}(\mathcal{R},q)_\chi\big)$ *and* $\mathrm{Mod}_{\mathrm{bor}}\big(\mathcal{H}(\mathcal{R},q)_\chi\big)$ *equal $r=\mathrm{rk}(R_0)$.*

Proof. a) Apply $\otimes_{\mathcal{H}}V$ to (2.10). The resulting differential complex is exact because $\mathcal{H}$ and $P_n(\mathcal{H})^\Omega$ are projective right $\mathcal{H}$-modules. For $V\in\mathrm{Mod}_{\mathrm{bor}}(\mathcal{H})$ this clearly gives a bornological differential complex. It is split exact because every contraction of $P_*(\mathcal{H})^\Omega$ yields a bounded splitting of $P_*(V)^\Omega$.
b) From

$$\mathcal{H}\otimes_{\mathcal{H}(W_f,q)\otimes\mathbb{C}[Z(W)]}V\cong\mathcal{H}(\mathcal{R},q)_\chi\otimes_{\mathcal{H}(W_f,q)}V\cong\mathrm{Ind}_{\mathcal{H}(W_f,q)}^{\mathcal{H}(\mathcal{R},q)_\chi}V \tag{2.19}$$

we see that this a projective $\mathcal{H}(\mathcal{R},q)_\chi$-module. Hence $P_n(V)$ also has this property. It follows from (2.14) that

$$P_n(V)^\Omega=R_\Omega\cdot P_n(V) \tag{2.20}$$

is a direct summand of $P_n(V)$.
c) By a) and b) these cohomological dimensions are at most r. On the other hand, we can easily find modules which do not have projective resolutions of length smaller than r. Note that

$$\mathcal{A}_\chi:=\mathcal{A}\otimes_{Z(W)}\mathbb{C}_\chi\cong\mathcal{O}(T_\chi),$$

where T_χ is the r-dimensional subtorus of T consisting of elements t such that $t\big|_{Z(W)}=\chi$. Pick $t\in T_\chi$ and consider the parabolically induced module

$$I_t=\mathrm{Ind}_{\mathcal{A}}^{\mathcal{H}}(\mathbb{C}_t)=\mathrm{Ind}_{\mathcal{A}_\chi}^{\mathcal{H}(\mathcal{R},q)_\chi}(\mathbb{C}_t). \tag{2.21}$$

With Theorem 1.3 we find

$$\mathrm{Ext}^r_{\mathcal{H}(\mathcal{R},q)_\chi}(I_t,I_t) \cong \mathrm{Ext}^r_{\mathcal{A}_\chi}(\mathbb{C}_t,I_t) \cong \mathrm{Ext}^r_{\mathcal{O}(T_\chi)}\Big(\mathbb{C}_t, \bigoplus_{w\in W_0} \mathbb{C}_{wt}\Big) \cong \bigoplus_{w\in W_0 : wt=t} \mathbb{C}_{wt}. \tag{2.22}$$

Since this space is not 0, any resolution of I_t by projective $\mathcal{H}(\mathcal{R},q)_\chi$-modules has length at least r.

This calculation goes through in the bornological setting, if we endow all spaces with the fine bornology. $\qquad \Box$

For purposes of homological algebra it would be useful if we could also construct projective resolutions for $\mathcal{H}$-modules that do not admit a $Z(W)$-character. Unfortunately the authors do not know how to achieve this in general. But we offer an alternative that comes quite close. Let

$$\widetilde{\mathcal{H}} := \mathcal{H}(\widetilde{\mathcal{R}},\tilde{q}) = \tilde{G} \ltimes \mathcal{H}(\mathcal{R},q)$$

be a semisimple affine Hecke algebra as in (1.3). Obviously $\widetilde{\mathcal{H}}$ is a free (left or right) $\mathcal{H}$-module with basis $\{T_g : g \in \widetilde{G}\}$. Moreover for $(\pi,V) \in \mathrm{Mod}(\mathcal{H})$ the $\widetilde{\mathcal{H}}$-module

$$\mathrm{Ind}_{\mathcal{H}}^{\widetilde{\mathcal{H}}} V = \widetilde{\mathcal{H}} \otimes_{\mathcal{H}} V \tag{2.23}$$

is isomorphic as an $\mathcal{H}$-module to $\bigoplus_{g\in\widetilde{G}} V_g$, where the $\mathcal{H}$-module structure on $V_g = (\pi_g, V)$ is given by

$$\pi_g(h)\, v = \pi(T_g^{-1} h T_g)\, v. \tag{2.24}$$

Clearly $V_g = V$ as an $\mathcal{H}(\mathcal{R}_{F_0},q)$-module. If V admits a $Z(W)$-character χ, then V_g differs only from V in the sense that its $Z(W)$-character is $g\chi$.

Applying the construction of Corollary 2.3.a) to $\widetilde{\mathcal{H}} \otimes_{\mathcal{H}} V$ as an $\widetilde{\mathcal{H}}$-module we get a resolution by modules that are projective in $\mathrm{Mod}\big(\widetilde{\mathcal{H}}\big)$ and in $\mathrm{Mod}(\mathcal{H})$. In several cases this might be used to find a resolution of (π,V) by projective $\mathcal{H}$-modules.

Proposition 2.4. *The cohomological dimensions of* $\mathrm{Mod}(\mathcal{H})$ *and* $\mathrm{Mod}_{\mathrm{bor}}(\mathcal{H})$ *are both equal to the rank of* X*.*

Proof. The cohomological dimension of $\mathrm{Mod}(\mathcal{H})$ is the least number $d \in \{0,1,2,\cdots,\infty\}$ such that

$$\mathrm{Ext}^n_{\mathcal{H}}(U,V) = 0 \qquad \forall\, U,V \in \mathrm{Mod}(\mathcal{H})\,, \forall n > d.$$

Let $t \in T$ and consider the module $I_t = \mathrm{Ind}_{\mathcal{A}}^{\mathcal{H}}(\mathbb{C}_t)$. In view of Theorem 1.3

$$\mathrm{Ext}^{\mathrm{rk}(X)}_{\mathcal{H}}(I_t,I_t) \cong \mathrm{Ext}^{\mathrm{rk}(X)}_{\mathcal{A}}(\mathbb{C}_t,I_t) \cong \mathrm{Ext}^{\mathrm{rk}(X)}_{\mathcal{O}(T)}\Big(\mathbb{C}_t, \bigoplus_{w\in W_0} \mathbb{C}_{wt}\Big) \cong \bigoplus_{w\in W_0 : wt=t} \mathbb{C}_{wt}.$$

Therefore $d \geq \mathrm{rk}(X)$. This argument also works in $\mathrm{Mod}_{\mathrm{bor}}(\mathcal{H})$, provided that we endow all spaces with the fine bornology.

On the other hand, let $U, V \in \mathrm{Mod}(\mathcal{H})$ be arbitrary and consider the $\widetilde{\mathcal{H}}$-modules $\mathrm{Ind}_{\mathcal{H}}^{\widetilde{\mathcal{H}}}(U)$ and $\mathrm{Ind}_{\mathcal{H}}^{\widetilde{\mathcal{H}}}(V)$.

$$\begin{aligned} \mathrm{Ext}^n_{\mathcal{H}}(U,V) \quad \subset \quad & \textstyle\bigoplus_{g \in \tilde{G}} \mathrm{Ext}^n_{\mathcal{H}}\big(U, V_g\big) \;\cong\; \mathrm{Ext}^n_{\mathcal{H}}\big(U, \mathrm{Ind}_{\mathcal{H}}^{\widetilde{\mathcal{H}}}(V)\big) \cong \\ & \mathrm{Ext}^n_{\widetilde{\mathcal{H}}}\big(\mathrm{Ind}_{\mathcal{H}}^{\widetilde{\mathcal{H}}}(U), \mathrm{Ind}_{\mathcal{H}}^{\widetilde{\mathcal{H}}}(V)\big). \end{aligned} \tag{2.25}$$

Assume $n > \mathrm{rk}(X)$. According to Corollary 2.3.c) the cohomological dimension of $\mathrm{Mod}(\widetilde{\mathcal{H}})$ is $\mathrm{rk}(X)$, so right hand side of (2.25) is 0. Hence $\mathrm{Ext}^n_{\mathcal{H}}(U,V) = 0$ and we conclude that $d \leq \mathrm{rk}(X)$. The same reasoning shows that the cohomological dimension of $\mathrm{Mod}_{\mathrm{bor}}(\mathcal{H})$ is $\mathrm{rk}(X)$. $\square$

Recall that a resolution (P_*, d_*) of a module V is of finite type if all the modules P_n are finitely generated, and moreover $P_n = 0$ for all n larger than some number.

Corollary 2.5. *Let V be a finitely generated $\mathcal{H}$-module. Then V admits a finite type projective resolution.*

Proof. Because $\mathcal{H}$ is Noetherian, every submodule of a finitely generated $\mathcal{H}$-module is itself finitely generated.

By assumption there exist a surjective $\mathcal{H}$-module map $d_0 : \mathcal{H}^{m_0} \to V$, for some $m_0 \in \mathbb{N}$. Then ker d_0 is again finitely generated, so we can find a surjection $d_1 : \mathcal{H}^{m_1} \to \ker d_0$. Continuing this process we construct a resolution $(P_n = \mathcal{H}^{m_n}, d_n)$ of V, consisting of free $\mathcal{H}$-modules of finite rank. Because the global dimension of $\mathcal{H}$ is $\mathrm{rk}(X)$, the module ker d_n must be projective $\forall n \geq \mathrm{rk}(X) - 1$ [CaEi, Proposition VI.2.1]. Hence

$$0 \leftarrow V \xleftarrow{d_0} P_0 \xleftarrow{d_1} \cdots \xleftarrow{d_{n-1}} P_{n-1} \leftarrow \ker d_{n-1} \leftarrow 0$$

is a finite type projective resolution of V. $\square$

2.3 Projective resolutions for Schwartz algebras

We will show that all the resolutions from the previous section can be induced from $\mathcal{H}$ to $\mathcal{S}$. Most importantly, we will construct a projective bimodule resolution of $\mathcal{S}$. This requires that we complete the $\mathcal{H}$-modules to Fréchet $\mathcal{S}$-modules. A convenient technique to achieve this in great generality is with completed bornological tensor products, and this is the viewpoint we chose to take in this section. The necessary background material is contained in the Appendix. However, for finite dimensional tempered modules it is not necessary to use bornologies. See the remark after Corollary 2.7.

Endow $\mathcal{S}$ with the precompact bornology and let V be a bornological $\mathcal{S}$-module. According to [Mey2, Theorem 42] we have

$$\mathcal{S}(Z(W)) \widehat{\otimes}_{\mathbb{C}[Z(W)]} V = \mathcal{S}(Z(W)) \widehat{\otimes}_{\mathcal{S}(Z(W))} V. \tag{2.26}$$

If V has finite dimension, then (2.26) also holds with algebraic tensor products. The reader is invited to check this, by reduction to the case where V admits a unique $Z(W)$-character.

Because $\mathcal{H}$ is a free $\mathcal{H}(W_f,q)\otimes\mathbb{C}[Z(W)]$-module, both algebraically and with the fine bornology, we have

$$\mathcal{H}\widehat{\otimes}_{\mathcal{H}(W_f,q)\otimes\mathbb{C}[Z(W)]}V=\mathcal{H}\otimes_{\mathcal{H}(W_f,q)\otimes\mathbb{C}[Z(W)]}V. \tag{2.27}$$

So if we induce $P_n(V)$ from $\mathcal{H}$ to $\mathcal{S}$ in the bornological fashion we get the module

$$\begin{aligned} P_n^t(V) \;:=\; \mathcal{S}\widehat{\otimes}_{\mathcal{H}}P_n(V) \;&=\; \mathcal{S}\widehat{\otimes}_{\mathcal{H}} \bigoplus_{f:\dim f=n} \mathcal{H}\widehat{\otimes}_{\mathcal{H}(W_f,q)\otimes\mathbb{C}[Z(W)]}V\otimes_{\mathbb{C}}\mathbb{C}\{f\} \\ &= \bigoplus_{f:\dim f=n} \mathcal{S}\widehat{\otimes}_{\mathcal{H}(W_f,q)\otimes\mathcal{S}(Z(W))}V\otimes_{\mathbb{C}}\mathbb{C}\{f\}. \end{aligned} \tag{2.28}$$

The maps $d_n:P_n(V)\to P_{n-1}(V)$ extend naturally to

$$d_n^t:P_n^t(V)\to P_{n-1}^t(V).$$

The action of Ω on $P_n(V)$ also extends to $P_n^t(V)$, so we can construct $P_n^t(V)^\Omega$. By (2.14)

$$P_n^t(V)^\Omega=R_\Omega\cdot P_n^t(V) \tag{2.29}$$

is a direct summand of $P_n^t(V)$. Clearly $P_n^t(V)$ and $P_n^t(V)^\Omega$ are finitely generated $\mathcal{S}$-modules if V has finite dimension.

We consider the important case $V=\mathcal{S}$. The topology and the bornology on $\mathcal{S}$ give rise to a topology and a bornology on $P_n^t(\mathcal{S})$. For $n,m,k\in\mathbb{N}\,,\,f\subset\overline{A_\emptyset}$ we have the continuous seminorms

$$\begin{aligned} &p_{m,k,f}:\mathcal{S}\widehat{\otimes}_{\mathcal{H}(W_f,q)\otimes\mathcal{S}(Z(W))}\mathcal{S}\otimes_{\mathbb{C}}\mathbb{C}\{f\}\to[0,\infty),\\ &p_{m,k,f}(y)=\inf\Big\{\sum_i p_m(h_i)p_k(h_i'):\sum_i h_i\otimes h_i'\otimes f=y\Big\}, \end{aligned}$$

which define a Fréchet topology on this space. The topology on $P_n^t(\mathcal{S})$ is defined by the norms $p_{m,k}:=\sum_f p_{m,k,f}$. We endow these modules with the precompact bornology. We note that d_n^t is continuous and bounded and that $P_n(\mathcal{S})$ is dense in $P_n^t(\mathcal{S})$.

In view of (2.18) we have

$$P_n^t(\mathcal{S})^\Omega=\bigoplus_{f:dim f=n}\mathcal{S}(\mathcal{R}_{F_0},q)\widehat{\otimes}_{\mathcal{H}(W_f,q)}\mathcal{S}(\mathcal{R},q)\cong\bigoplus_{f:dim f=n}\Big(\mathcal{S}(\mathcal{R}_{F_0},q)\widehat{\otimes}_{\mathbb{C}}\mathcal{S}(\mathcal{R},q)^{op}\Big)e_f.$$

Using Lemma 1.5 and Theorem 1.6 both for $\mathcal{S}(\mathcal{R}_{F_0},q)$ and for $\mathcal{S}(\mathcal{R},q)$ we see that there is a number $C_{m,k,f}>0$ such that

$$\begin{array}{ll} \sum_{w\in W^{\mathrm{aff}},w'\in W}|h_{w,w'}|(\mathcal{N}(w)+1)^m(\mathcal{N}(w')+1)^k & \leq \\ C_b^2\sup_{w\in W^{\mathrm{aff}},w'\in W}|h_{w,w'}|(\mathcal{N}(w)+1)^{m+b}(\mathcal{N}(w')+1)^{k+b} & \leq \\ C_b^2 p_{m+b,k+b}\Big(\sum_{w\in W^{\mathrm{aff}},w'\in W}h_{w,w'}(N_w\otimes N_w')e_f\otimes f\Big) & \leq \\ C_{m,k,f}\,p_{m+2b,k+2b,f}\big(\sum_{w\in W^{\mathrm{aff}}}\sum_{w'\in W}h_{w,w'}N_w\otimes N_{w'}\big). & \end{array} \tag{2.30}$$

Theorem 2.6. *Consider $\mathcal{S}$ as a $\mathcal{S}$-bimodule.*

$$0 \longleftarrow \mathcal{S} \xleftarrow{d_0^t} P_0^t(\mathcal{S})^\Omega \xleftarrow{d_1^t} P_1^t(\mathcal{S})^\Omega \longleftarrow \cdots \xleftarrow{d_r^t} P_r^t(\mathcal{S})^\Omega \longleftarrow 0 \tag{2.31}$$

is a $\mathcal{S}\widehat{\otimes}\mathcal{S}^{op}$-module resolution of $\mathcal{S}$, with a continuous bounded contraction. Every $P_n^t(\mathcal{S})$ is a bornologically projective $\mathcal{S}$-module, both from the left and from the right. If moreover $\mathcal{R}$ is semisimple, then $P_n^t(\mathcal{S})$ is also projective as a $\mathcal{S}\widehat{\otimes}\mathcal{S}^{op}$-module.

Proof. To show that the differential complex $\big(P_*^t(\mathcal{S})^\Omega, d_*^t\big)$ is contractible we use Proposition 2.1 and Theorem 2.2. The composition of (2.17) with (2.2) is the bijection

$$\begin{aligned}
&\tilde\phi : C_*(\Sigma) \otimes_{\mathbb{C}} \mathcal{S} \to P_*(\mathcal{S})^\Omega,\\
&\tilde\phi(\sigma \otimes h') = R_\Omega\big(T_w \otimes_{\mathcal{H}(W_f,q)\otimes\mathbb{C}[Z(W)]} T_w^{-1} h' \otimes f\big),
\end{aligned}$$

where $\sigma = w\overline{f}$ with $w \in W^{\mathrm{aff}}$. Let γ be as in Proposition 2.1. We claim that

$$\tilde\gamma := \tilde\phi(\gamma \otimes \mathrm{id}_{\mathcal{S}})\tilde\phi^{-1}$$

extends continuously to the required contraction. Suppose that $w' \in W$, $w \in W^{\mathrm{aff}} \cap w'\Omega$ and $\sigma = w'\overline{f} = w\overline{f}$. Then we have explicitly

$$\tilde\phi\big(R_\Omega(N_{w'} \otimes_{\mathcal{H}(W_f,q)\otimes\mathbb{C}[Z(W)]} h' \otimes f)\big) = \tilde\phi\big(\gamma(\sigma)\otimes N_w h'\big) = \tilde\phi\big(\textstyle\sum_\tau \gamma_{\sigma\tau}\tau\otimes N_w h'\big). \tag{2.32}$$

By Lemma 1.2 and condition 3) of Proposition 2.1 the coefficient $\gamma_{\sigma\tau}$ can only be nonzero if there exist $u \leq_A w$ and a facet f' of $A_\emptyset$ such that $\tau = u\overline{f'}$. This crucial for the following estimates. For every relevant τ we pick such a $u \in W^{\mathrm{aff}}$ and we write (a little sloppily) $\gamma_{wu} = \gamma_{\sigma\tau}$. Then (2.32) equals

$$\begin{array}{lr}
\tilde\phi\Big(\sum_{f'}\sum_{u\in W^{\mathrm{aff}}:u\leq_A w} \gamma_{wu}(u\overline{f'}) \otimes N_w h'\Big) & = \\
R_\Omega\Big(\sum_{f'}\sum_{u\in W^{\mathrm{aff}}:u\leq_A w} \gamma_{wu} N_u \otimes_{\mathcal{H}(W_{f'},q)\otimes\mathbb{C}[Z(W)]} N_u^{-1} N_w h' \otimes f'\Big) & = \\
\sum_{f'}\sum_{u\in W^{\mathrm{aff}}:u\leq_A w} R_\Omega\big(\gamma_{wu} N_u \otimes_{\mathcal{H}(W_{f'},q)\otimes\mathbb{C}[Z(W)]} N_{u^{-1}w} h' \otimes f'\big). &
\end{array} \tag{2.33}$$

Notice that we used $u \leq_A w$ in the last step. Every element of $P_n^t(\mathcal{S})^\Omega$ can be written as a finite sum (over facets f) of elements of the form

$$R_\Omega\, y = R_\Omega \sum_{w\in W^{\mathrm{aff}}} \sum_{w'\in W} h_{w,w'} N_w \otimes_{\mathcal{H}(W_{f'},q)\otimes\mathcal{S}(Z(W))} N_{w'} \otimes f,$$

with $(h_{w,w'}) \in \mathcal{S}(W^{\mathrm{aff}} \times W)$. According to the above calculation

$$\tilde\gamma(R_\Omega\, y) = R_\Omega \sum_{f'} \sum_{w'\in W} \sum_{u,w\in W^{\mathrm{aff}}:u\leq_A w} \gamma_{wu} h_{w,w'} N_u \otimes_{\mathcal{H}(W_{f'},q)\otimes\mathcal{S}(Z(W))} N_{u^{-1}w} N_{w'} \otimes f'.$$

Using (in this order) condition 4) of Proposition 2.1, Theorem 1.6, Lemma 1.1 and

(2.30) we estimate

$$\begin{aligned}
& p_{m,k}\Big(\sum_{w' \in W} \sum_{u,w \in W^{\mathrm{aff}} : u \leq_A w} \gamma_{wu} h_w N_u \otimes_{\mathcal{H}(W_{f'},q) \otimes \mathcal{S}(Z(W))} N_{u^{-1}w} h' \otimes f' \Big) && \leq \\
& \sum_{w' \in W} \sum_{w \in W^{\mathrm{aff}}} M_\gamma |h_{w,w'}| p_m \big(\sum_{u \in W^{\mathrm{aff}} : u \leq_A w} N_u \big) p_k \big(N_{u^{-1}w} N_{w'} \big) && \leq \\
& M_\gamma \sum_{w' \in W} \sum_{w \in W^{\mathrm{aff}}} |h_{w,w'}| (\mathcal{N}(w) + 1)^m C_q (\mathcal{N}(w) + 1)^{k+b} (\mathcal{N}(w') + 1)^{k+b} && \leq \\
& M_\gamma C_q C_{k+m+2b,k+b,f} p_{k+m+3b,k+2b}(y).
\end{aligned}$$

Since R_Ω is a continuous operator on $P_n^t(\mathcal{S})$, it follows that $\tilde{\gamma}$ is well-defined and continuous on $P_n^t(\mathcal{S})^\Omega$. Since $P_n^t(\mathcal{S})$ carries the precompact bornology, $\tilde{\gamma}$ is automatically bounded. Moreover

$$\tilde{\phi}(\delta_n \otimes \mathrm{id}_{\mathcal{S}}) \tilde{\phi}^{-1} = d_n,$$

so condition 1) of Proposition 2.1 assures that

$$\tilde{\gamma} d^t + d^t \tilde{\gamma} = \text{ id} \tag{2.34}$$

on $P_*(\mathcal{S})^\Omega$. Because $P_*(\mathcal{S})^\Omega$ is dense in $P_*^t(\mathcal{S})^\Omega$ and the maps in (2.34) are continuous, this relation holds on the whole of $P_*^t(\mathcal{S})^\Omega$. So the differential complex $\big(P_*^t(\mathcal{S})^\Omega, d_*^t\big)$ indeed has a bounded contraction.

For any facet f the space $\mathcal{S}$ is a bornologically free $\mathcal{H}(W_f, q) \otimes \mathcal{S}(Z(W))$-module. Hence $P_n^t(\mathcal{S})$ is a bornologically projective $\mathcal{S}$-module, both from the left and from the right. If $\mathcal{R}$ is semisimple, then by (2.18) $P_n^t(V)$ is direct sum of bimodules of the form $(\mathcal{S} \widehat{\otimes} \mathcal{S}^{op}) e_f$. Hence $P_n^t(V)$ is $\mathcal{S} \widehat{\otimes} \mathcal{S}^{op}$-projective.

By (2.29) $P_n^t(\mathcal{S})^\Omega$ enjoys the same projectivity properties. $\square$

Corollary 2.7. a) *Let V be any bornological $\mathcal{S}$-module.*

$$0 \longleftarrow V \xleftarrow{d_0^t} P_0^t(V)^\Omega \xleftarrow{d_1^t} P_1^t(V)^\Omega \longleftarrow \cdots \xleftarrow{d_r^t} P_r^t(V)^\Omega \longleftarrow 0$$

is a bornological resolution of V.

b) *If V admits the $Z(W)$-character χ, then every module $P_n^t(V)^\Omega$ is projective in* $\mathrm{Mod}_{\mathrm{bor}}\big(\mathcal{S}(\mathcal{R}, q)_\chi\big)$.

c) *If moreover V has finite dimension, then $P_n^t(V)^\Omega$ is also projective in* $\mathrm{Mod}\big(\mathcal{S}(\mathcal{R}, q)_\chi\big)$.

Proof. a) Apply $\otimes_{\mathcal{S}} V$ to (2.31) and use the projectivity of $P_n^t(\mathcal{S})^\Omega$ as a right $\mathcal{S}$-module.
b) From Corollary 2.3.b) we know that $P_n(V)^\Omega$ is projective in $\mathrm{Mod}_{\mathrm{bor}}\big(\mathcal{H}(\mathcal{R}, q)_\chi\big)$, so

$$P_n^t(V)^\Omega \cong \mathcal{S}(\mathcal{R}, q)_\chi \widehat{\otimes}_{\mathcal{H}(\mathcal{R},q)_\chi} P_n(V)^\Omega$$

is projective in $\mathrm{Mod}_{\mathrm{bor}}\big(\mathcal{S}(\mathcal{R},q)_\chi\big)$.
c) For any facet f

$$\mathcal{S}\widehat{\otimes}_{\mathcal{H}(W_f,q)\otimes\mathcal{S}(Z(W))}V = \mathcal{S}(\mathcal{R},q)_\chi\widehat{\otimes}_{\mathcal{H}(W_f,q)}V = \mathcal{S}(\mathcal{R},q)_\chi \otimes_{\mathcal{H}(W_f,q)} V = \mathrm{Ind}_{\mathcal{H}(W_f,q)}^{\mathcal{S}(\mathcal{R},q)_\chi}V$$

is a projective $\mathcal{S}(\mathcal{R},q)_\chi$-module. In view of (2.29) this implies that $P_n^t(V)$ and $P_n^t(V)^\Omega$ are also projective in $\mathrm{Mod}\big(\mathcal{S}(\mathcal{R},q)_\chi\big)$. $\square$

Remark.
If V is a finite dimensional tempered module with $Z(W)$-character χ, then the proof of Corollary 2.7 does not rely on the properties of bornology. Indeed, in this situation we may simply use the algebraic tensor product in the definition of $P_n^t(V)$, since the algebraic tensor product is already complete as a locally convex vector space. The continuity proof of the contraction is analogous to, and in fact somewhat simpler than, the above proof for the case $V=\mathcal{S}$. Hence the algebraic tensor product of the resolution of Corollary 2.3 a) by $\mathcal{S}(R,q)_\chi$ yields the resolution of Corollary 2.7.a).

2.4 Isocohomological inclusions

We will show that the inclusion $\mathcal{H}\to\mathcal{S}$ is isocohomological. As an intermediate step we do the same for algebras and modules corresponding to a fixed $Z(W)$-character.

Similar results for Schwartz algebras of reductive p-adic groups were proven by Meyer [Mey3, Theorems 21, 27 and 29] with highly sophisticated techniques. Maybe our bounded contraction from Section 2.1 can be used to simplify these proofs.

Theorem 2.8. *Let χ be a unitary $Z(W)$-character.*

a) The inclusion $\mathcal{H}(\mathcal{R},q)_\chi\to\mathcal{S}(\mathcal{R},q)_\chi$ is isocohomological.

b) The cohomological dimension of $\mathrm{Mod}_{\mathrm{bor}}\big(\mathcal{S}(\mathcal{R},q)_\chi\big)$ equals $r=\mathrm{rk}(R_0)$.

Proof. a) From (2.19) and (2.28) it follows that

$$\begin{aligned} P_n(\mathcal{H}(\mathcal{R},q)_\chi) &\cong \bigoplus_{f:\dim f=n}\mathcal{H}(\mathcal{R},q)_\chi\widehat{\otimes}_{\mathcal{H}(W_f,q)}\mathcal{H}(\mathcal{R},q)_\chi\otimes_{\mathbb{C}}\mathbb{C}\{f\},\\ P_n^t(\mathcal{S}(\mathcal{R},q)_\chi) &\cong \bigoplus_{f:\dim f=n}\mathcal{S}(\mathcal{R},q)_\chi\widehat{\otimes}_{\mathcal{H}(W_f,q)}\mathcal{S}(\mathcal{R},q)_\chi\otimes_{\mathbb{C}}\mathbb{C}\{f\}.\end{aligned}$$

Exactly as in the proof of Theorem 2.2 we can see that these are projective as bornological bimodules for $\mathcal{H}(\mathcal{R},q)_\chi$ respectively $\mathcal{S}(\mathcal{R},q)_\chi$. In view of (2.14) and (2.29) the same holds for $P_n(\mathcal{H}(\mathcal{R},q)_\chi)^\Omega$ and $P_n^t(\mathcal{S}(\mathcal{R},q)_\chi)^\Omega$. Combined with Corollaries 2.3.a) and 2.7.a) this yields condition 1) of Theorem A.1.
b) By Corollary 2.7 the cohomological dimension of $\mathrm{Mod}_{\mathrm{bor}}\big(\mathcal{S}(\mathcal{R},q)_\chi\big)$ is at most $r=\mathrm{rk}(R_0)$. If $t\in T$ is unitary then by [Opd1, Proposition 4.19] the module I_t from (2.21) is tempered. Together with (2.22) this gives

$$\mathrm{Ext}^r_{\mathcal{S}(\mathcal{R},q)_\chi}(I_t,I_t)\cong\mathrm{Ext}^r_{\mathcal{H}(\mathcal{R},q)_\chi}(I_t,I_t)\neq 0.$$

Hence this cohomological dimension is at least r. $\square$

Theorem 2.9. a) *The inclusion $\mathcal{H} \to \mathcal{S}$ is isocohomological.*

b) *The cohomological dimension of* $\mathrm{Mod}_{\mathrm{bor}}(\mathcal{S})$ *equals the rank of X.*

Proof. a) Let $(\widetilde{\mathcal{R}}, \tilde{q})$ be as in (1.3). Recall that

$$\begin{array}{rcccl} \mathcal{H}(\widetilde{\mathcal{R}}, \tilde{q}) & \cong & \tilde{G} \ltimes \mathcal{H}(\mathcal{R}, q) & = & \tilde{G} \ltimes \mathcal{H}, \\ \mathcal{S}(\widetilde{\mathcal{R}}, \tilde{q}) & \cong & \tilde{G} \ltimes \mathcal{S}(\mathcal{R}, q) & = & \tilde{G} \ltimes \mathcal{S}. \end{array}$$

We know from Theorem 2.8.a) that the inclusion $\mathcal{H}(\widetilde{\mathcal{R}}, \tilde{q}) \to \mathcal{S}(\widetilde{\mathcal{R}}, \tilde{q})$ is isocohomological. Therefore we can use an argument from the proof of [Mey2, Theorem 58]. The functor

$$\mathrm{Mod}(B) \to \mathrm{Mod}(\tilde{G} \ltimes B) \;:\; V \to \mathrm{Ind}_B^{\tilde{G} \ltimes B}(V) = (\tilde{G} \ltimes B) \otimes_B V \tag{2.35}$$

is exact for any $\tilde{G}$-algebra B. Hence in $\mathrm{Der}_{\mathrm{bor}}(\tilde{G} \ltimes \mathcal{S})$ we have

$$\begin{array}{rcl} \tilde{G} \ltimes \mathcal{S} & \cong & (\tilde{G} \ltimes \mathcal{S}) \widehat{\otimes}^{\mathbb{L}}_{\tilde{G} \ltimes \mathcal{S}} (\tilde{G} \ltimes \mathcal{S}) \\ & \cong & (\tilde{G} \ltimes \mathcal{S}) \widehat{\otimes}^{\mathbb{L}}_{\tilde{G} \ltimes \mathcal{H}} (\tilde{G} \ltimes \mathcal{S}) \\ & \cong & (\tilde{G} \ltimes \mathcal{S}) \widehat{\otimes}^{\mathbb{L}}_{\tilde{G} \ltimes \mathcal{H}} \tilde{G} \ltimes \mathcal{H} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{H}} \mathcal{S} \\ & \cong & (\tilde{G} \ltimes \mathcal{S}) \widehat{\otimes}^{\mathbb{L}}_{\mathcal{H}} \mathcal{S} \\ & \cong & \mathrm{Ind}_{\mathcal{S}}^{\tilde{G} \ltimes \mathcal{S}} \big(\mathcal{S} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{H}} \mathcal{S} \big). \end{array} \tag{2.36}$$

We want to show that this implies condition 2) of Theorem A.1 for the inclusion $\mathcal{H} \to \mathcal{S}$. However we have to be a little careful, as the functor (2.35) is not injective on objects. Namely, $\mathcal{H}$-modules like V and V_g in (2.24), which are conjugate by an element of $\tilde{G}$, have the same image under (2.35). It follows from (2.36) that

$$\mathbb{C}[\tilde{G}] \otimes_{\mathbb{C}} \mathrm{Tor}_n^{\mathcal{H}}(\mathcal{S}, \mathcal{S}) \cong \left\{ \begin{array}{lll} \tilde{G} \ltimes \mathcal{S} & \text{if} & n = 0 \\ 0 & \text{if} & n > 0. \end{array} \right. \tag{2.37}$$

Obviously the multiplication map

$$\mathrm{Tor}_0^{\mathcal{H}}(\mathcal{S}, \mathcal{S}) \cong \mathcal{S} \widehat{\otimes}_{\mathcal{H}} \mathcal{S} \to \mathcal{S}$$

is surjective. In view of (2.37) it must also be injective, and therefore

$$\mathrm{Tor}_n^{\mathcal{H}}(\mathcal{S}, \mathcal{S}) \cong \left\{ \begin{array}{lll} \mathcal{S} & \text{if} & n = 0 \\ 0 & \text{if} & n > 0. \end{array} \right.$$

Let

$$0 \leftarrow \mathcal{S} \leftarrow P_0 \leftarrow P_1 \leftarrow \cdots \tag{2.38}$$

be a bornological resolution of $\mathcal{S}$ by projective $\mathcal{H}$-modules. We already know that the homology of (2.38) vanishes in all degrees. Moreover $\mathrm{Ind}_{\mathcal{H}}^{\tilde{G} \ltimes \mathcal{H}}(P_*)$ is a resolution of $\tilde{G} \ltimes \mathcal{H}$. Theorems 2.8.a) and A.1 assure that the differential complex $\mathrm{Ind}_{\mathcal{H}}^{\tilde{G} \ltimes \mathcal{H}} \big(\mathcal{S} \widehat{\otimes}_{\mathcal{H}} P_* \big)$ is a bornological resolution of $\tilde{G} \ltimes \mathcal{S}$. In particular it admits a bounded $\mathbb{C}$-linear contraction. Hence $\mathcal{S} \widehat{\otimes}_{\mathcal{H}} P_*$ also admits a bounded contraction, in other words, it is an exact sequence in $\mathrm{Mod}_{\mathrm{bor}}(\mathcal{S})$. This shows that the natural map

$$\mathcal{S} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{H}} \mathcal{S} \to \mathcal{S} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{S}} \mathcal{S} \tag{2.39}$$

is an isomorphism. We conclude that $\mathcal{H} \to \mathcal{S}$ is indeed isocohomological.
b) In view of part a) and Proposition 2.4 the cohomological dimension of $\mathrm{Mod}_{\mathrm{bor}}(\mathcal{S})$ is at most $\mathrm{rk}(X)$. If $t \in T$ is unitary, then by [Opd1, Proposition 4.19] the module I_t from (2.21) is tempered. From a) and the proof of Proposition 2.4 we see that

$$\mathrm{Ext}^{\mathrm{rk}(X)}_{\mathcal{S}}(I_t, I_t) \cong \mathrm{Ext}^{\mathrm{rk}(X)}_{\mathcal{H}}(I_t, I_t) \neq 0.$$

Hence this cohomological dimension is at least $\mathrm{rk}(X)$. $\square$

Remark.
In the same way one can show that the cohomological dimension of the category $\mathrm{Mod}_{\mathrm{Fré}}(\mathcal{S})$ of continuous Fréchet $\mathcal{S}$-modules is the rank of X. To make this a meaningful statement we make this into an exact category as follows.

All morphisms are required to be continuous and $\widehat{\otimes}$ is the completed projective tensor product. Only extensions and resolutions that admit a continuous $\mathbb{C}$-linear splitting are called exact. This category has enough projective objects and has countable projective limits. However it does neither have enough injective objects, nor inductive limits.

Chapter 3

The Euler–Poincaré characteristic

3.1 Elliptic representation theory

Elliptic representation theory is a general notion that can be developed for many groups and algebras [Art, Kaz, Ree, ScSt, Wal]. The idea is that one considers all virtual representations of an algebra, modulo those that are induced from certain specified subalgebras. This should yield interesting equivalence classes of representations if the subalgebras are chosen cleverly.

For example, in a reductive p-adic group one can consider the collection of proper parabolic subgroups. The resulting space of representations contains among others all square integrable representations. It can be studied by means of certain integrals over the regular elliptic conjugacy classes, cf. [Kaz, Bez, ScSt].

In the context of the elliptic representation theory for Iwahori-spherical representations of a p-adic Chevalley group Reeder [Ree] was led to the general definition of elliptic representation theory for a finite group relative to a given representation. Let (ρ, E) be a real representation of a finite group Γ. We define an elliptic pairing on $\mathrm{Mod}_{\mathrm{fin}}(\Gamma)$ by

$$e_\Gamma(U,V) := \sum_{n=0}^{\infty} (-1)^n \dim \mathrm{Hom}_\Gamma\big(U \otimes {\textstyle\bigwedge^n} E, V\big). \tag{3.1}$$

We call an element $\gamma \in \Gamma$ elliptic (with respect to E) if $E^{\rho(\gamma)} = 0$. Since this property is preserved under conjugation, we can use the same terminology for conjugacy classes. Let $\mathcal{L}$ be the set of subgroups $H \subset \Gamma$ such that $E^{\rho(H)} \neq 0$. The space of elliptic trace functions on Γ is defined as

$$Ell(\Gamma) := G_{\mathbb{C}}(\Gamma) \Big/ \sum_{H \in \mathcal{L}} \mathrm{Ind}_H^\Gamma\big(G_{\mathbb{C}}(H)\big). \tag{3.2}$$

Theorem 3.1. [Ree, §2]

a) *The dimension of $Ell(\Gamma)$ equals the number of elliptic conjugacy classes of Γ.*

b) *e_Γ induces a Hermitian inner product on $\mathit{Ell}(\Gamma)$.*

c) *For all $\chi, \chi' \in G_{\mathbb{C}}(\Gamma)$ we have*

$$e_\Gamma(\chi, \chi') = \sum_{\gamma \in \Gamma} \frac{\det(\mathrm{id}_E - \rho(\gamma))}{|\Gamma|} \overline{\chi(\gamma)} \chi'(\gamma).$$

Assume now that X is a lattice in E (so $E = X \otimes_{\mathbb{Z}} \mathbb{R}$), which is stable under the action of Γ. We will show that Theorem 3.1 can be generalized to the group $\Gamma \ltimes X$. Of course affine Weyl groups are important examples of such groups.

In what follows an expression like γx always should be interpreted as the product in $\Gamma \ltimes X$. If we want to make γ act on x, then we write $\rho(\gamma)x$. We extend this to an action of $\Gamma \ltimes X$ on X by

$$\rho(y\gamma)x = y + \rho(\gamma)x.$$

Let $t \in T = \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{C}^\times)$. Clifford theory [Cli] tells us that there is a natural bijection between irreducible representations of $\Gamma_t = \{\gamma \in \Gamma : t \circ \rho(\gamma) = t\}$ and irreducible representations of $\Gamma \ltimes X$ with central character $\Gamma t \in T/\Gamma$. It is given explicitly by

$$\mathrm{Ind}_t : V \to \mathrm{Ind}_{\Gamma_t \ltimes X}^{\Gamma \ltimes X} V_t, \tag{3.3}$$

where V_t means that we regard V as a X-representation with character t.

We call an element $\gamma x \in \Gamma \ltimes X$ elliptic if it has an isolated fixpoint in E. It is easily seen that this is the case if and only if $\gamma \in \Gamma$ is elliptic. We have

$$x\, y\gamma\, (-x) = (x - \rho(\gamma)x)\, y\gamma \in \Gamma \ltimes X,$$

so all elements of $(y + (\mathrm{id}_E - \rho(\gamma))X)\gamma$ are conjugate in $\Gamma \ltimes X$. If γ is elliptic then the lattice $(\mathrm{id}_E - \rho(\gamma))X$ is of finite index in X. Consequently there are only finitely many elliptic conjugacy classes in $\Gamma \ltimes X$.

Let U and V be $\Gamma \ltimes X$ modules of finite length (which for this group means finite dimensional). We define the Euler–Poincaré characteristic

$$EP_{\Gamma \ltimes X}(U, V) := \sum_{n=0}^{\infty} (-1)^n \dim \mathrm{Ext}^n_{\Gamma \ltimes X}(U, V). \tag{3.4}$$

This kind of pairing stems from Schneider and Stuhler [ScSt, §III.4], who studied it for reductive p-adic groups. The space of elliptic trace functions on $\Gamma \ltimes X$ is

$$\mathit{Ell}(\Gamma \ltimes X) := G_{\mathbb{C}}(\Gamma \ltimes X) \Big/ \sum_{H \in \mathcal{L}} \mathrm{Ind}_{H \ltimes X}^{\Gamma \ltimes X} \big(G_{\mathbb{C}}(H \ltimes X)\big). \tag{3.5}$$

For every $t \in T$ we consider the elliptic representation theory of Γ_t with respect to the cotangent space to T at t. We note that Ind_t induces a map $\mathit{Ell}(\Gamma_t) \to \mathit{Ell}(\Gamma \ltimes X)$. Let H_{ell} denote the set of elliptic elements in a group H, and let $\sim_H$ be the equivalence relation "conjugate by an element of H".

Theorem 3.2. a) *The dimension of $\mathit{Ell}(\Gamma \ltimes X)$ equals the number of elliptic conjugacy classes of $\Gamma \ltimes X$.*

b) *$EP_{\Gamma \ltimes X}$ induces a Hermitian inner product on $Ell(\Gamma \ltimes X)$.*

c) *The map $\mathrm{Ind}_t : Ell(\Gamma_t) \to Ell(\Gamma \ltimes X)$ induced by (3.3) is an isometry:*

$$EP_{\Gamma \ltimes X}(\mathrm{Ind}_t U, \mathrm{Ind}_t V) = e_{\Gamma_t}(U, V)$$

for all finite dimensional Γ_t-representations and U and V.

d) *The map $\bigoplus_{t \in T/\Gamma} \mathrm{Ind}_t : \bigoplus_{t \in T/\Gamma} Ell(\Gamma_t) \to Ell(\Gamma \ltimes X)$ is an isomorphism.*

Proof. For U, V and t as above Frobenius reciprocity tells us that

$$\mathrm{Ext}^n_{\Gamma \ltimes X}(\mathrm{Ind}_t U, \mathrm{Ind}_t V) \cong \mathrm{Ext}^n_{\Gamma_t \ltimes X}(U_t, \mathrm{Ind}^{\Gamma \ltimes X}_{\Gamma_t \ltimes X} V_t). \tag{3.6}$$

Because two $\Gamma_t \ltimes X$-representations with different central characters admit only trivial extensions, (3.6) is isomorphic to $\mathrm{Ext}^n_{\Gamma_t \ltimes X}(U_t, V_t)$. Inside the group algebra

$$\mathcal{A} := \mathbb{C}[X] \cong \mathcal{O}(T)$$

we have the ideal of functions vanishing at $t \in T$:

$$I_t := \{f \in \mathcal{A} : f(t) = 0\}.$$

Let us denote the completion of $\mathcal{A}$ with respect to the powers of this ideal by $\hat{\mathcal{A}}_t$. Clearly

$$(\Gamma_t \ltimes \hat{\mathcal{A}}_t) \otimes_{\Gamma_t \ltimes X} U_t = U_t$$

as $\Gamma_t \ltimes X$-modules. Completing is an exact functor, so (3.6) becomes

$$\mathrm{Ext}^n_{\mathbb{C}[\Gamma_t \ltimes X]}(U_t, V_t) \cong \mathrm{Ext}^n_{\Gamma_t \ltimes \hat{\mathcal{A}}_t}(U_t, V_t). \tag{3.7}$$

Because the Γ_t-module I_t^2 has finite codimension in $\mathcal{A}$, there exists a Γ_t-module $E_t \subset \mathcal{A}$ such that

$$\mathcal{A} = \mathbb{C} \oplus E_t \oplus I_t^2. \tag{3.8}$$

As a Γ_t-module E_t is the cotangent space to T at t. Since $\mathcal{A}_t$ is a local ring we have $\hat{\mathcal{A}}_t E_t = \hat{\mathcal{A}}_t I_t$, by Nakayama's Lemma. Any finite dimensional Γ_t-module is projective, so

$$U \otimes \bigwedge\nolimits^n E_t \otimes \hat{\mathcal{A}}_t = \mathrm{Ind}^{\Gamma_t \ltimes \hat{\mathcal{A}}_t}_{\Gamma_t} \big(U \otimes \bigwedge\nolimits^n E_t \big)$$

is a projective $\Gamma_t \ltimes \hat{\mathcal{A}}_t$-module for all $n \in \mathbb{N}$. With these modules we construct a resolution of U_t. Define $\Gamma_t \ltimes \hat{\mathcal{A}}_t$-module maps

$$\begin{array}{lcl}
\delta_n : U \otimes \bigwedge^n E_t \otimes \hat{\mathcal{A}}_t & \to & U \otimes \bigwedge^{n-1} E_t \otimes \hat{\mathcal{A}}_t, \\
\delta_n \left(u \otimes e_1 \wedge \cdots \wedge e_n \otimes f \right) & = & \sum\limits_{i=1}^{n} (-1)^{i-1} u \otimes e_1 \wedge \cdots \wedge e_{i-1} \wedge e_{i+1} \wedge \cdots \wedge e_{j_n} \otimes e_i f, \\
\delta_0 : U \otimes \hat{\mathcal{A}}_t & \to & U_t, \\
\delta_0 \left(u \otimes f \right) & = & f(t) u.
\end{array}$$

This makes

$$\big(U \otimes \bigwedge\nolimits^* E_t \otimes \hat{\mathcal{A}}_t, \delta_* \big) \tag{3.9}$$

into an augmented differential complex. Notice that in $\mathrm{Mod}\big(\hat{\mathcal{A}}_t\big)$ this just the Koszul resolution of

$$U_t \otimes \hat{\mathcal{A}}_t / I_t \hat{\mathcal{A}}_t = U_t .$$

Therefore (3.9) is the required projective resolution of U_t and

$$\begin{aligned} EP_{\Gamma \ltimes X}(\mathrm{Ind}_t U, \mathrm{Ind}_t V) &= \sum_{n=0}^{\infty} (-1)^n \dim \mathrm{Ext}^n_{\Gamma_t \ltimes \hat{\mathcal{A}}_t}(U_t, V_t) \\ &= \sum_{n=0}^{r} (-1)^n \dim H^n \big(\mathrm{Hom}_{\Gamma_t \ltimes \hat{\mathcal{A}}_t} \big(U \otimes \textstyle\bigwedge^* E_t \otimes \hat{\mathcal{A}}_t, V_t \big), \mathrm{Hom}(\delta_*, \mathrm{id}_{V_t}) \big) \\ &= \sum_{n=0}^{r} (-1)^n \dim \mathrm{Hom}_{\Gamma_t \ltimes \hat{\mathcal{A}}_t} \big(U \otimes \textstyle\bigwedge^n E_t \otimes \hat{\mathcal{A}}_t, V_t \big) \\ &= \sum_{n=0}^{r} (-1)^n \dim \mathrm{Hom}_{\Gamma_t} \big(U \otimes \textstyle\bigwedge^n E_t, V \big) \quad = \quad e_{\Gamma_t}(U, V). \end{aligned}$$

This proves c). According to Theorem 3.1, e_{Γ_t} induces an inner product on $Ell(\Gamma_t)$ and by definition $\mathrm{Ind}_t(Ell(\Gamma_t)) \subset Ell(\Gamma \ltimes X)$ is precisely of the span of the $\Gamma \ltimes X$-modules with central character Γt. Two $\Gamma \ltimes X$-representations with different $Z(\Gamma \ltimes \mathcal{A})$-characters are orthogonal for $EP_{\Gamma \ltimes X}$, so b) and d) follow.

Now let us count the elliptic conjugacy classes in $\Gamma \ltimes X$. Two sets

$$(x + (\mathrm{id}_E - \rho(\gamma))X)\gamma \quad \text{and} \quad (y + (\mathrm{id}_E - \rho(\gamma))X)\gamma$$

are conjugate if and only if there is a $w \in Z_\Gamma(\gamma)$ such that $\rho(w)x - y \in (\mathrm{id}_E - \rho(\gamma))X$. As Γ-sets we have $T^\gamma = \mathrm{Hom}\big(X/(\mathrm{id}_E - \rho(\gamma))X, \mathbb{C}^\times\big)$. Therefore

$$\begin{aligned} \#\big((\Gamma \ltimes X)_{\mathrm{ell}} / \sim_{\Gamma \ltimes X}\big) &= \sum_{\gamma \in \Gamma_{\mathrm{ell}} / \sim_\Gamma} \#\big((X/(1-\gamma)X)/Z_\Gamma(\gamma)\big) \\ &= \sum_{\gamma \in \Gamma_{\mathrm{ell}} / \sim_\Gamma} \#\big(T^\gamma / Z_\Gamma(\gamma)\big) \\ &= \#\big(\{(\gamma, t) : \gamma \in \Gamma_{\mathrm{ell}}, t \in T^\gamma\} / Z_\Gamma(\gamma)\big) \\ &= \#\big(\{(\gamma, t) : t \in T, \gamma \in \Gamma_{t,ell}\} / Z_\Gamma(\gamma)\big) \\ &= \sum_{t \in T/\Gamma} \#\big(\Gamma_{t,ell} / \sim_{\Gamma_t}\big) \\ &= \sum_{t \in T/\Gamma} \dim Ell(\Gamma_t) \\ &= \dim Ell(\Gamma \ltimes X), \end{aligned}$$

where we let Γ act on $\Gamma_{\mathrm{ell}} \times T$ by $w \cdot (\gamma, t) = (w\gamma w^{-1}, wt)$. $\square$

From the above proof we see that part c) of Theorem 3.2 remains valid in the following more general settings:

- T is a nonsingular complex affine variety, $\mathcal{A} = \mathcal{O}(T)$ and Γ acts on T by algebraic automorphisms,
- T is a smooth manifold, $\mathcal{A} = C^\infty(T)$ and Γ acts on T by diffeomorphisms.

3.2 The elliptic measure

It is shown in [ScSt, Theorem III.4.21] and [Bez, Theorem 0.20] that the Euler–Poincaré characteristic for semisimple p-adic groups agrees with the elliptic integral introduced in [Kaz, p. 5].

For the group $\Gamma \ltimes X$ this relation can be made even more explicit. We endow it with the σ-algebra generated by the sets

$$L_w := \{xw(-x) : x \in X\} \qquad w \in \Gamma \ltimes X \tag{3.10}$$

Let χ_V denote the character of a representation V.

Theorem 3.3. a) *There exists a unique conjugation-invariant "elliptic" measure μ_{ell} on the measurable space $\Gamma \ltimes X$ such that*

$$EP_{\Gamma \ltimes X}(U,V) = \int_{\Gamma \ltimes X} \overline{\chi_U} \chi_V \, d\mu_{\text{ell}} \qquad \forall U,V \in \text{Mod}_{\text{fin}}(\Gamma \ltimes X).$$

b) *The support of μ_{ell} is the set of elliptic elements.*

c) *Let $e \in E$ be an isolated fixpoint of an elliptic element $c \in \Gamma \ltimes X$ and let $C \subset \Gamma \ltimes X$ be the conjugacy class of c. Then*

$$\begin{array}{lll} \mu_{\text{ell}}(L_c) & = & |\Gamma|^{-1}, \\ \mu_{\text{ell}}(C) & = & \dfrac{\#\{w \in C : \rho(w)e = e\}}{\#\{w \in \Gamma \ltimes X : \rho(w)e = e\}}, \\ \mu_{\text{ell}}(\Gamma \ltimes X) & = & \sum\limits_{n=0}^{\infty} (-1)^n \dim \left(\bigwedge^n E \right)^{\Gamma}. \end{array}$$

Proof. Suppose we have a trace function $f \in G_{\mathbb{C}}(\Gamma \ltimes X)$ such that $f(w) = 0 \; \forall w \in (\Gamma \ltimes X)_{\text{ell}}$. Write $f = \sum_{t \in T/\Gamma} \text{Ind}_t f_t$. This is a finite sum because $G_{\mathbb{C}}(\Gamma \ltimes X)$ is built from finite dimensional representations. If $\gamma \in \Gamma_{t,\text{ell}}$ then we have $f(x\gamma) = 0 \; \forall x \in X$, so $[\Gamma : \Gamma_t] f_t(\gamma) = \text{Ind}_t(f_t)(\gamma) = 0$.

Hence by Theorem 3.1.b) $[f_t] = 0 \in \mathit{Ell}(\Gamma_t)$. By Theorem 3.2.d) $[f] = 0 \in \mathit{Ell}(\Gamma \ltimes X)$. Now parts and a) and b) follow automatically, since there are only finitely many elliptic conjugacy classes in $\Gamma \ltimes X$ and every conjugacy class contains only finitely many L_w's.

To find the explicit form of μ_{ell}, we consider a possibly different measure μ on $\Gamma \ltimes X$, defined by $\mu(L_c) := |\Gamma|^{-1}$ for any elliptic element $c \in \Gamma \ltimes X$. We will show that μ satisfies the properties attributed to μ_{ell}. It will follow from the just proven uniqueness that $\mu = \mu_{\text{ell}}$.

Let U and V be irreducible $\Gamma \ltimes X$-representations, with central characters Γt and $\Gamma t'$, respectively. By (3.3) there are characters χ of Γ_t and $\chi_{t'}$ of $\Gamma_{t'}$ such that

$\chi_U = \mathrm{Ind}_t \chi$ and $\chi_V = \mathrm{Ind}_t \chi'$. Extend χ and χ' to functions on Γ by making them zero on $\Gamma \setminus \Gamma_t$ and on $\Gamma \setminus \Gamma_{t'}$, respectively. For $\gamma \in \Gamma_{\mathrm{ell}}$ we have

$$\chi_U(x\gamma) = \sum_{h \in \Gamma/\Gamma_t} t(\rho(h)^{-1}x)\chi(h^{-1}\gamma h).$$

This can only be nonzero if $\chi(h^{-1}\gamma h) \neq 0$, which forces $h^{-1}\gamma h$ to be an elliptic element of Γ_t. Therefore

$$\int_{\Gamma \ltimes X} \overline{\chi_U} \chi_V \, d\mu = 0$$

if either $Ell(\Gamma_t) = 0$ or $Ell(\Gamma_{t'}) = 0$, which is in agreement with Theorem 3.1.b).

Hence we assume that Γ_t and $\Gamma_{t'}$ do contain elliptic elements. This forces all elements of $\Gamma\{t, t'\}$ to have finite order in the group T. Now

$$X' := \bigcap_{t'' \in \Gamma\{t,t'\}} \ker t'' \cap \bigcap_{\gamma \in \Gamma_{\mathrm{ell}}} (\mathrm{id}_E - \rho(\gamma))X$$

is a lattice of finite index in X and the map

$$X/X' \to \mathbb{C} : x \mapsto t(\rho(h)^{-1}x)\chi(h^{-1}\gamma h)$$

is well defined for all $h, \gamma \in \Gamma$. For a fixed $\gamma \in \Gamma_{\mathrm{ell}}$ we have

$$\begin{aligned}
[(\mathrm{id}_E - \rho(\gamma))X : X'] \sum_{x \in X/(\mathrm{id}_E - \rho(\gamma))X} \overline{\chi_U(x\gamma)} \chi_V(x\gamma) &= \sum_{x \in X/X'} \overline{\chi_U(x\gamma)} \chi_V(x\gamma) \\
= \sum_{h \in \Gamma/\Gamma_t} \sum_{g \in \Gamma/\Gamma_{t'}} \sum_{x \in X/X'} \overline{t(\rho(h)^{-1}x)\chi(h^{-1}\gamma h)} \, & t'(\rho(g)^{-1}x)\chi'(g^{-1}\gamma g).
\end{aligned} \tag{3.11}$$

By the orthogonality relations for characters of the group X/X', the only nonzero contributions to this sum come from pairs (g, h) for which $h(t) = g(t')$. In particular

$$\int_{\Gamma \ltimes X} \overline{\chi_U} \chi_V \, d\mu = 0$$

if $\Gamma t \neq \Gamma t'$. This leaves the case $t = t'$. From (3.11) we see that

$$\begin{aligned}
\sum_{x \in X/(\mathrm{id}_E - \rho(\gamma))X} \overline{\chi_U(x\gamma)} \chi_V(x\gamma) &= \sum_{h,g \in \Gamma/\Gamma_t} \sum_{x \in X/X'} \frac{\overline{t(\rho(h)^{-1}x)\chi(h^{-1}\gamma h)} \, t(\rho(g)^{-1}x)\chi'(g^{-1}\gamma g)}{[(\mathrm{id}_E - \rho(\gamma))X : X']} \\
&= \sum_{h \in \Gamma/\Gamma_t} \sum_{x \in X/X'} \frac{\overline{t(\rho(h)^{-1}x)\chi(h^{-1}\gamma h)} \, t(\rho(h)^{-1}x)\chi'(h^{-1}\gamma h)}{[(\mathrm{id}_E - \rho(\gamma))X : X']} \\
&= [X : (\mathrm{id}_E - \rho(\gamma))X] \sum_{h \in \Gamma/\Gamma_t} \overline{\chi(h^{-1}\gamma h)} \chi'(h^{-1}\gamma h).
\end{aligned}$$

Now we can compute

$$\begin{aligned}
\int_{\Gamma \ltimes X} \overline{\chi_U} \chi_V \, d\mu &= \sum_{\gamma \in \Gamma_{\text{ell}}} \sum_{x \in X / (\text{id}_E - \rho(\gamma)) X} \frac{\overline{\chi_U(x\gamma)} \chi_V(x\gamma)}{|\Gamma|} \\
&= \sum_{\gamma \in \Gamma_{\text{ell}}} \frac{[X : (\text{id}_E - \rho(\gamma))X]}{|\Gamma|} \sum_{h \in \Gamma / \Gamma_t} \overline{\chi(h^{-1}\gamma h)} \chi'(h^{-1}\gamma h) \\
&= \sum_{\gamma \in \Gamma_{t,ell}} \frac{\det(\text{id}_E - \rho(\gamma))}{|\Gamma|} [\Gamma : \Gamma_t] \overline{\chi(\gamma)} \chi'(\gamma) \\
&= \sum_{\gamma \in \Gamma_{t,ell}} \frac{\det(\text{id}_E - \rho(\gamma))}{|\Gamma_t|} \overline{\chi(\gamma)} \chi'(\gamma) \quad = \quad e_{\Gamma_t}(\chi, \chi').
\end{aligned}$$

Thus indeed $\mu = \mu_{\text{ell}}$.

Let e, c and C be as above. To determine $\mu_{\text{ell}}(C)$ we must count the number n_C of sets L_w that are contained in C. Consider the map

$$\begin{aligned}
&\psi_e : C \to E/X, \\
&\psi_e(wcw^{-1}) = \rho(w)e + X.
\end{aligned}$$

It is easily seen that ψ_e is well-defined and that

$$\psi_e(xwcw^{-1}(-x)) = \psi_e(wcw^{-1}) \qquad \forall x \in X,\, w \in \Gamma \ltimes X.$$

The image of ψ_e is $\rho(\Gamma \ltimes X)e/X$ and

$$\psi_e^{-1}(\rho(w)e + X) = \{xwvcv^{-1}w^{-1}(-x) : x \in X, v \in \Gamma \ltimes X, \rho(v)e = e\}.$$

The number of L_w's contained in $\psi_e^{-1}(\rho(w)e + X)$ is

$$\#\{vcv^{-1} : v \in \Gamma \ltimes X, \rho(v)e = e\} = \#\{v \in C : \rho(v)e = e\}.$$

Consequently

$$\begin{aligned}
n_C &= |\rho(\Gamma \ltimes X)e/X| \, \#\{v \in C : \rho(v)e = e\} = \frac{|\Gamma| \, \#\{v \in C : \rho(v)e = e\}}{\#\{w \in \Gamma \ltimes X : \rho(w)e = e\}}, \\
\mu_{\text{ell}}(C) &= \frac{n_C}{|\Gamma|} = \frac{\#\{v \in C : \rho(v)e = e\}}{\#\{w \in \Gamma \ltimes X : \rho(w)e = e\}}.
\end{aligned}$$

Finally, using Theorem 3.2.c) we compute

$$\begin{aligned}
\mu_{\text{ell}}(\Gamma \ltimes X) &= EP_{\Gamma \ltimes X}(\text{triv}_{\Gamma \ltimes X}, \text{triv}_{\Gamma \ltimes X}) \\
&= EP_{\Gamma \ltimes X}\big(\text{Ind}_1(\text{triv}_\Gamma), \text{Ind}_1(\text{triv}_\Gamma)\big) \\
&= e_\Gamma\big(\text{triv}_\Gamma, \text{triv}_\Gamma\big) \\
&= \sum_{n=0}^{\infty} (-1)^n \dim \text{Hom}_\Gamma\big(\textstyle\bigwedge^n E, \text{triv}_\Gamma\big) \\
&= \sum_{n=0}^{\infty} (-1)^n \dim \big(\textstyle\bigwedge^n E\big)^\Gamma. \qquad \Box
\end{aligned}$$

3.3 Example: the Weyl group of type B_2

Let R_0 be the root system B_2 in $E = \mathbb{R}^2$, with positive roots

$$\alpha_1 = (1, -1),\ \alpha_2 = (0, 1),\ \alpha_3(1, 0),\ \alpha_4 = (1, 1).$$

Denote the rotation of E over an angle θ by ρ_θ and the reflection corresponding to α_i by s_i. Then

$$W_0 = \{e, s_1, s_2, s_3, s_4, \rho_{\pi/2}, \rho_\pi, \rho_{-\pi/2}\}$$

is isomorphic to the dihedral group D_4. This group has four irreducible representations of dimension one, defined by

π	$\pi(s_1)$	$\pi(s_2)$
ϵ_0	1	1
ϵ_1	-1	1
ϵ_2	1	-1
ϵ_3	-1	-1

(3.12)

The one remaining irreducible representation is just E.

The elliptic conjugacy classes in W_0 (with respect to the defining representation E) are $\{\rho_\pi\}$ and $\{\rho_{\pi/2}, \rho_{-\pi/2}\}$.

$$\begin{aligned}
\mathrm{Ind}_{W_\emptyset}^{W_0}(G_\mathbb{C}\{e\}) &= \mathbb{C}\{\epsilon_0 \oplus \epsilon_1 \oplus \epsilon_2 \oplus \epsilon_3 \oplus E \oplus E\},\\
\mathrm{Ind}_{W_{\{1\}}}^{W_0}(G_\mathbb{C}\{e, s_1\}) &= \mathbb{C}\{\epsilon_0 \oplus \epsilon_2 \oplus E\,, \epsilon_1 \oplus \epsilon_3 \oplus E\},\\
\mathrm{Ind}_{W_{\{2\}}}^{W_0}(G_\mathbb{C}\{e, s_2\}) &= \mathbb{C}\{\epsilon_0 \oplus \epsilon_1 \oplus E\,, \epsilon_2 \oplus \epsilon_3 \oplus E\}.
\end{aligned}$$

We see that $Ell(W_0)$ has dimension two and is spanned for example by $[\epsilon_0]$ and $[\epsilon_1]$. With Theorem 3.1.c) we can easily write down a complete table for e_{W_0} :

e_{W_0}	ϵ_0	ϵ_1	ϵ_2	ϵ_3	E
ϵ_0	1	0	0	1	-1
ϵ_1	0	1	1	0	-1
ϵ_2	0	1	1	0	-1
ϵ_3	1	0	0	1	-1
E	-1	-1	-1	-1	2

(3.13)

Since $\overline{A_\emptyset}$ is a fundamental domain for the action of W on E, every point of E that is fixed by an elliptic element of W must be in the W-orbit of some vertex of the fundamental alcove $A_\emptyset$. This leads to the following list of elliptic conjugacy classes:

vertex $e = c(e)$	conjugacy class $[c]$	elliptic measure $\mu_{\text{ell}}([c])$
$(0, 0)$	$[\rho_\pi]$	1/8
$(0, 0)$	$[\rho_{\pi/2}]$	1/4
$(1/2, 1/2)$	$[t_{(1,1)}\rho_\pi]$	1/8
$(1/2, 1/2)$	$[t_{(1,0)}\rho_{\pi/2}]$	1/4
$(1/2, 0)$	$[t_{(1,0)}\rho_\pi]$	1/4

(3.14)

In particular $\dim Ell(W) = 5$.

For $t \in T$ we write $t = (t(1,0), t(0,1))$. The following points of T are fixed by an elliptic element of W_0:

- $(1,1)$ is fixed by all $w \in W_0$. Thus we get a two dimensional subspace $\mathrm{Ind}_{(1,1)}\big(Ell(W_0)\big)$ of $Ell(W)$.
- $(-1,-1)$ is also fixed by the whole group W_0. This gives another two dimensional subspace $\mathrm{Ind}_{(-1,-1)}\big(Ell(W_0)\big) \subset Ell(W)$.
- $(-1,1)$ has isotropy group $V_4 = \{e, s_2, s_3, \rho_\pi\} \subset W_0$. The only elliptic element is ρ_π so $\dim Ell(V_4) = 1$.
- $(1,-1)$ also has isotropy group V_4. But $(-1,1)$ and $(1,-1)$ are in the same W_0-orbit so $\mathrm{Ind}_{(1,-1)}\big(Ell(V_4)\big) = \mathrm{Ind}_{(-1,1)}\big(Ell(V_4)\big)$. This one dimensional subspace of $Ell(W)$ is spanned for example by the two dimensional representation $\mathrm{Ind}_{(1,-1)}\big(\mathrm{triv}_{V_4}\big)$.

Now we have three subspaces of $Ell(W)$, they are mutually orthogonal for EP_W and their dimensions add up to 5. Since this is exactly the number of elliptic conjugacy classes in W, we found all of $Ell(W)$.

3.4 The Euler–Poincaré characteristic

Following Schneider and Stuhler [ScSt, §III.4] we introduce an Euler–Poincaré characteristic for affine Hecke algebras. For finite dimensional $\mathcal{H}$-modules U and V we define

$$EP_{\mathcal{H}}(U,V) = \sum_{n=0}^{\infty} (-1)^n \dim \mathrm{Ext}^n_{\mathcal{H}}(U,V). \tag{3.15}$$

By Proposition 2.4 the sum is actually finite, so this is well-defined. With standard homological algebra (see for instance [CaEi]) one can show that this extends to a bilinear pairing on $G(\mathcal{H})$. Reeder [Ree] studied this pairing for affine Hecke algebras with equal parameters, via p-adic groups.

Proposition 3.4. a) *Let $I \subset F_0$ be a proper subset of simple roots and let $V \in \mathrm{Mod}_{\mathrm{fin}}\big(\mathcal{H}(\mathcal{R}^I, q^I)\big)$. Then*

$$EP_{\mathcal{H}}\big(U, \mathrm{Ind}^{\mathcal{H}}_{\mathcal{H}(\mathcal{R}^I,q^I)} V\big) = 0 \qquad \forall\, U \in \mathrm{Mod}_{\mathrm{fin}}(\mathcal{H}).$$

b) *If the root datum $\mathcal{R}$ is not semisimple then, $EP_{\mathcal{H}} \equiv 0$.*

Proof. This result is the translation of [ScSt, Lemma III.4.18.ii] to affine Hecke algebras. The proof is similar and based on an argument due to Kazhdan.

We may assume that (π, V) is irreducible with $Z(\mathcal{H}(\mathcal{R}^I, q^I))$-character $W_I t \in T/W_I$. If $W_0 t$ is not an $Z(\mathcal{H})$-weight of U then $\mathrm{Ext}^{\mathcal{H}}_n\big(U, \mathrm{Ind}^{\mathcal{H}}_{\mathcal{H}(\mathcal{R}^I,q^I)} V\big) = 0$, so certainly $EP_{\mathcal{H}}\big(U, \mathrm{Ind}_{\mathcal{H}(\mathcal{R}^I,q^I)}^{\mathcal{H}} V\big) = 0$. Therefore we may also assume that U is irreducible with $Z(\mathcal{H})$-character $W_0 t \in T/W_0$.

Recall the groups $\widetilde{G} \subset W_0(\widetilde{\mathcal{R}})$ from (1.4). Objects constructed from $\widetilde{\mathcal{R}}$ will be denoted by the same symbol as the corresponding objects for $\mathcal{R}$, but with additional tilde. Like on 13 we have the fundamental alcove $\widetilde{A_\emptyset}$, the subgroup $\widetilde{\Omega} \subset W(\widetilde{\mathcal{R}})$ of elements of length zero, a facet $\tilde{f}$ of $\widetilde{A_\emptyset}$, and its stabilizer $\widetilde{\Omega}_{\tilde{f}}$. Let $\mathcal{F}_n$ be a set of representatives for the action of $\widetilde{\Omega}$ on these facets. We abbreviate

$$\begin{array}{rcl} m_t & = & \#\{g \in \widetilde{G} : g W_0 t = W_0 t\}, \\ \widetilde{U} & = & \mathrm{Ind}_{\mathcal{H}(\mathcal{R},q)}^{\mathcal{H}(\widetilde{\mathcal{R}},\tilde{q})}(U). \end{array}$$

From (2.24), Corollary 2.3.a) and the Euler–Poincaré principle we deduce that

$$\begin{aligned} m_t\, EP_{\mathcal{H}}\big(U, \mathrm{Ind}_{\mathcal{H}(\mathcal{R}^I,q^I)}^{\mathcal{H}} V\big) \; &= \; EP_{\mathcal{H}}\big(U, \mathrm{Ind}_{\mathcal{H}(\mathcal{R}^I,q^I)}^{\mathcal{H}(\widetilde{\mathcal{R}},\tilde{q})} V\big) \\ &= EP_{\mathcal{H}(\widetilde{\mathcal{R}},\tilde{q})}\big(\widetilde{U}, \mathrm{Ind}_{\mathcal{H}(\mathcal{R}^I,q^I)}^{\mathcal{H}(\widetilde{\mathcal{R}},\tilde{q})} V\big) \\ &= \sum_{n=0}^{\infty} (-1)^n \dim \mathrm{Ext}^n_{\mathcal{H}(\widetilde{\mathcal{R}},\tilde{q})}\big(\widetilde{U}, \mathrm{Ind}_{\mathcal{H}(\mathcal{R}^I,q^I)}^{\mathcal{H}(\widetilde{\mathcal{R}},\tilde{q})} V\big) \\ &= \sum_{n=0}^{\mathrm{rk}(X)} (-1)^n \dim \Big(\mathrm{Hom}_{\mathcal{H}(\widetilde{\mathcal{R}},\tilde{q})}\big(P_n(\widetilde{U})^{\widetilde{\Omega}}, \mathrm{Ind}_{\mathcal{H}(\mathcal{R}^I,q^I)}^{\mathcal{H}(\widetilde{\mathcal{R}},\tilde{q})} V\big)\Big) \qquad (3.16) \\ &= \sum_{n=0}^{\mathrm{rk}(X)} (-1)^n \dim \Big(\mathrm{Hom}_{\mathcal{H}(\widetilde{\mathcal{R}},\tilde{q})}\big(\bigoplus_{\tilde{f} \in \mathcal{F}_n} \mathrm{Ind}_{\mathcal{H}(\widetilde{\mathcal{R}},\tilde{f},\tilde{q})}^{\mathcal{H}(\widetilde{\mathcal{R}},\tilde{q})} (\widetilde{U} \otimes \epsilon_{\tilde{f}}), \mathrm{Ind}_{\mathcal{H}(\mathcal{R}^I,q^I)}^{\mathcal{H}(\widetilde{\mathcal{R}},\tilde{q})} V\big)\Big) \\ &= \sum_{n=0}^{\mathrm{rk}(X)} (-1)^n \dim \Big(\bigoplus_{\tilde{f} \in \mathcal{F}_n} \mathrm{Hom}_{\mathcal{H}(\widetilde{\mathcal{R}},\tilde{f},\tilde{q})}\big(\widetilde{U} \otimes \epsilon_{\tilde{f}}, \mathrm{Ind}_{\mathcal{H}(\mathcal{R}^I,q^I)}^{\mathcal{H}(\widetilde{\mathcal{R}},\tilde{q})} V\big)\Big) \\ &= \sum_{n=0}^{\mathrm{rk}(X)} (-1)^n \sum_{\tilde{f} \in \mathcal{F}_n} \dim \mathrm{Hom}_{\mathcal{H}(\widetilde{\mathcal{R}},\tilde{f},\tilde{q})}\big(\widetilde{U} \otimes \epsilon_{\tilde{f}}, \mathrm{Ind}_{\mathcal{H}(\mathcal{R}^I,q^I)}^{\mathcal{H}(\widetilde{\mathcal{R}},\tilde{q})} V\big). \end{aligned}$$

Because V is irreducible there exist a $\mathcal{H}(\mathcal{R}_I, q_I)$-representation (π_1, V) and a $Z\big(W(\mathcal{R}^I)\big)$-character t_1, such that

$$(\pi, V) = (\pi_1 \circ \phi_{t_1}, V)$$

with ϕ_{t_1} as in (1.8). Note that $Z\big(W(\mathcal{R}^I)\big) = (I^\vee)^\perp \cap X \neq 0$ because $I \neq F_0$. Let t_2 be an arbitrary $Z\big(W(\mathcal{R}^I)\big)$-character and consider the integer

$$\dim \mathrm{Hom}_{\mathcal{H}(\widetilde{\mathcal{R}},\tilde{f},\tilde{q})}\big(\widetilde{U} \otimes \epsilon_{\tilde{f}}, \mathrm{Ind}_{\mathcal{H}(\mathcal{R}^I,q^I)}^{\mathcal{H}(\widetilde{\mathcal{R}},\tilde{q})}(\pi_1 \circ \phi_{t_2}, V)\big).$$

According to Lemma 1.4 $\mathcal{H}(\widetilde{\mathcal{R}}, \tilde{f}, \tilde{q})$ is a finite dimensional semisimple algebra. Therefore the above integer is invariant under continuous deformations of t_2, and hence independent of t_2. Pick t_2 such that the central character of

$\mathrm{Ind}_{\mathcal{H}(\mathcal{R}^I,q^I)}^{\mathcal{H}(\widetilde{\mathcal{R}},\tilde{q})}(\pi_1 \circ \phi_{t_2}, V)$ is not $W_0(\widetilde{\mathcal{R}})t \in T/W_0(\widetilde{\mathcal{R}})$. Then

$$\begin{aligned} 0 &= m_t EP_{\mathcal{H}}\big(U, \mathrm{Ind}_{\mathcal{H}(\mathcal{R}^I,q^I)}^{\mathcal{H}(\widetilde{\mathcal{R}},\tilde{q})}(\pi_1 \circ \phi_{t_2}, V)\big) \\ &= \sum_{n=0}^{\mathrm{rk}(X)} (-1)^n \sum_{\tilde{f} \in \mathcal{F}_n} \dim \mathrm{Hom}_{\mathcal{H}(\widetilde{\mathcal{R}},\tilde{f},\tilde{q})}\big(\widetilde{U} \otimes \epsilon_{\tilde{f}}, \mathrm{Ind}_{\mathcal{H}(\mathcal{R}^I,q^I)}^{\mathcal{H}(\widetilde{\mathcal{R}},\tilde{q})}(\pi_1 \circ \phi_{t_2}, V)\big) \\ &= \sum_{n=0}^{\mathrm{rk}(X)} (-1)^n \sum_{\tilde{f} \in \mathcal{F}_n} \dim \mathrm{Hom}_{\mathcal{H}(\widetilde{\mathcal{R}},\tilde{f},\tilde{q})}\big(\widetilde{U} \otimes \epsilon_{\tilde{f}}, \mathrm{Ind}_{\mathcal{H}(\mathcal{R}^I,q^I)}^{\mathcal{H}(\widetilde{\mathcal{R}},\tilde{q})}(\pi_1 \circ \phi_{t_1}, V)\big) \\ &= m_t EP_{\mathcal{H}}\big(U, \mathrm{Ind}_{\mathcal{H}(\mathcal{R}^I,q^I)}^{\mathcal{H}}(\pi, V)\big). \end{aligned} \tag{3.17}$$

To prove b) we suppose that $\mathcal{R}$ is not semisimple and that $U', V' \in \mathrm{Mod}_{\mathrm{fin}}(\mathcal{H})$. We have to show that

$$EP_{\mathcal{H}}(U', V') = 0.$$

We may assume that U' and V' admit the same central character $W_0 t$. From the proof of part a) we see that

$$m_t \, EP_{\mathcal{H}}(U', V') = EP_{\mathcal{H}(\widetilde{\mathcal{R}},\tilde{q})}\big(\mathrm{Ind}_{\mathcal{H}}^{\mathcal{H}(\widetilde{\mathcal{R}},\tilde{q})} U', \mathrm{Ind}_{\mathcal{H}}^{\mathcal{H}(\widetilde{\mathcal{R}},\tilde{q})} V'\big) = 0. \qquad \Box$$

We can use the scaling maps

$$\tilde{\sigma}_\epsilon : \mathrm{Mod}_{\mathrm{fin}}(\mathcal{H}(\mathcal{R}, q)) \to \mathrm{Mod}_{\mathrm{fin}}(\mathcal{H}(\mathcal{R}, q^\epsilon))$$

from Theorem 1.7 to relate $EP_{\mathcal{H}}$ to EP_W.

Theorem 3.5. a) *The pairing $EP_{\mathcal{H}}$ is symmetric and positive semidefinite.*

b) *If $U, V \in \mathrm{Mod}_{\mathrm{fin}}(\mathcal{H})$ then*

$$EP_{\mathcal{H}}(U, V) = EP_{\mathcal{H}(\mathcal{R},q^\epsilon)}\big(\tilde{\sigma}_\epsilon(U), \tilde{\sigma}_\epsilon(V)\big) \qquad \forall \epsilon \in [-1, 1].$$

Proof. In view of Proposition 3.4.b) we may assume that $\mathcal{R}$ is semisimple. For every $\epsilon \in [-1, 1]$ Theorem 1.7 gives us the $\mathcal{H}(\mathcal{R}, q^\epsilon)$-representations

$$\tilde{\sigma}_\epsilon(\rho, U) = (\rho_\epsilon, U) \qquad \text{and} \qquad \tilde{\sigma}_\epsilon(\pi, V) = (\pi_\epsilon, V).$$

As a vector space $\mathcal{H}(\mathcal{R}, f, q^\epsilon)$ is just $\mathbb{C}[W_f \rtimes \Omega_f]$. As an algebra it is semisimple and the multiplication varies continuously with ϵ, so by Tits' deformation theorem it is independent of ϵ. Furthermore for any $w \in W_f \rtimes \Omega_f$ the maps

$$\epsilon \mapsto \rho_\epsilon(N_w) \qquad \text{and} \qquad \epsilon \mapsto \pi_\epsilon(N_w)$$

are continuous. In view of (3.16) this implies that

$$EP_{\mathcal{H}(\mathcal{R},q^\epsilon)}\big(\tilde{\sigma}_\epsilon(U), \tilde{\sigma}_\epsilon(V)\big)$$

depends continuously on ϵ. But this expression is integer valued, so it is actually independent of ϵ. In particular

$$EP_{\mathcal{H}}(U,V) = EP_W\big(\tilde{\sigma}_0(U), \tilde{\sigma}_0(V)\big). \tag{3.18}$$

Now Theorem 3.2.b) assures that $EP_{\mathcal{H}}$ is symmetric and positive semidefinite. $\Box$

For semisimple root data we can also compute the Euler–Poincaré characteristic in another way, as the character value of a certain index function.

According to Lemma 1.4 the algebra $\mathcal{H}(\mathcal{R}, f, q)$ is finite dimensional and semisimple for all facets f of the fundamental alcove $A_\emptyset$. In particular the collection $\mathrm{Irr}(\mathcal{H}(\mathcal{R}, f, q))$ of irreducible representations is finite. Let $e_\sigma \in \mathcal{H}(\mathcal{R}, f, q)$ denote the primitive central idempotent corresponding to an irreducible $\mathcal{H}(\mathcal{R}, f, q)$-module σ. For $U \in \mathrm{Mod}(\mathcal{H}(\mathcal{R}, f, q))$ let $[U : \sigma]$ be the multiplicity of σ in U.

In the spirit of Kottwitz [Kot, §2], Schneider and Stuhler [ScSt, III.4] we define an Euler–Poincaré function

$$f_{EP}^U := \sum_{f \subset \overline{A_\emptyset}} \frac{(-1)^{\dim f}}{[\Omega : \Omega_f]} \sum_{\sigma \in \mathrm{Irr}(\mathcal{H}(\mathcal{R},f,q))} \frac{[U \otimes \epsilon_f : \sigma]}{\dim \sigma} e_\sigma. \tag{3.19}$$

Proposition 3.6. *Let $\mathcal{R}$ be a semisimple root datum and $U, V \in \mathrm{Mod}_{\mathrm{fin}}(\mathcal{H})$. Then*

$$EP_{\mathcal{H}}(U,V) = \chi_V\big(f_{EP}^U\big).$$

Proof. Exactly like in (3.16) we can calculate that

$$\begin{aligned} EP_{\mathcal{H}}(U,V) &= \sum_{n=0}^{\mathrm{rk}(X)} (-1)^n \dim \mathrm{Hom}_{\mathcal{H}}(P_n(U)^\Omega, V) \\ &= \sum_{n=0}^{\mathrm{rk}(X)} \sum_{f : \dim f = n} \frac{(-1)^n}{[\Omega : \Omega_f]} \dim \mathrm{Hom}_{\mathcal{H}(\mathcal{R},f,q)}(U \otimes \epsilon_f, V) \\ &= \sum_{f \subset \overline{A_\emptyset}} \frac{(-1)^{\dim f}}{[\Omega : \Omega_f]} \sum_{\sigma \in \mathrm{Irr}(\mathcal{H}(\mathcal{R},f,q))} [U \otimes \epsilon_f : \sigma]\,[V : \sigma] \\ &= \sum_{f \subset \overline{A_\emptyset}} \frac{(-1)^{\dim f}}{[\Omega : \Omega_f]} \sum_{\sigma \in \mathrm{Irr}(\mathcal{H}(\mathcal{R},f,q))} \frac{[U \otimes \epsilon_f : \sigma]}{\dim \sigma} \chi_V(e_\sigma) \\ &= \chi_V\big(f_{EP}^U\big). \end{aligned}$$

$\Box$

We will use this result in [OpSo] to show that the Plancherel measure of a discrete series representation is a rational function in q, with rational coefficients.

3.5 Extensions of tempered modules

We apply the results of Chapter 2 to relate the bornological Tor and Ext functors over $\mathcal{H}$ with those over $\mathcal{S}$. That is more interesting than it looks at first sight, because $\mathcal{S}$ is not flat over $\mathcal{H}$ (unless $q \equiv 1$).

Corollary 3.7. *Take* $n \in \mathbb{N}$.

a) *For all* $U_b, V_b \in \mathrm{Mod}_{\mathrm{bor}}(\mathcal{S})$ *the inclusion* $\mathcal{H} \to \mathcal{S}$ *induces isomorphisms*

$$\begin{array}{lclcl} \mathrm{Tor}_n^{\mathcal{H}}(\mathcal{S}, V_b) & \cong & \mathrm{Tor}_n^{\mathcal{S}}(\mathcal{S}, V_b) & \cong & \left\{ \begin{array}{lll} V_b & \text{if} & n = 0 \\ 0 & \text{if} & n > 0, \end{array} \right. \\ \mathrm{Ext}^n_{\mathcal{H}}(U_b, V_b) & \cong & \mathrm{Ext}^n_{\mathcal{S}}(U_b, V_b). & & \end{array}$$

b) *For all finite dimensional tempered* $\mathcal{H}$*-modules* U *and* V *there is a natural isomorphism* $\mathrm{Ext}^n_{\mathcal{H}}(U, V) \cong \mathrm{Ext}^n_{\mathcal{S}}(U, V)$.

c) $EP_{\mathcal{H}}(U, V) = EP_{\mathcal{S}}(U, V)$.

Proof. a) follows directly from Theorems 2.9.a) and A.1.
b) In this setting the bornological functor $\mathrm{Ext}_n^{\mathcal{H}}$ agrees with its purely algebraic counterpart, as discussed in the Appendix. The same holds for $\mathrm{Ext}_n^{\mathcal{S}}$, because the resolution from Corollary 2.7 consists of $\mathcal{S}$-modules that are projective in both the algebraic and the bornological sense. Hence b) is a special case of a).

However, for semisimple root data this can be proved more directly, without the use of bornological techniques. Namely, we can simply compare the projective resolutions from Corollaries 2.3.a) and 2.7.a). If we use these to compute the Ext-groups and we apply Frobenius reciprocity, then we see that $\mathrm{Ext}^n_{\mathcal{H}}(U, V)$ and $\mathrm{Ext}^n_{\mathcal{S}}(U, V)$ are the homologies of isomorphic differential complexes. See also the remark at the end of Section 2.3.
c) is a trivial consequence of b). $\square$

We remark that the corresponding results for reductive p-adic groups were proved in [Mey3, §7] and [ScZi, §9]. These proofs are much more involved however, in particular no shortcut like the one described in our proof seems available.

Notice that we have to take the derived functors with respect to bornological tensor products and bounded maps, if we want to get Corollary 3.7.a) for infinite dimensional modules. If we would work purely algebraically this would already fail for $U = V = \mathcal{S}$.

The main use of Corollary 3.7 is the next theorem.

Theorem 3.8. *Suppose that* U *and* V *are irreducible tempered* $\mathcal{H}$*-modules. If* U *or* V *belongs to the discrete series then*

$$\mathrm{Ext}^n_{\mathcal{H}}(U, V) \cong \left\{ \begin{array}{ll} \mathbb{C} & \text{if } U \cong V \text{ and } n = 0 \\ 0 & \text{otherwise.} \end{array} \right.$$

Proof. The assertion for $n = 0$ follows directly from Schur's lemma and the general isomorphism $\mathrm{Ext}^0 \cong \mathrm{Hom}$.

Let δ be a discrete series representation of $\mathcal{H}$. According to [DeOp, Corollary 3.13] $\mathrm{End}_{\mathbb{C}}(\delta)$ is a direct summand of $\mathcal{S}$, as algebras. Therefore δ is both injective and projective as a $\mathcal{S}$-module. Using Corollary 3.7.b) we find that for any tempered $\mathcal{H}$-module V and any $n > 0$

$$\mathrm{Ext}^n_{\mathcal{H}}(V, \delta) = \mathrm{Ext}^n_{\mathcal{S}}(V, \delta) = 0 \tag{3.20}$$

because δ is injective, and

$$\mathrm{Ext}^n_{\mathcal{H}}(\delta, V) = \mathrm{Ext}^n_{\mathcal{S}}(\delta, V) = 0 \tag{3.21}$$

because δ is projective. □

Let us introduce the space of "elliptic trace functions"

$$Ell(\mathcal{H}) := G_{\mathbb{C}}(\mathcal{H}) \Big/ \sum_{I \subset F_0, I^\perp \neq 0} \mathrm{Ind}^{\mathcal{H}}_{\mathcal{H}(\mathcal{R}^I, q^I)} G_{\mathbb{C}}\big(\mathcal{H}(\mathcal{R}^I, q^I)\big), \tag{3.22}$$

where $I^\perp = \{y \in Y : \langle \alpha\, , y \rangle = 0 \, \forall \alpha \in I\}$. Notice that this space is zero whenever $\mathcal{R}$ is not semisimple. From Proposition 3.4 and Theorem 3.5 we see that the Euler–Poincaré characteristic induces a semidefinite Hermitian form on $Ell(\mathcal{H})$:

$$EP_{\mathcal{H}}(\lambda[U], \mu[V]) := \bar{\lambda}\mu \, EP_{\mathcal{H}}(U, V) \qquad U, V \in \mathrm{Mod}_{\mathrm{fin}}(\mathcal{H}),\ \lambda, \mu \in \mathbb{C}.$$

Proposition 3.9.

a) *The scaling map $\tilde{\sigma}_0$ induces a linear map $Ell(\mathcal{H}) \to Ell(W)$ which is an isometry with respect to the (semidefinite) Hermitian forms $EP_{\mathcal{H}}$ and EP_W.*

b) *The number of inequivalent discrete series representations of $\mathcal{H}$ is at most the number of elliptic conjugacy classes in W.*

Proof. a) follows directly from Theorem 3.5.b).
b) According to Theorem 3.8 the inequivalent discrete series representations form an orthonormal set in $Ell(\mathcal{H})$. By part a) the same holds for their images in $Ell(W)$. From Theorem 3.2.a) we know that the dimension of $Ell(W)$ is precisely the number of elliptic conjugacy classes in W. □

Remark.
A lower bound for the number of discrete series representations can be obtained from counting their central characters. In turns out that for the crucial irreducible non-simply laced cases $C_n^{(1)}, F_4$ and G_2 this lower bound equals the above upper bound, for *generic* parameters. We will exploit this in [OpSo] to give a classification of the irreducible discrete series characters for any irreducible non-simply laced affine Hecke algebra, with *arbitrary* positive parameters.

Example.
Let $R_0 = A_1 = \{1, -1\} = \{\pm\alpha\}$ and $X = \mathbb{Z}$. Then $W_0 = \{e, s_{\alpha^\vee}\}$ and W is generated by $s_{\alpha^\vee}$ and $s_{1+\alpha^\vee}$. Take a label function such that $q(s_{\alpha^\vee}) = q(s_{1+\alpha^\vee}) = q > 1$. The affine Hecke algebra $\mathcal{H} = \mathcal{H}(A_1, q)$ has a unique discrete series representation called the Steinberg representation. It has dimension one and is defined simply by

$$\mathrm{St}(N_w) = (-1)^{\ell(w)} q(w)^{-1/2} = (-q^{-1/2})^{\ell(w)}.$$

On the other hand we have the "trivial" $\mathcal{H}$-representation, defined by

$$\mathrm{triv}_{\mathcal{H}}(N_w) = q(w)^{1/2} = q^{\ell(w)/2}.$$

It is unitary but not tempered. From Theorem 3.8 we see that

$$EP_{\mathcal{H}}(\mathrm{St}, \mathrm{St}) = 1,$$

but is not immediately clear how many extensions of St by $\mathrm{triv}_{\mathcal{H}}$ there are. There certainly is an extension

$$0 \leftarrow \mathrm{St} \leftarrow \mathrm{Ind}_{\mathcal{A}}^{\mathcal{H}}(\phi_{q^{-1}}) \leftarrow \mathrm{triv}_{\mathcal{H}} \leftarrow 0, \tag{3.23}$$

so $\big[\mathrm{Ind}_{\mathcal{A}}^{\mathcal{H}}(\phi_{q^{-1}})\big] = [\mathrm{St}] + [\mathrm{triv}_{\mathcal{H}}]$ in $G(\mathcal{H})$. Therefore

$$\begin{aligned} EP_{\mathcal{H}}(\mathrm{St}, \mathrm{triv}_{\mathcal{H}}) &= EP_{\mathcal{H}}\big(\mathrm{St}, [\mathrm{triv}_{\mathcal{H}}] - \big[\mathrm{Ind}_{\mathcal{A}}^{\mathcal{H}}(\phi_{q^{-1}})\big]\big) \\ &= EP_{\mathcal{H}}\big(\mathrm{St}, -[\mathrm{St}]\big) = -1. \end{aligned}$$

From Corollary 2.3.d) we know that the cohomological dimension of $\mathrm{Mod}(\mathcal{H})$ is 1, so in particular

$$\mathrm{Ext}^n_{\mathcal{H}}(\mathrm{St}, \mathrm{triv}_{\mathcal{H}}) = 0 \quad \text{for } n > 1.$$

Therefore (3.23) is up to a scalar factor the only nontrivial extension of St by $\mathrm{triv}_{\mathcal{H}}$.

Appendix A

Bornological algebras

In Chapter 2 we induce several modules from $\mathcal{H}$ to $\mathcal{S}$. From an analytical point of view this operation is trivial for finite dimensional modules, since in that case all involved tensor products are purely algebraic. However for infinite dimensional modules we have to take the topology into account. For Fréchet $\mathcal{S}$-modules we can use the complete projective tensor product. But for tensor products over $\mathcal{H}$ this is problematic, because there is no canonical topology on $\mathcal{H}$.

Consider for example the trivial onedimensional root datum $(\mathbb{Z}, \emptyset, \mathbb{Z}, \emptyset)$. Then

$$\mathcal{H} = \mathbb{C}[\mathbb{Z}] \cong \mathcal{O}\big(\mathbb{C}^\times\big), \quad \mathcal{S} = \mathcal{S}(\mathbb{Z}) \cong C^\infty\big(S^1\big)\,.$$

For $t \in S^1$ the ideal

$$J_t := \{f \in C^\infty\big(S^1\big) : f(t) = 0\} \subset C^\infty\big(S^1\big)$$

is generated by $J_t \cap \mathcal{O}\big(\mathbb{C}^\times\big)$. It follows that for any finite dimensional $\mathcal{S}(\mathbb{Z})$-module V we have

$$\mathcal{S}(\mathbb{Z}) \otimes_{\mathbb{C}[\mathbb{Z}]} V \cong \mathcal{S}(\mathbb{Z}) \otimes_{\mathcal{S}(\mathbb{Z})} V = V\,.$$

This property does not readily generalize to infinite dimensional modules, for example

$$\mathcal{S}(\mathbb{Z}) \otimes_{\mathbb{C}[\mathbb{Z}]} \mathcal{S}(\mathbb{Z}) \not\cong \mathcal{S}(\mathbb{Z}) \otimes_{\mathcal{S}(\mathbb{Z})} \mathcal{S}(\mathbb{Z}) = \mathcal{S}(\mathbb{Z})\,.$$

The right technique to fix this is bornology. On many vector spaces bornological and topological analysis are equivalent, but bornologies combine well with homological algebra in larger classes. Bornologies are not so well-known, so we provide a brief introduction. See also [Mey1, Mey2].

A bornology on a complex vector space is a certain collection of subsets that are called bounded. This collection has to satisfy some axioms that generalize obvious properties of bounded sets in Banach spaces. A morphism of bornological vector spaces is a linear map that sends bounded sets to bounded sets. There is a natural notion of completeness of bornological vector spaces, similar to that of completeness of locally convex spaces.

On any vector space V we can define a more or less trivial bornology, the fine bornology. A subset $X \subset V$ belongs to this bornology if and only X is a bounded (in the usual sense) subset of some finite dimensional subspace of V. In this case

V is bornologically complete and any linear map from V to another bornological vector space is bounded. By default we equip vector spaces with a countable basis with the fine bornology.

More interestingly, if V is a complete topological vector space (e.g. a Fréchet space) we can define the precompact bornology on V as follows. We call $X \subset V$ bounded if and only if its closure $\overline{X}$ is compact. Under these assumptions V is bornologically complete and any continuous map between such vector spaces is bounded. Conversely, any bounded linear map between two Fréchet spaces with the precompact bornology is continuous [Mey1, Lemma 2.2].

The category of bornological vector spaces is not abelian, but it does have enough injective and projective objects. It also possesses inductive and projective limits.

Let V be a bornological vector space and $\mathrm{End}_{\mathrm{bor}}(V)$ the algebra of bounded linear maps $V \to V$. A subset $L \subset \mathrm{End}_{\mathrm{bor}}(V)$ is equibounded if $L(X) := \{l(x) : l \in L, x \in X\}$ is bounded for any bounded set $X \subset V$. This gives $\mathrm{End}_{\mathrm{bor}}(V)$ the structure of a bornological algebra.

Let A be a unital bornological algebra. By definition a bornological A-module structure on V is the same as a bounded bilinear map $A \times V \to V$, or as a bounded algebra homomorphism $A \to \mathrm{End}_{\mathrm{bor}}(V)$. Let $\mathrm{Mod}_{\mathrm{bor}}(A)$ be the category of bornological A-modules.

The A-balanced completed bornological tensor product $\widehat{\otimes}_A$ is defined by the following universal property. Bounded linear maps $V_1 \widehat{\otimes}_A V_2 \to V_3$ with V_3 complete correspond bijectively to bounded bilinear maps $b : V_1 \times V_2 \to V_3$ that satisfy $b(v_1 a, v_2) = b(v_1, a v_2)$.

In case V_1, V_2 and A have the fine bornology this is just the algebraic tensor product over A. On the other hand, if V_1, V_2 and A are Fréchet spaces with the precompact bornology, then this agrees with the completed projective tensor product over A.

By definition a sequence

$$0 \to V_1 \to V_2 \to V_3 \to 0$$

in $\mathrm{Mod}_{\mathrm{bor}}(A)$ is a bornological extension if the maps are bounded A-module homomorphisms and the sequence is split exact in the category of bornological vector spaces. We call a differential complex of bornological A-modules exact if it admits a bounded $\mathbb{C}$-linear contraction. These notions of extensions and exactness make $\mathrm{Mod}_{\mathrm{bor}}(A)$ into an exact category, whose derived category we denote by $\mathrm{Der}_{\mathrm{bor}}(A)$. Let $\widehat{\otimes}_A^{\mathbb{L}}$ and $\mathbb{R}\mathrm{Hom}_A$ denote the total derived functors of $\widehat{\otimes}_A$ and Hom_A. Thus $U \widehat{\otimes}_A^{\mathbb{L}} V$ is an object of $\mathrm{Der}_{\mathrm{bor}}(A)$ whose homology is $\mathrm{Tor}_*^A(U, V)$, and the (co)homology of $\mathbb{R}\mathrm{Hom}_A(U, V)$ is $\mathrm{Ext}_A^*(U, V)$. However, the total derived functors contain somewhat more information, as the passage to homology forgets the bornological properties of these differential complexes.

Suppose that A, U and V have the fine bornology. Then the bornological functors $\widehat{\otimes}_A$ and Hom_A agree with their algebraic counterparts. Hence $\mathrm{Tor}_n^A(U, V)$ and $\mathrm{Ext}_A^n(U, V)$ are the same in the algebraic and the bornological sense.

Let $f : A \to B$ be a morphism of unital complete bornological algebras and

$$0 \leftarrow A \leftarrow P_0 \leftarrow P_1 \leftarrow \cdots$$

a resolution of A by projective $A\widehat{\otimes}A^{op}$-modules.

Theorem A.1. [Mey2, Theorem 35]
The following are equivalent:

1) $B\widehat{\otimes}_A P_* \widehat{\otimes}_A B$ *is a projective* $B\widehat{\otimes}B^{op}$*-module resolution of* B.

2) $(f^*B)\widehat{\otimes}^{\mathbb{L}}_A(f^*B) \to B\widehat{\otimes}^{\mathbb{L}}_B B\ (\cong B)$ *is an isomorphism.*

3) $(f^*U)\widehat{\otimes}^{\mathbb{L}}_A(f^*V) \to U\widehat{\otimes}^{\mathbb{L}}_B V$ *is an isomorphism* $\forall\, U \in \mathrm{Mod}_{\mathrm{bor}}(B^{op}), V \in \mathrm{Mod}_{\mathrm{bor}}(B)$.

4) $\mathbb{R}\mathrm{Hom}_B(U,V) \to \mathbb{R}\mathrm{Hom}_A(f^*U, f^*V)$ *is an isomorphism* $\forall\, U, V \in \mathrm{Mod}_{\mathrm{bor}}(B)$.

5) *The functor* $f^* : \mathrm{Der}_{\mathrm{bor}}(B) \to \mathrm{Der}_{\mathrm{bor}}(A)$ *is fully faithful.*

We call f isocohomological if these conditions hold.

Direct consequences of conditions 3) and 4) are

$$\begin{aligned} \mathrm{Tor}^B_*(U,V) &\cong \mathrm{Tor}^A_*(f^*U, f^*V), \\ \mathrm{Ext}^*_B(U,V) &\cong \mathrm{Ext}^*_A(f^*U, f^*V), \end{aligned} \tag{A.1}$$

where we mean are the derived functors in the bornological category.

We equip $\mathcal{H}$ with the fine bornology and let $\mathrm{Mod}_{\mathrm{bor}}(\mathcal{H})$ be the category of all bornological $\mathcal{H}$-modules. Notice that any $\mathcal{H}$-module can be made bornological by endowing it with the fine bornology. This identifies $\mathrm{Mod}(\mathcal{H})$ with a full subcategory of $\mathrm{Mod}_{\mathrm{bor}}(\mathcal{H})$. An $\mathcal{H}$-module is bornologically projective if and only if it is algebraically projective, namely if and only if it is a direct summand of an (algebraically) free $\mathcal{H}$-module. So as long as we are working in a purely algebraic setting the bornological structure does not give much extra, but neither is it a restriction.

We endow $\mathcal{S}$ with the precompact bornology, so that any finite dimensional $\mathcal{S}$-module is bornological. We denote the category of all bornological $\mathcal{S}$-modules by $\mathrm{Mod}_{\mathrm{bor}}(\mathcal{S})$. Probably there exist $\mathcal{S}$-modules that do not admit the structure of a bornological $\mathcal{S}$-module, but they seem to be rather far-fetched. We note that a projective object of $\mathrm{Mod}_{\mathrm{bor}}(\mathcal{S})$ is usually not a projective $\mathcal{S}$-module in the algebraic sense, rather a completion of the latter.

A bornological $\mathcal{H}$-module (π, V) is called tempered if it extends to $\mathcal{S}$, that is, if the following equivalent conditions hold:

1) π extends to a bounded algebra homomorphism $\mathcal{S} \to \mathrm{End}_{\mathrm{bor}}(V)$,

2) π induces a bounded bilinear map $\mathcal{S} \times V \to V$.

A (sub-)linear functional $f : \mathcal{H} \to \mathbb{C}$ is tempered if there exist $C, N \in (0, \infty)$ such that

$$|f(N_w)| \leq C(1 + \mathcal{N}(w))^N \qquad \forall w \in W.$$

The collection of all tempered linear functionals is the continuous dual space of $\mathcal{S}(\mathcal{R}, q)$.

Proposition A.2. *Let V be a Fréchet space endowed with the precompact bornology. An $\mathcal{H}$-module (π, V) is bornological if and only if $\pi(h) : V \to V$ is continuous $\forall h \in \mathcal{H}$. Moreover it is tempered if and only if the following equivalent conditions hold:*

3) *π induces a jointly continuous map $\mathcal{S} \times V \to V$,*

4) *π induces a separately continuous map $\mathcal{S} \times V \to V$,*

5) *for every $v \in V$ and every continuous seminorm p on V the sublinear functional*

$$\mathcal{H} \to [0, \infty) : h \mapsto p(\pi(h)v)$$

is tempered,

6) *for every $v \in V$ and every $f \in V^*$ the linear functional*

$$\mathcal{H} \to \mathbb{C} : h \mapsto f(\pi(h)v)$$

is tempered.

In particular the category $\mathrm{Mod}_{Fr\acute{e}}(\mathcal{S})$ of continuous Fréchet $\mathcal{S}$-modules is a full subcategory of $\mathrm{Mod}_{\mathrm{bor}}(\mathcal{S})$.

Proof. We already noted that $\pi(h) : V \to V$ is continuous if and only if it is bounded. Since $\mathcal{H}$ carries the fine bornology this is equivalent to the first assertion.

For the same reason $\mathrm{Mod}_{\mathrm{Fré}}(\mathcal{S})$ forms a full subcategory of $\mathrm{Mod}_{\mathrm{bor}}(\mathcal{S})$.

It is clear that condition 3) implies the other five. Conversely 3) follows from 2) by [Mey1, Lemma 2.2] and from 4) by the Banach-Steinhaus theorem.

If $f \in V^*$ then $|f|$ is a continuous seminorm on V, so 5) implies 6).

Finally we show that 6) implies 4). Endow $\mathcal{H}$ with the induced topology from $\mathcal{S}$ and fix $v \in V$. By assumption the linear map

$$\mathcal{H} \to V : h \mapsto \pi(h)v \tag{A.2}$$

is continuous for the weak topology on V. Since V is Fréchet (A.2) is also continuous for the metric topology on V [KeNa, 21.4.i]. Hence (A.2) extends continuously to the metric completion $\mathcal{S}$ of $\mathcal{H}$.

Now we fix $h = \sum_{w \in W} h_w N_w \in \mathcal{S}$ and we write $h_n = \sum_{w : \mathcal{N}(w) \leq n} h_w N_w$. We assumed that V is a Fréchet $\mathcal{H}$-module, so $(\pi(h_n))_{n=1}^{\infty}$ is a sequence of continuous linear operators on V. We just showed that for fixed $v \in V$ the sequence $(\pi(h_n)v)_{n=1}^{\infty}$ converges to $\pi(h)v$. The Banach-Steinhaus theorem (see e.g. [KeNa, p. 104-105]) assures that $\pi(h)$ is continuous.

We conclude that $(h, v) \mapsto \pi(h)v$ is separately continuous. $\square$

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