

A proof of the multiplicity one conjecture for GL_n in GL_{n+1}

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Abstract

Let \mathbb{F} be a non-archimedean local field of characteristic zero. We consider distributions on $GL(n+1, \mathbb{F})$ which are invariant under the adjoint action of $GL(n, \mathbb{F})$. We prove that any such distribution is invariant with respect to transposition. This implies that the restriction to $GL(n)$ of any irreducible smooth representation of $GL(n+1)$ is multiplicity free.

Our paper is based on the recent work [RS] of Steve Rallis and Gérard Schiffmann where they made a remarkable progress on this problem.

In [RS], they also show that our result implies multiplicity one theorem for restrictions from the orthogonal group $O(V \oplus \mathbb{F})$ to $O(V)$.

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1 Introduction

Let \mathbb{F} be a non-archimedean local field of characteristic zero. Consider the standard imbedding $\mathrm{GL}(n, \mathbb{F}) \subset \mathrm{GL}(n+1, \mathbb{F})$. Let $\mathrm{GL}(n, \mathbb{F})$ act on $\mathrm{GL}(n+1, \mathbb{F})$ by conjugation. The goal of this paper is to prove the following theorem.

Theorem 1.1 *Every $\mathrm{GL}(n, \mathbb{F})$ -invariant distribution on $\mathrm{GL}(n+1, \mathbb{F})$ is invariant with respect to transposition.*

This theorem is important in representation theory, since it implies the following multiplicity one theorem (see e.g. [RS], section 2).

Theorem 1.2 *Let π be an irreducible smooth representation of $\mathrm{GL}(n+1, \mathbb{F})$ and ρ be an irreducible smooth representation of $\mathrm{GL}(n, \mathbb{F})$. Then*

$$\dim \mathrm{Hom}_{\mathrm{GL}(n, \mathbb{F})}(\pi|_{\mathrm{GL}(n, \mathbb{F})}, \rho) \leq 1.$$

In their recent paper [RS], in part II, Steve Rallis and Gérard Schiffmann have shown that theorem 1.1 also implies similar theorems for distributions on unitary and orthogonal groups, which in turn imply multiplicity one results for those groups.

In [RS], Rallis and Schiffmann have also made a remarkable progress in proving the above theorem 1.1, and our paper is based on their results. For the benefit of the reader we present the proofs of all the statements from [RS] that we use.

Theorem 1.1 also gives another proof of Bernstein's theorem about P -invariant distributions on $\mathrm{GL}(n)$ (see [Ber]) which proves Kirillov's conjecture in the non-archimedean case.

1.1 Reformulation of the main theorem

Let $G := G_n := \mathrm{GL}(n, \mathbb{F})$. Consider the action of the 2-element group S_2 on G given by the involution $g \mapsto {}^t g^{-1}$. It defines a semidirect product $\tilde{G} := \tilde{G}_n := G_n \rtimes S_2$. Let $V := V_n = \mathbb{F}^n$ and $X := X_n := \mathrm{sl}(V_n) \times V_n \times V_n^*$ where $\mathrm{sl}(V) \subset \mathrm{End}(V)$ is the space of operators with zero trace.

The group \tilde{G} acts on X by

$$g(A, v, \phi) := (gAg^{-1}, gv, g^{-1*} \phi) \text{ and}$$

$$T(A, v, \phi) := ({}^t A, {}^t \phi, {}^t v)$$

where $g \in G$ and T is the generator of S_2 . Here, ${}^t A$ denotes the transposed matrix in sl_n , ${}^t \phi \in V_n$ denotes the column vector corresponding to the row vector $\phi \in V_n^*$, and ${}^t v$ denotes the row vector corresponding to the column vector $v \in V_n$. Also for any operator $g : V \rightarrow V$, we denote by $g^* : V^* \rightarrow V^*$ the adjoint operator.

Note that \tilde{G} acts separately on $\mathrm{sl}(V)$ and on $V \times V^*$. Define a character χ of \tilde{G} by $\chi(g, s) := \mathrm{sign}(s)$.

Theorem 1.1 can be deduced from the following theorem.

Theorem 1.3 *Any (\tilde{G}, χ) -equivariant distribution on X is zero.*

The deduction was first proven in [RS] (section 5). We prove it in section 3 in a slightly different way. In section 3 we also give a coordinate-free definition of the group \tilde{G} and its action on X .

1.2 Sketch of our proof

The theorem will be proved by induction on n . Let S denote the closure of the union of the supports of all (\tilde{G}, χ) -equivariant distributions on X . We would like to prove that $S = \emptyset$.

Let $pr_1 : X \rightarrow \mathfrak{sl}(V)$ and $pr_2 : X \rightarrow V \oplus V^*$ be the natural projections. Rallis and Schiffman have shown that the induction hypothesis implies:

- (i) $pr_2(S)$ is contained in $Y := \{(v, \phi) \in V \oplus V^* \mid \langle \phi, v \rangle = 0\}$
- (ii) $pr_1(S)$ is contained in the nilpotent cone \mathcal{N} .

Part (i) follows from the localization principle and Frobenius reciprocity.

Part (ii) is proven using Harish-Chandra descent method.

We will present a proof of this statement in the appendix (section 7).

For any vector v and covector ϕ let $v \otimes \phi$ denote the operator of rank one defined by them. Let $\lambda \in \mathbb{F}$ be a scalar. We use a family of automorphisms ν_λ of X defined by

$$\nu_\lambda(A, v, \phi) := (A + \lambda v \otimes \phi - \lambda \frac{\langle \phi, v \rangle}{n} \text{Id}, v, \phi).$$

Note that the automorphisms ν_λ commute with the action of \tilde{G} and hence preserve S .

Let $\mathcal{N}_i \subset \mathfrak{sl}(V)$ be the union of all nilpotent orbits of dimensions $\leq i$. We prove by downward induction that $S \subset \mathcal{N}_i \times Y$ for all i . Suppose $S \subset \mathcal{N}_i \times Y$. We have to show $S \subset \mathcal{N}_{i-1} \times Y$. Since $\nu_\lambda(S) = S$, S is contained in the intersection $\bigcap \nu_\lambda(\mathcal{N}_i \times Y)$.

We have to show that for any nilpotent orbit O of dimension i , the restriction of any (\tilde{G}, χ) -equivariant distribution ξ to $O \times Y$ is zero. As we have seen, the support of $\xi|_{O \times Y}$ is contained in $(O \times Y) \cap (\bigcap \nu_\lambda(\mathcal{N}_i \times Y))$, which we denote by \tilde{O} . Using the fact that the Fourier transform of a (\tilde{G}, χ) -equivariant distribution is also (\tilde{G}, χ) -equivariant, the theorem boils down to the following key lemma.

Lemma 1.4 (Key lemma) *Let O be a nilpotent orbit. Let $\zeta \in \mathcal{S}^*(O \times V \times V^*)^{\tilde{G}, \chi}$. Suppose that both ζ and $\mathcal{F}(\zeta)$ are supported in \tilde{O} . Then $\zeta = 0$.*

Using Frobenius reciprocity, the key lemma reduces to a statement about distributions on $V \oplus V^*$.

Namely, fix $A \in O$. Let R_A denote the fiber over A of the projection $\tilde{O} \rightarrow O$. Then ζ corresponds to a distribution η on $V \oplus V^*$ with the following properties:

- (i) η is supported in R_A
- (ii) $\mathcal{F}(\eta)$ is supported in R_A
- (iii) η is χ -equivariant with respect to the stabilizer of A in \tilde{G} .

We have to show $\eta = 0$. We will prove that R_A is contained in

$$Q_A := \{(v, \phi) \in V \oplus V^* \mid v \otimes \phi \in [A, \mathfrak{gl}(V)]\}.$$

It is convenient to work with Q_A since its description is linear.

For example, we will show that $Q_{A_1 \oplus A_2} \subset Q_{A_1} \times Q_{A_2}$. This will allow us to decompose the problem into Jordan blocks (see section 5.1).

Then we will solve the case of one Jordan block (in section 5.2) using an important result by Rallis and Schiffmann which is proven using the Weil representation.

2 Preliminaries and notations

We will use the standard terminology of l -spaces introduced in [BZ], section 1. We denote by $\mathcal{S}(Z)$ the space of Schwartz functions on an l -space Z , and by $\mathcal{S}^*(Z)$ the space of distributions on Z equipped with the weak topology.

We fix a nontrivial additive character ψ of \mathbb{F} .

Notation 2.1 (Fourier transform) *Let W be a finite dimensional vector space over \mathbb{F} . Let B be a nondegenerate symmetric bilinear form on W . We denote by $\mathcal{F}_B : \mathcal{S}^*(W) \rightarrow \mathcal{S}^*(W)$ the Fourier transform defined using B and the self-dual measure on W .*

By abuse of notation, we also denote by \mathcal{F}_B the partial Fourier transform $\mathcal{F}_B : \mathcal{S}^(Z \times W) \rightarrow \mathcal{S}^*(Z \times W)$ for any l -space Z .*

If $W = U \oplus U^$ then it has a canonical symmetric bilinear form given by the quadratic form $Q((v, \phi)) := \langle \phi, v \rangle := \phi(v)$. We will denote the Fourier transform defined by it simply by \mathcal{F}_W . If there is no ambiguity, we will denote it simply by \mathcal{F} .*

Proposition 2.2 *Let $W_1 \oplus W_2$ be finite dimensional vector spaces. Let B_1 and B_2 be nondegenerate symmetric bilinear forms on W_1 and W_2 respectively. Let $Z \subset W_1$ be a closed subset. Let $\xi \in \mathcal{S}^*(W_1 \oplus W_2)$ be a distribution. Suppose that $\mathcal{F}_{B_1 \oplus B_2}(\xi)$ is supported in $Z \times W_2$. Then $\mathcal{F}_{B_1}(\xi)$ is also supported in $Z \times W_2$.*

Proof. Let p_1 denote the projection $W_1 \oplus W_2 \rightarrow W_1$. Since \mathcal{F}_{B_2} does not change the projection of the support of a distribution to W_1 ,

$$p_1(\text{Supp}(\mathcal{F}_{B_1}(\xi))) = p_1(\text{Supp}(\mathcal{F}_{B_2} \circ \mathcal{F}_{B_1}(\xi))) = p_1(\mathcal{F}_{B_1 \oplus B_2}(\xi)) \subset Z$$

□

We will use the localization principle, formulated in [Ber], section 1.4.

Theorem 2.3 (Localization principle) *Let $q : Z \rightarrow T$ be a continuous map of l -spaces. Denote $Z_t := q^{-1}(t)$. Consider $\mathcal{S}^*(Z)$ as $\mathcal{S}(T)$ -module. Let M be a closed subspace of $\mathcal{S}^*(Z)$ which is an $\mathcal{S}(T)$ -submodule. Then $M = \bigoplus_{t \in T} (M \cap \mathcal{S}^*(Z_t))$.*

Informally, it means that in order to prove a certain property of distributions on Z it is enough to prove that distributions on every fiber Z_t have this property.

Corollary 2.4 *Let $q : Z \rightarrow T$ be a continuous map of l -spaces. Let an l -group H act on an l -space Z preserving the fibers of q . Let μ be a character of H . Suppose that for any $t \in T$, $\mathcal{S}^*(q^{-1}(t))^{H, \mu} = 0$. Then $\mathcal{S}^*(Z)^{H, \mu} = 0$*

Corollary 2.5 *Let $H_i \subset \tilde{H}_i$ be l -groups acting on l -spaces Z_i for $i = 1, \dots, k$. Suppose that $\mathcal{S}^*(Z_i)^{H_i} = \mathcal{S}^*(Z_i)^{\tilde{H}_i}$ for all i . Then $\mathcal{S}^*(\prod Z_i)^{\prod H_i} = \mathcal{S}^*(\prod Z_i)^{\prod \tilde{H}_i}$.*

We will use the following version of the Frobenius reciprocity. It can be easily deduced from the Frobenius reciprocity described in [Ber], section 1.5.

Theorem 2.6 (Frobenius reciprocity) *Let a unimodular l -group H act transitively on an l -space Z . Let $\varphi : E \rightarrow Z$ be an H -equivariant map of l -spaces. Let*

$x \in Z$. Suppose that its stabilizer $\text{Stab}_H(x)$ is unimodular. Let W be the fiber of x . Let μ be a character of H . Then

- (i) There exists a canonical isomorphism $\text{Fr} : \mathcal{S}^*(E)^{H,\mu} \rightarrow \mathcal{S}^*(W)^{\text{Stab}_H(x),\mu}$.
- (ii) For any distribution $\xi \in \mathcal{S}^*(E)^{H,\mu}$, $\text{Supp}(\text{Fr}(\xi)) = \text{Supp}(\xi) \cap W$.
- (iii) Frobenious reciprocity commutes with Fourier transform.

Namely, let W be a finite dimensional linear space over \mathbb{F} with a nondegenerate bilinear form B . Let H act on W linearly preserving B .

Then for any $\xi \in \mathcal{S}^*(Z \times W)^{H,\mu}$, we have $\mathcal{F}_B(\text{Fr}(\xi)) = \text{Fr}(\mathcal{F}_B(\xi))$ where Fr is taken with respect to the projection $Z \times W \rightarrow Z$.

Definition 2.7 Let W be a finite dimensional vector space over \mathbb{F} . We call a distribution $\xi \in \mathcal{S}^*(W)$ **abs-homogeneous of degree d** if for any function $f \in \mathcal{S}(W)$, $|\xi(h_{t^{-1}}(f))| = |t|^{-d}|\xi(f)|$ where $(h_{t^{-1}}(f))(v) = f(tv)$.

For example, a Haar measure on W is abs-homogeneous of degree $\dim W$ and the δ -distribution is abs-homogeneous of degree 0.

A crucial step in the proof of the main theorem is the following special case of a result by Rallis and Schiffmann ([RS], lemma 8.1).

Theorem 2.8 (Rallis-Schiffmann) Let W be a finite dimensional vector space over \mathbb{F} and B be a nondegenerate symmetric bilinear form on W . Denote $Z(B) := \{v \in W \mid B(v, v) = 0\}$. Let ξ be a distribution on W . Suppose that both ξ and $\mathcal{F}_B(\xi)$ are supported in Z_B .

Then ξ is abs-homogeneous of degree $\frac{1}{2} \dim W$.

For the benefit of the reader we reproduce the proof of this theorem in section 6.

Remark 2.9 Let Z be an l -space and $Q \subset Z$ be a closed subset. We will identify $\mathcal{S}^*(Q)$ with the space of all distributions on Z supported on Q . In particular, we can restrict a distribution ξ to any open subset of the support of ξ .

3 Reformulations of the problem

In this section we will prove the following proposition.

Proposition 3.1 *Theorem 1.3 implies theorem 1.1.*

We will divide this reduction to several propositions. Consider the action of \tilde{G}_n on G_{n+1} and on \mathfrak{gl}_{n+1} where G_n acts by conjugation and the generator of S_2 acts by transposition.

Proposition 3.2 *If $\mathcal{S}^*(G_{n+1})^{\tilde{G}_n, \chi} = 0$ then theorem 1.1 holds.*

The proof is straightforward.

Proposition 3.3 *If $\mathcal{S}^*(\mathfrak{gl}_{n+1})^{\tilde{G}_n, \chi} = 0$ then $\mathcal{S}^*(G_{n+1})^{\tilde{G}_n, \chi} = 0$.*

*Proof.*¹ Let $\xi \in \mathcal{S}^*(G_{n+1})^{\tilde{G}_{n,\chi}}$. We have to prove $\xi = 0$. Assume the contrary. Take $p \in \text{Supp}(\xi)$. Let $t = \det(p)$. Let $f \in \mathcal{S}(\mathbb{F})$ be such that $f(0) = 0$ and $f(t) \neq 0$. Consider the determinant map $\det : G_{n+1} \rightarrow \mathbb{F}$. Consider $\xi' := (f \circ \det) \cdot \xi$. It is easy to check that $\xi' \in \mathcal{S}^*(G_{n+1})^{\tilde{G}_{n,\chi}}$ and $p \in \text{Supp}(\xi')$. However, we can extend ξ' by zero to $\xi'' \in \mathcal{S}^*(\mathfrak{gl}_{n+1})^{\tilde{G}_{n,\chi}}$, which is zero by the assumption. Hence ξ' is also zero. Contradiction. \square

Proposition 3.4 *If $\mathcal{S}^*(\mathfrak{sl}_{n+1})^{\tilde{G}_{n,\chi}} = 0$ then $\mathcal{S}^*(\mathfrak{gl}_{n+1})^{\tilde{G}_{n,\chi}} = 0$*

Proof. Consider the trace map $\text{tr} : \mathfrak{gl}_{n+1} \rightarrow \mathbb{F}$. By the localization principle (Corollary 2.4), it is enough to prove that for any $t \in \mathbb{F}$, $\mathcal{S}^*(\text{tr}^{-1}(t))^{\tilde{G}_{n,\chi}} = 0$. However, all $\text{tr}^{-1}(t)$ are isomorphic as \tilde{G}_n -equivariant l -spaces to \mathfrak{sl}_{n+1} by $A \mapsto A - \frac{\text{tr}(A)}{n}\text{Id}$. \square

Proposition 3.5 *If $\mathcal{S}^*(X_n)^{\tilde{G}_{n,\chi}} = 0$ then $\mathcal{S}^*(\mathfrak{sl}_{n+1})^{\tilde{G}_{n,\chi}} = 0$.*

Proof. Consider the map $q : \mathfrak{sl}_{n+1} \rightarrow \mathbb{F}$ given by $q(B) := B_{n+1,n+1}$. By the localization principle (corollary 2.4), it is enough to prove that for any $t \in \mathbb{F}$, $\mathcal{S}^*(q^{-1}(t))^{\tilde{G}_{n,\chi}} = 0$. However, all $q^{-1}(t)$ are isomorphic as \tilde{G}_n -equivariant l -spaces to X_n by

$$\begin{pmatrix} A_{n \times n} & v_{n \times 1} \\ \phi_{1 \times n} & \lambda \end{pmatrix} \mapsto (A + \frac{\lambda}{n}\text{Id}, v, \phi)$$

\square

This finishes the proof of proposition 3.1.

Remark 3.6 *One can give a coordinate free description of \tilde{G} and of its action on X . Namely, \tilde{G} is isomorphic to the disjoint union of the group $G = \text{Aut}(V)$ of automorphisms of V and the set $\text{Iso}(V, V^*)$ of isomorphisms between V and V^* with the following multiplication. Let $g, g' \in \text{Aut}(V)$ and $h, h' \in \text{Iso}(V, V^*)$.*

$$\begin{aligned} g \times g' &:= g \circ g' & h \times g &:= h \circ g \\ g \times h &:= g^{*-1} \circ h & h \times h' &:= h^{*-1} \circ h' \end{aligned}$$

The action of \tilde{G} on X is given by

$$g(A, v, \phi) = (gA, gv, (g^*)^{-1}\phi) \quad \text{and} \quad h(A, v, \phi) = ((hAh^{-1})^*, (h^*)^{-1}\phi, hv)$$

4 Proof of the main theorem

Rallis and Schiffmann have proven various properties of the support of (\tilde{G}, χ) -equivariant distributions on X . We will now summarize those that we need in our proof.

Notation 4.1 *Denote the cone of nilpotent operators in $\mathfrak{sl}(V)$ by \mathcal{N} . Denote also*

$$Y := \{(v, \phi) \in V \oplus V^* \mid \langle \phi, v \rangle = 0\}.$$

¹This proof is analogous to the proof of an analogous statement in [Ber], section 2.2.

Theorem 4.2 (Rallis-Schiffmann) *Suppose that the main theorem holds for all dimensions smaller than n for all finite extensions \mathbb{E} of \mathbb{F} . Let ξ be a (\tilde{G}, χ) -equivariant distribution on X . Then ξ is supported in $\mathcal{N} \times Y$.*

We reproduce the proof of this theorem in the appendix (section 7).

Now we will stratify the nilpotent cone and reduce the support of the distribution stratum by stratum.

Notation 4.3 *For any i we denote by \mathcal{N}_i the union of all nilpotent orbits of dimensions $\leq i$. Note that \mathcal{N}_i are Zariski closed, $\mathcal{N}_i = \mathcal{N}$ for i large enough and $\mathcal{N}_{-1} = \emptyset$.*

In order to excise the support of the distribution we will use a family of automorphisms of the problem, which play a crucial role in our proof.

Notation 4.4 *For any $\lambda \in \mathbb{F}$ we denote by $\nu_\lambda : X \rightarrow X$ the homeomorphism defined by*

$$\nu_\lambda(A, v, \phi) := (A + \lambda v \otimes \phi - \lambda \frac{\langle \phi, v \rangle}{n} \text{Id}, v, \phi).$$

A simple but important observation is that ν_λ commutes with the action of \tilde{G} .

Notation 4.5 *Let O be a nilpotent orbit of dimension i . We set*

$$\tilde{O} := (O \times Y) \cap \bigcap_{\lambda \in \mathbb{F}} \nu_\lambda^{-1}(\mathcal{N}_i \times Y).$$

To proceed stratum by stratum we will need the following key lemma.

Lemma 4.6 (Key lemma) *Let O be a nilpotent orbit. Note that $O \times V \times V^*$ is \tilde{G} -invariant. Let $\zeta \in \mathcal{S}^*(O \times V \times V^*)^{\tilde{G}, \chi}$. Suppose that $\text{Supp}(\zeta) \subset \tilde{O}$ and $\text{Supp}(\mathcal{F}_{V \oplus V^*}(\zeta)) \subset \tilde{O}$. Then $\zeta = 0$.*

The proof will be given in section 5 below.

Now we are ready to prove the main theorem.

Theorem 4.7 *Any (\tilde{G}, χ) -equivariant distribution on X is zero.*

Proof. We prove by downward induction the following statement: for any i , any $\xi \in \mathcal{S}^*(X)^{(\tilde{G}, \chi)}$ is supported in $\mathcal{N}_i \times Y$. For i large enough it is theorem 4.2. Suppose that the statement is true for i and let us prove it for $i - 1$. Let $\xi \in \mathcal{S}^*(X)^{\tilde{G}, \chi}$. We need to show that $\xi_{(\mathcal{N}_i \setminus \mathcal{N}_{i-1}) \times Y} = 0$. For this it is enough to show that for any nilpotent orbit O of dimension i , we have $\xi|_{O \times Y} = 0$.

Denote $\zeta = \xi|_{O \times V \times V^*}$. We know that $\text{Supp}(\xi) \subset \mathcal{N}_i \times Y$. On the other hand, $\nu_\lambda(\xi)$ is also (\tilde{G}, χ) -equivariant for any λ . Therefore $\text{Supp}(\xi) \subset \bigcap_{\lambda \in \mathbb{F}} \nu_\lambda^{-1}(\mathcal{N}_i \times Y)$.

Thus

$$\text{Supp}(\zeta) \subset (O \times Y) \cap \left(\bigcap_{\lambda \in \mathbb{F}} \nu_\lambda^{-1}(\mathcal{N}_i \times Y) \right) = \tilde{O}.$$

Since the action of \tilde{G} preserves the standard bilinear form on $V \oplus V^*$, $\mathcal{F}_{V \oplus V^*}(\xi)$ is also (\tilde{G}, χ) -equivariant. Note that $\mathcal{F}_{V \oplus V^*}(\zeta) = \mathcal{F}_{V \oplus V^*}(\xi)|_{O \times V \times V^*}$ and hence $\text{Supp}(\mathcal{F}_{V \oplus V^*}(\zeta))$ is also contained in \tilde{O} . Thus by the key lemma $\zeta = 0$. \square

5 Proof of the key lemma

Notation 5.1 Let $A \in \mathfrak{sl}(V)$ be a nilpotent element. Let O be the orbit of A and i be the dimension of O . We denote by R_A the fiber at A of the projection $\tilde{O} \rightarrow O$. We consider R_A as a subset of $V \oplus V^*$.

Note that $R_A \subset Y$.

Notation 5.2 Let $A \in \mathfrak{sl}(V)$ be a nilpotent element. We denote

$$Q_A := \{(v, \phi) \in V \oplus V^* \mid v \otimes \phi \in [A, \mathfrak{gl}(V)]\}.$$

Lemma 5.3 $R_A \subset Q_A$

Proof. Let $(v, \phi) \in R_A$. Let O be the orbit of A and i be the dimension of O . Consider the Zariski tangent space $T_A \mathcal{N}_i$ to \mathcal{N}_i at A . It coincides with $T_A O = [A, \mathfrak{gl}(V)]$. Since $\langle \phi, v \rangle = 0$, we see that \mathcal{N}_i contains the line $\{A + \lambda v \otimes \phi\}$. Thus $v \otimes \phi \in T_A \mathcal{N}_i = [A, \mathfrak{gl}(V)]$ and hence $(v, \phi) \in Q_A$. \square

Notation 5.4 Let $A \in \mathfrak{sl}(V)$ be a nilpotent element. We denote by C_A the stabilizer of A in G and by \tilde{C}_A the stabilizer of A in \tilde{G} .

It is known that C_A is unimodular and hence \tilde{C}_A is also unimodular.

The key lemma follows now from Frobenius reciprocity and the following proposition.

Proposition 5.5 Let $A \in \mathfrak{sl}(V)$ be a nilpotent element. Let $\eta \in \mathcal{S}^*(V \oplus V^*)^{C_A}$. Suppose that both η and $\mathcal{F}(\eta)$ are supported in Q_A . Then $\eta \in \mathcal{S}^*(V \oplus V^*)^{\tilde{C}_A}$.

We will call a nilpotent element $A \in \mathfrak{sl}(V_k)$ 'nice' if the previous proposition holds for A . Namely, A is 'nice' if any distribution $\eta \in \mathcal{S}^*(V_k \oplus V_k^*)^{C_A}$ such that both η and $\mathcal{F}(\eta)$ are supported in Q_A is also \tilde{C}_A -invariant.

Proposition 5.5 clearly follows from the following two lemmas and Jordan decomposition.

Lemma 5.6 Let $A_1 \in \mathfrak{sl}(V_k)$ and $A_2 \in \mathfrak{sl}(V_l)$ be nice nilpotent elements. Then $A_1 \oplus A_2 \in \mathfrak{sl}(V_{k+l})$ is nice.

Lemma 5.7 Let $A \in \mathfrak{sl}(V_r)$ be a nilpotent Jordan block. Then A is nice.

5.1 Proof of lemma 5.6

We will need the following simple lemma.

Lemma 5.8 $Q_{A_1 \oplus A_2} \subset Q_{A_1} \times Q_{A_2}$.

Proof. Let $(v, \phi) \in Q_{A_1 \oplus A_2}$. This means that $v \otimes \phi = [A_1 \oplus A_2, B]$. Let $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$, $v = v_1 + v_2$ and $\phi = \phi_1 + \phi_2$ be the decompositions corresponding

to the blocks of $A_1 \oplus A_2$. Then $v_1 \otimes \phi_1 = [A_1, B_{11}]$ and $v_2 \otimes \phi_2 = [A_2, B_{22}]$. Hence $(v_1, \phi_1) \in Q_{A_1}$ and $(v_2, \phi_2) \in Q_{A_2}$. \square

Now let us prove lemma 5.6. Let $A_1 \in \mathfrak{sl}(V_{k_1})$ and $A_2 \in \mathfrak{sl}(V_{k_2})$ be nice operators. Let $\eta \in \mathcal{S}^*(V_{k_1} \oplus V_{k_2} \oplus V_{k_1}^* \oplus V_{k_2}^*)^{C_{A_1 \oplus A_2}}$. Suppose that both η and $\mathcal{F}_{V_{k_1+k_2} \oplus V_{k_1+k_2}^*}(\eta)$ are supported in $Q_{A_1} \times Q_{A_2}$. We need to show that η is $\tilde{C}_{A_1 \oplus A_2}$ -invariant. Note that \tilde{C}_{A_1} acts on $V_{k_1} \oplus V_{k_2} \oplus V_{k_1}^* \oplus V_{k_2}^*$. Denote

$$M := \{\alpha \in \mathcal{S}^*(V_{k_1} \oplus V_{k_1}^* \oplus V_{k_2} \oplus V_{k_2}^*)^{C_{A_1}} \mid \text{both } \alpha \text{ and } \mathcal{F}_{V_{k_1} \oplus V_{k_1}^*}(\alpha) \\ \text{are supported in } Q_{A_1} \times V_{k_2} \times V_{k_2}^*\}.$$

By proposition 2.2, $\eta \in M$. The following lemma follows from the fact that A_1 is nice using the localization principle (theorem 2.3).

Lemma 5.9 $M = M^{\tilde{C}_{A_1}}$.

Therefore, η is \tilde{C}_{A_1} -invariant. By similar reasons, η is \tilde{C}_{A_2} -invariant. Since η is $C_{A_1 \oplus A_2}$ -invariant, we get that η is invariant with respect to $\tilde{C}_{A_1 \oplus A_2}$. This completes the proof of lemma 5.6. \square

5.2 Proof of lemma 5.7

Let $A \in \mathfrak{sl}_r$ be the standard nilpotent Jordan block.

Notation 5.10 Denote

$$F^i := \text{Ker} A^i = \text{Im} A^{r-i}; \quad L^i := (F^{r-i})^\perp = \text{Im}(A^*)^{r-i} = \text{Ker}(A^*)^i \subset V_r^*$$

$$\text{and } Z := \bigcup_{i=0}^r F^i \oplus L^{r-i}.$$

We will first prove the following lemma from linear algebra.

Lemma 5.11 $Q(A) \subset Z$.

Proof. Let $(v, \phi) \in Q(A)$. Note that for any $i \geq 0$ and any element $B \in \mathfrak{gl}_r$, $\text{tr}(A^i[A, B]) = \text{tr}([A, A^i B]) = 0$. Hence $\langle \phi, A^i v \rangle = \text{tr}(A^i v \otimes \phi) = 0$. Therefore the spaces $W := \text{Span}\{A^i v\}$ and $\Psi := \text{Span}\{(A^*)^i \phi\}$ are orthogonal and thus $\dim W + \dim \Psi \leq r$. Denote $k := \dim W$ and $l := \dim \Psi$.

Consider the set of all non-zero vectors of the form $A^i v$. Since A is nilpotent, it is easy to see that this set is linearly independent. Hence $v \in \text{Ker} A^k$ and by the same reasoning $\phi \in \text{Ker}(A^*)^l$. But since $l \leq r - k$, $\text{Ker}(A^*)^l \subset \text{Ker}(A^*)^{r-k}$. Hence $(v, \phi) \in F^k \oplus L^{r-k} \subset Z$. \square

Now let $T : V_r \rightarrow V_r^*$ be the symmetric nondegenerate bilinear form defined by $T(e_i) = e_{r+1-i}^*$. By remark 3.6, T can be viewed as an element of \tilde{G}_r . Since $(TAT^{-1})^* = A$, $T \in \tilde{C}_A$.

In order to finish the proof of lemma 5.7 it remains to prove the following lemma.

Lemma 5.12 Consider the action of \mathbb{F}^\times on $V_r \oplus V_r^*$ defined by $\rho(\lambda)(v, \phi) := (\lambda v, \lambda^{-1} \phi)$. Let $\eta \in \mathcal{S}^*(V_r \oplus V_r^*)^{\mathbb{F}^\times}$. Suppose that $T(\eta) = -\eta$ and that both η and $\mathcal{F}(\eta)$ are supported in Z . Then $\eta = 0$.

*Proof.*² We will prove this lemma by induction on r . The case $r = 0$ is trivial. Suppose now that $r \geq 1$ and the lemma is true for all smaller r . By theorem 2.8, η is abs-homogeneous of degree r . Consider $\eta|_{(V_r \oplus V_r^*) \setminus (F^{r-1} \oplus V_r^*)}$. Since $Z \setminus (F^{r-1} \oplus V_r^*) = (V_r \setminus F^{r-1}) \oplus \{0\}$, on $Z \setminus (F^{r-1} \oplus V_r^*)$, the action of \mathbb{F}^\times coincides with homothety. Therefore $\eta|_{(V_r \oplus V_r^*) \setminus (F^{r-1} \oplus V_r^*)}$ is homothety invariant. On the other hand, it is abs-homogeneous of degree r . Hence it is zero. So η is supported in $F^{r-1} \oplus V_r^*$. By the same reasons η is supported in $V_r \oplus L^{r-1}$. Hence it is supported in $F^{r-1} \oplus L^{r-1}$.

By the same reasoning $\mathcal{F}(\eta)$ is supported in $F^{r-1} \oplus L^{r-1}$. Hence η is invariant with respect to translations in $(F^{r-1} \oplus L^{r-1})^\perp$ which is equal to $F^1 \oplus L^1$. If $r = 1$ it implies $\eta = 0$. Otherwise it implies that η is the pull back of a distribution α on the space $(F^{r-1} \oplus L^{r-1}) / (F^1 \oplus L^1)$ which can be identified with $V_{r-2} \oplus V_{r-2}^*$.

It is easy to see that α satisfies the conditions of the lemma for dimension $r - 2$. Hence by the induction hypothesis $\alpha = 0$. \square

6 Weil representation and proof of theorem 2.8

The goal of this section is to prove the following theorem.

Theorem 6.1 (Rallis-Schiffmann) Let W be a finite dimensional vector space over \mathbb{F} and B be a nondegenerate symmetric bilinear form on W . Denote $Z(B) := \{v \in W | B(v, v) = 0\}$. Let ξ be a distribution on W . Suppose that both ξ and $\mathcal{F}_B(\xi)$ are supported in Z_B .

Then ξ is abs-homogeneous of degree $\frac{1}{2} \dim W$.

For the proof we will need the Weil representation.³

Notation 6.2 Let $t \in \mathbb{F}^\times$ be a scalar. We denote

$$a_t := \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \quad n_t := \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad \bar{n}_t := \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \quad J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The following standard lemma follows from Gauss elimination.

Lemma 6.3 The families n_t and \bar{n}_t generate $\mathrm{SL}(2, \mathbb{F})$.

The following theorem is well known.

Theorem 6.4 (Weil representation) Let W be a vector space over \mathbb{F} of dimension d . Let B be a symmetric nondegenerate bilinear form on W . Then there exists a projective representation $\pi_B : \mathrm{SL}(2, \mathbb{F}) \rightarrow \mathrm{Aut}(\mathcal{S}(W))$ such that
(i) for any $g, h \in \mathrm{SL}(2, \mathbb{F})$, $\pi_B(gh) = u\pi_B(g)\pi_B(h)$ for some $u \in \mathbb{C}$ such that $|u| = 1$

²The ideas of the proof of Lemma 5.12 appeared also in [RS].

³We summarize here those properties of Weil representation that we need. For a more complete description see for example [Gel], sections 2.3 and 2.5.

- (ii) $\pi_B(\bar{n}_t)(f)(v) = \psi(tB(v, v))f(v)$
- (iii) $\pi_B(a_t)(f)(v) = |t|^{-\frac{d}{2}}f(t^{-1}v)$
- (iv) $\pi_B(J) = \mathcal{F}_B$.

We denote the dual representation on $\mathcal{S}^*(W)$ by π_B^* .

6.1 Proof of theorem 2.8.

Since $\text{Supp}(\xi) \subset Z(B)$, we have $\pi_B^*(\bar{n}_t)\xi = \xi$ for all t .

Since $\text{Supp}(\mathcal{F}_B(\xi)) \subset Z(B)$, we have

$$\pi_B^*(J^{-1}\bar{n}_tJ)\xi = u_1\xi \text{ where } |u_1| = 1.$$

Thus $\pi_B^*(n_{-t})\xi = u_1\xi$.

Since the families n_t and \bar{n}_t generate $\text{SL}(2, \mathbb{F})$, we have

$$\pi_B^*(a_t)\xi = u_2\xi \text{ where } |u_2| = 1.$$

Thus ξ is abs-homogeneous of degree $\frac{\dim W}{2}$. □

7 Appendix: Proof of theorem 4.2

The goal of this section is to prove the following theorem.

Theorem 7.1 (Rallis-Schiffmann) *Suppose that the main theorem holds for all dimensions smaller than n for all finite extensions \mathbb{E} of \mathbb{F} . Let ξ be a (\tilde{G}, χ) -equivariant distribution on X . Then ξ is supported in $\mathcal{N} \times Y$.*

Lemma 7.2 $\text{Supp}(\xi) \subset \mathfrak{sl}(V) \times Y$.

Proof. Let $U := X \setminus (\mathfrak{sl}(V) \times Y)$. We have to show $\mathcal{S}^*(U)^{\tilde{G}} = 0$. Consider the map $p : U \rightarrow \mathbb{F}^\times$ given by $p(A, v, \phi) = \langle \phi, v \rangle$. By the localization principle (corollary 2.4), it is enough to show that $\mathcal{S}^*(p^{-1}(\lambda))^{\tilde{G}} = 0$ for any $\lambda \in \mathbb{F}^\times$.

Fix $\lambda \in \mathbb{F}^\times$. Denote $Z_\lambda := \{(v, \phi) \in V \oplus V^* \mid \langle \phi, v \rangle = \lambda\}$. Note that Z_λ is a transitive \tilde{G} -equivariant l -space. Define $pr_2 : p^{-1}(\lambda) \rightarrow Z_\lambda$ by $pr_2(A, v, \phi) := (v, \phi)$. Note that pr_2 is \tilde{G} -equivariant. Let $z_0 = (e_n, \lambda e_n^*) \in Z_\lambda$ where e_n is the last element of the standard basis of V_n and e_n^* is the last element of the dual basis of V_n^* .

Note that the stabilizer of z_0 is isomorphic to \tilde{G}_{n-1} and the fiber $pr_2^{-1}(z_0)$ is isomorphic to $\mathfrak{sl}(V_n)$ as a \tilde{G}_{n-1} -equivariant l -space. Hence by Frobenius reciprocity (theorem 2.6), $\mathcal{S}^*(p^{-1}(\lambda))^{\tilde{G}_{n, \chi}} = \mathcal{S}^*(\mathfrak{sl}(V_n))^{\tilde{G}_{n-1, \chi}}$. By proposition 3.3, the main theorem for dimension $n - 1$ implies that $\mathcal{S}^*(\mathfrak{sl}(V_n))^{\tilde{G}_{n-1, \chi}} = 0$. □

Lemma 7.3 *Let $A \in \mathfrak{sl}(V)$ be a non-zero semisimple element. Let C_A be the stabilizer of A in G , and \tilde{C}_A be the stabilizer of A in \tilde{G} . Let $\mathcal{N}_A \subset \mathcal{N}$ be the subset of all nilpotent operators that commute with A . Then $\mathcal{S}^*(\mathcal{N}_A \times V \times V^*)^{C_A} = \mathcal{S}^*(\mathcal{N}_A \times V \times V^*)^{\tilde{C}_A}$.*

Proof. It is known that a centralizer of a semisimple element is a Levi subgroup. Hence C_A is isomorphic to $\prod_i G_{k_i}(\mathbb{E}_i)$ where $k_i < n$ are certain natural numbers and \mathbb{E}_i are certain finite extensions of \mathbb{F} . It is easy to see (e.g. using remark 3.6) that \tilde{C}_A can be identified with a subgroup of $\prod_i \tilde{G}_{k_i}(\mathbb{E}_i)$.

Therefore, the main theorem for k_i and \mathbb{E}_i and corollary 2.5 of the localization principle imply $\mathcal{S}^*(\prod_i X_{k_i}(\mathbb{E}_i))^{C_A} = \mathcal{S}^*(\prod_i X_{k_i}(\mathbb{E}_i))^{\tilde{C}_A}$.

The lemma follows now from the fact that $\mathcal{N}_A \times V \times V^*$ can be identified with a closed subset of $\prod_i X_{k_i}(\mathbb{E}_i)$. \square

Proof of theorem 4.2. By lemma 7.2, ξ is supported in $sl(V) \times Y$. Hence it is left to show that ξ is supported in $\mathcal{N} \times V \times V^*$.

Let $\mathfrak{X} := \mathfrak{X}_n$ be the set of all monic polynomials of degree n in variable λ . Consider the map $\Delta := \Delta_n : X_n \rightarrow \mathfrak{X}_n$ that maps (A, v, ϕ) to the characteristic polynomial of A . By the localization principle (corollary 2.4) it is enough to show $\mathcal{S}^*(\Delta^{-1}(P))^{\tilde{G}, \chi} = 0$ for any $P \neq \lambda^n \in \mathfrak{X}_n$.

Let $\mathcal{R} \subset sl(V)$ be the set of all semisimple operators with characteristic polynomial P . Note that \tilde{G} acts transitively on \mathcal{R} . We recall that by the Jordan decomposition theorem any operator A can be decomposed in a unique way into a sum of commuting operators A_s and A_n such that A_s is semi-simple and A_n is nilpotent. This defines a map $\mathcal{J} : \Delta^{-1}(P) \rightarrow \mathcal{R}$ by $\mathcal{J}(A) := A_s$. It is easy to see that \mathcal{J} is continuous and \tilde{G} -equivariant.

The theorem follows now from lemma 7.3 by Frobenius reciprocity. \square

References

- [Ber] J. Bernstein: *P-invariant Distributions on $GL(N)$ and the classification of unitary representations of $GL(N)$ (non-archimedean case)* Lie group representations, II (College Park, Md., 1982/1983), 50–102, Lecture Notes in Math., **1041**, Springer, Berlin (1984).
- [BZ] J. Bernstein, A.V. Zelevinsky: *Representations of the group $GL(n, F)$, where F is a local non-Archimedean field.* Uspekhi Mat. Nauk **10**, No.3, 5-70 (1976).
- [Gel] S. Gelbart: *Weil's Representation and the Spectrum of the Metaplectic Group*, Lecture Notes in Math., **530**, Springer, Berlin-New York (1976).
- [RS] S. Rallis, G. Schiffmann: *Multiplicity one Conjectures*, arXiv:0705.2168v1 [math.RT]

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