Higher order cohomology of arithmetic groups

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Abstract: Higher order cohomology of arithmetic groups is expressed in terms of $(\mathfrak{g}, K)$-cohomology. Generalizing results of Borel, it is shown that the latter can be computed using functions of (uniform) moderate growth. A higher order version of Borel's conjecture is stated, asserting that the cohomology can be computed using automorphic forms.

Contents

Introduction

In [2] we have defined higher order group cohomology in the following general context: Let Γ be a group and Σ a normal subgroup. For a ring R we define a sequence of functors H_q^0 from the category of $R[\Gamma]$-modules to the category of R-modules. First, for an $R[\Gamma]$-module V, one defines $H_0^0(\Gamma, \Sigma, V) = H^0(\Gamma, V) = V^\Gamma$ as the fixed point module. Inductively, $H_{q+1}^0(\Gamma, \Sigma, V)$ is the module of all $v \in V$ such that $\sigma v = v$ for every $\sigma \in \Sigma$ and $\gamma v - v$ is in $H_q^1(\Gamma, \Sigma, V)$ for every $\gamma \in \Gamma$. For every $q \geq 0$ the functor $H_q^0(\Gamma, \Sigma, \cdot)$ is left-exact and we define the higher order group cohomology as the right derived functor

$$H_q^p = R^p H_q^0.$$

In the case of a Fuchsian group the choice $\Sigma = \Gamma_{\text{par}}$ = the subgroup generated by all parabolic elements, turned out to be the adequate choice for an Eichler-Shimura isomorphy result to hold, see [2]. (Note, however, that we have performed a shift by one in the parameter q here.) For general arithmetic groups $\Gamma \subset G$, where G is a reductive linear group over $\mathbb{Q}$, a replacement for the Eichler-Shimura isomorphism is the isomorphism to $(\mathfrak{g}, K)$-cohomology,

$$H^p(\Gamma, E) \;\cong\; H^p_{\mathfrak{g},K}(C^\infty(\Gamma\backslash G) \otimes E),$$

where E is a finite dimensional representation of G. In this paper we present a higher order analogue of this result, i.e., we will show isomorphy of higher order cohomology to $(\mathfrak{g}, K)$-cohomology,

$$H_q^p(\Gamma, \Sigma, E) \;\cong\; H^p_{\mathfrak{g},K}(H_q^0(\Gamma, \Sigma, C^\infty(G)) \otimes E).$$

We will prove higher order versions of results of Borel by which one can compute the cohomology using spaces of functions with growth restrictions. We also state a higher order version of the Borel conjecture, proved by Franke [3], that the cohomology can be computed using automorphic forms.

Note that if $\text{Hom}(\Gamma, \mathbb{C}) = 0$, then $H_q^p = H_0^p = H^p$ for every $q \geq 0$. Consequently, in the case of arithmetic groups, higher order cohomology is of interested only for rank-one groups.

1 General groups

Let R be a commutative ring with unit. Let Γ be a group and $\Sigma \subset \Gamma$ a normal subgroup. Let I denote the augmentation ideal in the group algebra $A = R[\Gamma]$. Let I_Σ denote the augmentation ideal of $R[\Sigma]$. As Σ is normal in Γ, the set AI_Σ is a 2-sided ideal in A. For $q \geq 1$ consider the ideal

$$J_q \stackrel{\text{def}}{=} I^q + AI_\Sigma.$$

So in particular, for $\Sigma = \{1\}$ one has $J_q = I^q$. On the other end, for $\Sigma = \Gamma$ one gets $J_q = I$ for every $q \geq 1$. For an A-module V define

$$H_q^p(\Gamma, \Sigma, V) = \operatorname{Ext}_A^p(A/J_{q+1}, V).$$

This is the higher order cohomology of the module V, see [2]. Note that in the case $q = 0$, we get the ordinary group cohomology, so

$$H_0^p(\Gamma, \Sigma, V) = H^p(\Gamma, V).$$

For convenience, we will sometimes suppress the Σ in the notation, so we simply write $H_q^p(\Gamma, V)$ or even $H_q^p(V)$ for $H_q^p(\Gamma, \Sigma, V)$.

For an R-module M and a set S we write M^S for the R-module of all maps from S to M. Then $M^\emptyset$ is the trivial module 0. Up to isomorphy, the module M^S depends only on the cardinality of S. It therefore makes sense to define M^c for any cardinal number c in this way. Note that J_q/J_{q+1} is a free R-module. Define

$$N_{\Gamma,\Sigma}(q) \stackrel{\text{def}}{=} \dim_R J_q/J_{q+1}.$$

Then $N_{\Gamma,\Sigma}(q)$ is a possibly infinite cardinal number.

Lemma 1.1 (a) *For every $q \geq 1$ there is a natural exact sequence*

$$\begin{aligned} 0 \to H_{q-1}^0(\Gamma, V) \to H_q^0(\Gamma, V) \to H^0(\Gamma, V)^{N_{\Gamma,\Sigma}(q)} \to \\ \to H_{q-1}^1(\Gamma, V) \to H_q^1(\Gamma, V) \to H^1(\Gamma, V)^{N_{\Gamma,\Sigma}(q)} \to \dots \\ \dots \to H_{q-1}^p(\Gamma, V) \to H_q^p(\Gamma, V) \to H^p(\Gamma, V)^{N_{\Gamma,\Sigma}(q)} \to \dots \end{aligned}$$

(b) *Suppose that for a given $p \geq 0$ one has $H^p(\Gamma, V) = 0$. Then it follows $H_q^p(\Gamma, V) = 0$ for every $q \geq 1$. In particular, if V is acyclic as Γ-module, then $H_q^p(\Gamma, V) = 0$ for all $p \geq 1$, $q \geq 0$.*

Proof: Consider the exact sequence

$$0 \to J_q/J_{q+1} \to A/J_{q+1} \to A/J_q \to 0.$$

As an A-module, J_q/J_{q+1} is isomorphic to a direct sum $\bigoplus_\alpha R_\alpha$ of copies of $R = A/I$. So we conclude that for every $p \geq 0$,

$$\mathrm{Ext}^p_A(J_q/J_{q+1}, V) \;\cong\; \prod_\alpha \mathrm{Ext}^p_A(R, V) \;\cong\; H^p(\Gamma, V)^{N_{\Gamma,\Sigma}(q)}.$$

The long exact Ext-sequence induced by the above short sequence is

$$\begin{aligned} 0 \to \mathrm{Hom}_A(A/J_q, V) \to \mathrm{Hom}_A(A/J_{q+1}, V) \to \mathrm{Hom}_A(J_q/J_{q+1}, V) \to \\ \to \mathrm{Ext}^1_A(A/J_q, V) \to \mathrm{Ext}^1_A(A/J_{q+1}, V) \to \mathrm{Ext}^1_A(J_q/J_{q+1}, V) \to \\ \to \mathrm{Ext}^2_A(A/J_q, V) \to \mathrm{Ext}^2_A(A/J_{q+1}, V) \to \mathrm{Ext}^2_A(J_q/J_{q+1}, V) \to \dots \end{aligned}$$

This is the claim (a). For (b) we proceed by induction on q. For $q = 0$ the claim follows from $H^p_0(\Gamma, V) = H^p(\Gamma, V)$. Inductively, assume the claim proven for $q - 1$ and $H^p(\Gamma, V) = 0$. As part of the above exact sequence, we have the exactness of

$$H^p_{q-1}(\Gamma, V) \to H^p_q(\Gamma, V) \to H^p(\Gamma, V)^{N_{\Gamma,\Sigma}(q)}.$$

By assumption, we have $H^p(\Gamma, V)^{N_{\Gamma,\Sigma}(q)} = 0$ and by induction hypothesis the module $H^p_{q-1}(\Gamma, V)$ vanishes. This implies $H^p_q(\Gamma, V) = 0$ as well. □

Lemma 1.2 (Cocycle representation) *The module* $\mathrm{H}^1_q(\Gamma, V)$ *is naturally isomorphic to*

$$\mathrm{Hom}_A(J_{q+1}, V)/\alpha(V),$$

where $\alpha : V \to \mathrm{Hom}_A(J_{q+1}, V)$ *is given by* $\alpha(v)(m) = mv$.

Proof: This is Lemma 1.3 of [2]. □

2 Higher order cohomology of sheaves

Let Y be a topological space which is path-connected and locally simply connected. Let $C \to Y$ be a normal covering of Y. Let Γ be the fundamental group of Y and let $X \stackrel{\pi}{\longrightarrow} Y$ be the universal covering. The fundamental group Σ of C is a normal subgroup of Γ.

For a sheaf $\mathcal{F}$ on Y define

$$H_q^0(Y, C, \mathcal{F}) \stackrel{\text{def}}{=} H_q^0(\Gamma, \Sigma, H^0(X, \pi^*\mathcal{F})).$$

Let $\mathrm{Mod}(R)$ be the category of R-modules, let $\mathrm{Mod}_R(Y)$ be the category of sheaves of R-modules on Y, and let $\mathrm{Mod}_R(X)_\Gamma$ be the category of sheaves over X with an equivariant Γ-action. Then $H_q^0(Y, C, \cdot)$ is a left exact functor from $\mathrm{Mod}_R(Y)$ to $\mathrm{Mod}(R)$. We denote its right derived functors by $H_q^p(Y, C, \cdot)$ for $p \geq 0$.

Lemma 2.1 *Assume that the universal cover X is contractible.*

(a) *For each $p \geq 0$ one has a natural isomorphism $H_0^p(Y, C, \mathcal{F}) \cong H^p(Y, \mathcal{F})$.*

(b) *If a sheaf $\mathcal{F}$ is $H^0(Y, \cdot)$-acyclic, then it is $H_q^0(Y, C, \cdot)$-acyclic for every $q \geq 0$.*

Note that part (b) allows one to use flabby or fine resolutions to compute higher order cohomology.

Proof: We decompose the functor $H^0(Y, C, \cdot)$ into the functors

$$\mathrm{Mod}_R(Y) \stackrel{\pi^*}{\longrightarrow} \mathrm{Mod}_R(X)_\Gamma \stackrel{H^0(X,\cdot)}{\longrightarrow} \mathrm{Mod}(R[\Gamma]) \stackrel{H^0(\Gamma,\Sigma,\cdot)}{\longrightarrow} \mathrm{Mod}(R).$$

The functor π^* is exact and maps injectives to injectives. We claim that $H^0(X, \cdot)$ has the same properties. For the exactness, consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Mod}_R(X)_\Gamma & \stackrel{H^0}{\longrightarrow} & \mathrm{Mod}(R[\Gamma]) \\ \downarrow f & & \downarrow f \\ \mathrm{Mod}_R(X) & \stackrel{H^0}{\longrightarrow} & \mathrm{Mod}(R), \end{array}$$

where the vertical arrows are the forgetful functors. As X is contractible, the functor H^0 below is exact. The forgetful functors have the property, that a sequence upstairs is exact if and only if its image downstairs is exact.

This implies that the above H^0 is exact. It remains to show that H^0 maps injective objects to injective objects. Let $\mathcal{J} \in \mathrm{Mod}_R(X)_\Gamma$ be injective and consider a diagram with exact row in $\mathrm{Mod}(R[\Gamma])$,

$$\begin{array}{ccccc} 0 & \longrightarrow & M & \longrightarrow & N \\ & & \downarrow{\scriptstyle \varphi} & & \\ & & H^0(X, \mathcal{J}). & & \end{array}$$

The morphism φ gives rise to a morphism $\phi : M \times X \to \mathcal{J}$, where $M \times X$ stands for the constant sheaf with stalk M. Note that $H^0(X, \phi) = \varphi$. As $\mathcal{J}$ is injective, there exists a morphism $\psi : N \times X \to \mathcal{J}$ making the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & M \times X & \longrightarrow & N \times X \\ & & \downarrow{\scriptstyle \phi} & \swarrow{\scriptstyle \psi} & \\ & & \mathcal{J} & & \end{array}$$

commutative. This diagram induces a corresponding diagram on the global sections, which implies that $H^0(X, \mathcal{J})$ is indeed injective.

For a sheaf $\mathcal{F}$ on Y it follows that

$$H^p(Y, \mathcal{F}) = R^p(H^0(Y, \mathcal{F})) = R^p H^0(\Gamma, \Sigma, \mathcal{F}) \circ H^0_\Gamma \circ \pi^* = H^p_0(Y, C, \mathcal{F}).$$

Now let $\mathcal{F}$ be acyclic. Then we conclude $H^p_0(\mathcal{F}) = 0$ for every $p \geq 1$, so the Γ-module $V = H^0(X, \pi^* \mathcal{F})$ is Γ-acyclic. The claim follows from Lemma 1.1. □

3 Arithmetic groups

Let G be a semisimple Lie group with compact center and let $X = G/K$ be its symmetric space. Let $\Gamma \subset G$ be an arithmetic subgroup which is torsion-free, and let $\Sigma \subset \Gamma$ be a normal subgroup. Let $Y = \Gamma \backslash X$, then Γ is the fundamental group of the manifold Y, and the universal covering X of Y is contractible. This means that we can apply the results of the last section.

Theorem 3.1 *Let (σ, E) be a finite dimensional representation of G. There is a natural isomorphism*

$$H_q^p(\Gamma, \Sigma, E) \;\cong\; H_{\mathfrak{g},K}^p(H_q^0(\Gamma, \Sigma, C^\infty(G)) \otimes E),$$

where the right hand side is the $(\mathfrak{g}, K)$-cohomology.

Note that, as a consequence of the definition of $(\mathfrak{g}, K)$-cohomology and the fact that E is finite dimensional, one can replace the module $H_q^0(\Gamma, \Sigma, C^\infty(G))$ by its subspace of K-finite vectors

$$H_q^0(\Gamma, \Sigma, C^\infty(G))_K = H_q^0(\Gamma, \Sigma, C^\infty(G)_K),$$

where K acts on $C^\infty(G)$ by right translations.

Proof: Let $\mathcal{F}_E$ be the locally constant sheaf on Y corresponding to E. Let Ω_Y^p be the sheaf of complex valued p-differential forms on Y. Then $\Omega_Y^p \otimes \mathcal{F}_E$ is the sheaf of $\mathcal{F}_E$-valued differential forms. These form a fine resolution of $\mathcal{F}_E$:

$$0 \to \mathcal{F}_E \to \mathbb{C}^\infty \otimes \mathcal{F}_E \xrightarrow{d\otimes 1} \Omega_Y^1 \otimes \mathcal{F}_E \to \dots$$

Since $\pi^*\Omega_Y^\bullet = \Omega_X^\bullet$, we conclude that $H_q^p(\Gamma, \Sigma, E)$ is the cohomology of the complex $H_q^0(\Gamma, \Sigma, H^0(X, \Omega_X^\bullet \otimes E))$. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K respectively, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition. Then $H^0(X, \Omega^p \otimes \mathcal{F}_E) = (C^\infty(G) \otimes \bigwedge^p \mathfrak{p})^K \otimes E$. Mapping a form ω in this space to $(1 \otimes x^{-1})\omega(x)$ one gets an isomorphism to $(C^\infty(G) \otimes \bigwedge^p \mathfrak{p} \otimes E)^K$, where K acts diagonally on all factors and Γ now acts on $C^\infty(G)$ alone. The claim follows. □

Let $U(\mathfrak{g})$ act on $C^\infty(G)$ as algebra of left invariant differential operators. Let $\|\cdot\|$ be a norm on G, see [4], Section 2.A.2. Recall that a function $f \in C^\infty(G)$ is said to be *of moderate growth*, if for every $D \in U(\mathfrak{g})$ one has $Df(x) = O(\|x\|^a)$ for some $a > 0$. The function f is said to be of *uniform moderate growth*, if the exponent a above can be chosen independent of D, Let $C_{\text{mg}}^\infty(G)$ and $C_{\text{umg}}^\infty(G)$ denote the spaces of functions of moderate growth and uniform moderate growth respectively.

Let $\mathfrak{z}$ be the center of the algebra $U(\mathfrak{g})$. Let $\mathcal{A}(G)$ denote the space of functions $f \in C^\infty(G)$ such that

- f is of moderate growth,
- f is right K-finite, and
- f is $\mathfrak{z}$-finite.

Proposition 3.2 (a) *For* $\Omega = C^\infty_{\rm umg}(G), C^\infty_{\rm mg}(G), C^\infty(G)$ *one has*

$$H^1_q(\Gamma, \Sigma, \Omega) = 0$$

for every $q \geq 0$.

(b) *If* $\operatorname{Hom}(\Gamma, \mathbb{C}) \neq 0$*, then one has*

$$H^1(\Gamma, \mathcal{A}(G)) \neq 0.$$

Proof: In order to prove (a), it suffices by Lemma 1.1 (b), to consider the case $q = 0$. A 1-cocycle is a map $\alpha : \Gamma \to \Omega$ such that $\alpha(\gamma\tau) = \gamma\alpha(\tau) + \alpha(\gamma)$ holds for all $\gamma, \tau \in \Gamma$. We have to show that for any given such map α there exists $f \in \Omega$ such that $\alpha(\tau) = \tau f - f$. To this end consider the symmetric space $X = G/K$ of G. Let $d(xK, yK)$ for $x, y \in G$ denote the distance in X induced by the G-invariant Riemannian metric. For $x \in G$ we also write $d(x) = d(xK, eK)$. Then the functions $\log \|x\|$ and $d(x)$ are equivalent in the sense that there exists a constant $C > 1$ such that

$$\frac{1}{C} d(x) \ \leq \ \log \|x\| \ \leq \ C d(x)$$

or

$$\|x\| \ \leq \ e^{Cd(x)} \ \leq \ \|x\|^{C^2}$$

holds for every $x \in G$. We define

$$\mathcal{F} = \{y \in G : d(y) < d(\gamma y) \ \forall \gamma \in \Gamma \smallsetminus \{e\}\}.$$

As Γ is torsion-free, this is a fundamental domain for the Γ left translation action on G. In other words, $\mathcal{F}$ is open, its boundary is of measure zero, and there exists a set of representatives $R \subset G$ for the Γ-action such that $\mathcal{F} \subset R \subset \overline{\mathcal{F}}$. Next let $\varphi \in C^\infty_c(G)$ with $\varphi \geq 0$ and $\int_G \varphi(x)\, dx = 1$. Then set $u = \mathbf{1}_{\mathcal{F}} * \varphi$, where $\mathbf{1}_{\mathcal{F}}$ is the characteristic function of the set $\mathcal{F}$ and $*$ is the convolution product $f * g(x) = \int_G f(y) g(y^{-1}x)\, dy$. Let C be the support of φ, then the support of u is a subset of $\overline{\mathcal{F}}C$ and the sum $\sum_{\tau\in\Gamma} u(\tau^{-1}x)$ is locally finite in x. More sharply, for a given compact unit-neighborhood V there exists $N \in \mathbb{N}$ such that for every $x \in G$ one has

$$\#\{\tau \in \Gamma : u(\tau^{-1}xV) \nsubseteq \{0\}\} \ \leq \ N.$$

This is to say, the sum is uniformly locally finite. For a function h on G and $x, y \in G$ we write $L_y h(x) = h(y^{-1}x)$. Then for a convolution product one has $L_y(f * g) = (L_y f) * g$, and so

$$\sum_{\tau\in\Gamma} u(\tau^{-1}x) = \left(\sum_{\tau\in\Gamma} L_\tau \mathbf{1}_{\mathcal{F}}\right) * \varphi.$$

The sum in parenthesis is equal to one on the complement of a nullset. Therefore,

$$\sum_{\tau\in\Gamma} u(\tau^{-1}x) \ \equiv \ 1.$$

Set

$$f(x) = \ -\sum_{\tau\in\Gamma} \alpha(\tau)(x)\, u(\tau^{-1}x).$$

Lemma 3.3 *The function f lies in the space Ω.*

Proof: Since the sum is uniformly locally finite, it suffices to show that for each $\tau \in \Gamma$ we have $\alpha(\tau)(x)u(\tau^{-1}x) \in \Omega$ where the $O(\|\cdot\|^d)$ estimate is uniform in τ. By the Leibniz-rule it suffices to show this separately for the two factors $\alpha(\tau)$ and $L_\tau u$. For $D \in U(\mathfrak{g})$ we have

$$D(L_\tau u) = (L_\tau \mathbf{1}_{\mathcal{F}}) * (D\varphi).$$

This function is bounded uniformly in τ, hence $L_\tau u \in C^\infty_{\text{umg}}(G)$. Now $\alpha(\tau) \in \Omega$ by definition, but we need uniformity of growth in τ. We will treat the case $\Omega = C^\infty_{\text{umg}}(G)$ here, the case C^∞_{mg} is similar and the case $C^\infty(G)$ is trivial, as no growth bounds are required.

So let $\Omega = C^\infty_{\text{umg}}(G)$ and set

$$S = \{\gamma \in \Gamma \smallsetminus \{e\} : \gamma\overline{\mathcal{F}} \cap \overline{\mathcal{F}} \neq \emptyset\}.$$

Then S is a finite symmetric generating set for Γ. For $\gamma \in \Gamma$, let $\mathcal{F}_\gamma$ be the set of all $x \in G$ with $d(x) < d(\gamma x)$. Then

$$\mathcal{F} = \bigcap_{\gamma \in \Gamma \smallsetminus \{e\}} \mathcal{F}_\gamma$$

Let $\tilde{\mathcal{F}} = \bigcap_{s \in S} \mathcal{F}_s$. We claim that $\mathcal{F} = \tilde{\mathcal{F}}$. As the intersection runs over fewer elements, one has $\mathcal{F} \subset \tilde{\mathcal{F}}$. For the converse note that for every $s \in S$ the set $s\overline{\mathcal{F}}/K$ lies in $X \smallsetminus \tilde{\mathcal{F}}/K$, therefore $\mathcal{F}/K$ is a connected component of $\tilde{\mathcal{F}}/K$. By the invariance of the metric, we conclude that $x \in \mathcal{F}_\gamma$ if and only if $d(xK, eK) < d(xK, \gamma^{-1}K)$. This implies that $\mathcal{F}_\gamma/K$ is a convex subset of X. Any intersection of convex sets remains convex, therefore $\tilde{\mathcal{F}}/K$ is convex and hence connected, and so $\tilde{\mathcal{F}}/K = \mathcal{F}/K$, which means $\tilde{\mathcal{F}} = \mathcal{F}$.

Likewise we get $\overline{\mathcal{F}} = \bigcap_{s \in S} \overline{\mathcal{F}_s}$. The latter implies that for each $x \in G \smallsetminus \overline{\mathcal{F}}$ there exists $s \in S$ such that $d(s^{-1}x) < d(x)$. Iterating this and using the fact that the set of all $d(\gamma x)$ for $\gamma \in \Gamma$ is discrete, we find for each $x \in G \smallsetminus \overline{\mathcal{F}}$ a chain of elements $s_1, \dots, s_n \in S$ such that $d(x) > d(s_1^{-1}x) > \cdots > d(s_n^{-1} \cdots s_1^{-1}x)$ and $s_n^{-1} \cdots s_1^{-1}x \in \overline{\mathcal{F}}$. The latter can be written as $x \in s_1 \dots s_n \overline{\mathcal{F}}$. Now let $\tau \in \Gamma$ and suppose $u(\tau^{-1}x) \neq 0$. Then $x \in \overline{\mathcal{F}}C$, so, choosing C small enough, we can assume $x \in s\tau\overline{\mathcal{F}}$ for some $s \in S \cap \{e\}$. As the other case is similar, we can assume $s = e$. It suffices to assume $x \in \tau\mathcal{F}$, as we only need the estimates on the dense open set $\Gamma\mathcal{F}$. So then it follows $\tau = s_1 \dots s_n$.

Let $D \in U(\mathfrak{g})$. As α maps to $\Omega = C^\infty_{\text{umg}}(G)$, for every $\gamma \in \Gamma$ there exist $C(D, \gamma), a(\gamma) > 0$ such that

$$|D\alpha(\gamma)(x)| \ \le \ C(D, \gamma)\, \|x\|^{a(\gamma)}.$$

The cocycle relation of α implies

$$\alpha(\tau)(x) = \sum_{j=1}^{n} \alpha(\gamma_j)(s_{j-1}^{-1} \dots s_1^{-1}x).$$

We get

$$\begin{aligned}
|D\alpha(\tau)(x)| &\leq \sum_{j=1}^{n} C(D,s_j)\|s_{j-1}^{-1}\dots s_1^{-1}x\|^{a(s_j)} \\
&\leq \sum_{j=1}^{n} C(D,s_j)\, e^{Cd(s_{j-1}^{-1}\dots s_1^{-1}x)a(s_j)} \\
&\leq \sum_{j=1}^{n} C(D,s_j)\, e^{Cd(x)a(s_j)} \\
&\leq \sum_{j=1}^{n} C(D,s_j)\|x\|^{C^2a(s_j)} \\
&\leq nC_0(D)\|x\|^{a_0},
\end{aligned}$$

where $C(D) = \max_j C(D,s_j)$ and $a_0 = C^2 \max_j d(s_j)$. It remains to show that n only grows like a power of $\|x\|$. To this end let for $r > 0$ denote $N(r)$ the number of $\gamma \in \Gamma$ with $d(\gamma) \leq r$. Then a simple geometric argument shows that

$$N(r) = \frac{1}{\mathrm{vol}\mathcal{F}}\mathrm{vol}\left(\bigcup_{\gamma:d(\gamma)\leq r} \gamma\mathcal{F}/K\right) \;\leq\; C_1\mathrm{vol}(B_{2r}),$$

where B_{2r} is the ball of radius $2r$ around eK. Note that for the homogeneous space X there exists a constant $C_2 > 0$ such that $\mathrm{vol}B_{2r} \leq e^{C_2 r}$. Now $n \leq N(d(x))$ and therefore

$$n \;\leq\; C_1\mathrm{vol}B_{2d(x)} \;\leq\; C_1 e^{C_2 d(x)} \;\leq\; C_1\|x\|^{C_3}$$

for some $C_3 > 0$. Together it follows that there exists $C(D) > 0$ and $a > 0$ such that

$$|D\alpha(\tau)(x)| \;\leq\; C(D)\,\|x\|^a.$$

This is the desired estimate which shows that $f \in \Omega$. The lemma is proven. □

To finish the proof of part (a) of the proposition, we now compute for $\gamma \in \Gamma$,

$$\begin{aligned}\gamma f(x) - f(x) &= f(\gamma^{-1}x) - f(x) \\ &= \sum_{\tau\in\Gamma} \alpha(\tau x)u(\tau^{-1}x) - \alpha(\tau)(\gamma^{-1}x)u(\tau^{-1}\gamma^{-1}x) \\ &= \sum_{\tau\in\Gamma} \alpha(\tau)(x)u(\tau^{-1}x) + \alpha(\gamma)(x)\sum_{\tau\in\Gamma} u((\gamma\tau)^{-1}x) \\ &\quad - \sum_{\tau\in\Gamma} \alpha(\gamma\tau)(x)u((\gamma\tau)^{-1}x)\end{aligned}$$

The first and the last sum cancel and the middle sum is $\alpha(\gamma)(x)$. Therefore, part (a) of the proposition is proven.

We now prove part (b). Let $Q = C^\infty(G)/\mathcal{A}(G)$. We have an exact sequence of Γ-modules

$$0 \to \mathcal{A}(G) \to C^\infty(G) \to Q \to 0.$$

This results in the exact seuence

$$0 \to \mathcal{A}(G)^\Gamma \to C^\infty(\Gamma\backslash G) \xrightarrow{\phi} Q^\Gamma \to H^1(\Gamma, \mathcal{A}(G)) \to 0.$$

The last zero comes by part (a) of the proposition. We have to show that the map ϕ is not surjective. So let $\chi : \Gamma \to \mathbb{C}$ be a non-zero group homomorphism and let $u \in C^\infty(G)$ as above with $\sum_{\tau\in\Gamma} u(\tau^{-1}x) = 1$, and u is supported in $\overline{\mathcal{F}}C$ for a small unit-neighborhood C. Set

$$h(x) = -\sum_{\tau\in\Gamma} \chi(\tau)u(\tau^{-1}x).$$

Then for every $\gamma \in \Gamma$ the function

$$h(\gamma^{-1}x) - h(x) = \chi(\gamma)$$

is constant and hence lies in $\mathcal{A}(G)^\Gamma$. This means that the class $[h]$ of h in Q lies in the Γ-invariants Q^Γ. As $\chi \neq 0$, the function h is not in $C^\infty(\Gamma\backslash G)$, and therefore ϕ is indeed not surjective. $\square$

Proposition 3.4 *For every $q \geq 1$ there is an exact sequence of continuous G-homomorphisms,*

$$0 \to H^0_{q-1}(\Gamma, \Sigma, C^\infty_*(G)) \xrightarrow{\phi} H^0_q(\Gamma, \Sigma, C^\infty_*(G)) \xrightarrow{\psi} C^\infty_*(\Gamma\backslash G)^{N_{\Gamma,\Sigma}(q)} \to 0,$$

where ϕ is the inclusion map and $*$ can be $\emptyset$, umg, or mg.

Proof: This follows from Lemma 1.1 together with Propostion 3.2 (a). □

The space $C^\infty(G)$ carries a natural topology which makes it a nuclear topological vector space. For every $q \geq 0$, the space $H_q^0(\Gamma, \Sigma, C^\infty(G))$ is a closed subspace. If Γ is cocompact, then one has the isotypical decomposition

$$H_1^0(\Gamma, \Sigma, C^\infty(G)) = C^\infty(\Gamma\backslash G) = \overline{\bigoplus_{\pi \in \hat{G}}} C^\infty(\Gamma\backslash G)(\pi),$$

and $C^\infty(\Gamma\backslash G)(\pi) \cong m_\Gamma(\pi)\pi^\infty$, where the sum runs over the unitary dual $\hat{G}$ of G, and for $\pi \in \hat{G}$ we write π^∞ for the space of smooth vectors in π. The multiplicity $m_\Gamma(\pi) \in \mathbb{N}_0$ is the multiplicity of π as a subrepresentation of $L^2(\Gamma\backslash G)$, i.e.,

$$m_\Gamma(\pi) = \dim \operatorname{Hom}_G\left(\pi, L^2(\Gamma\backslash G)\right).$$

Finally, the direct sum $\overline{\bigoplus}$ means the closure of the algebraic direct sum in $C^\infty(G)$. We write $\hat{G}_\Gamma$ for the set of all $\pi \in \hat{G}$ with $m_\Gamma(\pi) \neq 0$.

Let $\pi \in \hat{G}$. A smooth representation (β, V_β) of G is said to be *of type* π^∞, if it is of finite length and every irreducible subquotient is isomorphic to π^∞. For a smooth representation (η, V_η) we define the π^∞*-isotype* as

$$V_\eta(\pi^\infty) \stackrel{\text{def}}{=} \overline{\bigoplus_{\substack{V_\beta \subset V_\eta \\ \beta \text{ of type } \pi}}} V_\beta,$$

where the sum runs over all subrepresentations V_β of type π^∞.

Theorem 3.5 *Suppose* Γ *is cocompact and let* $* \in \{\emptyset, \mathrm{mg}, \mathrm{umg}\}$. *We write* $V_q = V$. *For every* $q \geq 1$ *there is an isotypical decomposition*

$$V_q = \overline{\bigoplus_{\pi \in \hat{G}_\Gamma}} V_q(\pi),$$

and each $V_q(\pi)$ *is of type* π *itself. The exact sequence of Proposition 3.4 induces an exact sequence*

$$0 \to V_q(\pi) \to V_{q+1}(\pi) \to (\pi^\infty)^{m_\Gamma(\pi) N_{\Gamma,\Sigma}(q)} \to 0$$

for every $\pi \in \hat{G}_\Gamma$.

Proof: We will prove the theorem by reducing to a finite dimensional situation by means of considering infinitesimal characters and K-types. For this let $\hat{\mathfrak{z}} = \mathrm{Hom}(\mathfrak{z}, \mathbb{C})$ be the set of all algebra homomorphisms from $\mathfrak{z}$ to $\mathbb{C}$. For a $\mathfrak{z}$-module V and $\chi \in \hat{\mathfrak{z}}$ let

$$V(\chi) \stackrel{\text{def}}{=} \{v \in V : \forall z \in \mathfrak{z}\ \exists n \in \mathbb{N}\ (z - \chi(z))^n v = 0\}$$

be the *generalized* χ*-eigenspace*. Since $\mathfrak{z}$ is finitely generated, one has

$$V(\chi) = \{v \in V : \exists n \in \mathbb{N}\ \forall z \in \mathfrak{z}\ (z - \chi(z))^n v = 0\}.$$

For $\chi \neq \chi'$ in $\mathfrak{z}$ one has $V(\chi) \cap V(\chi') = 0$. Recall that the algebra $\mathfrak{z}$ is free in r generators, where r is the absolute rank of G. Fix a set of generators $z_1, \dots, z_r$. The map $\chi \mapsto (\chi(z_1), \dots, \chi(z_r))$ is a bijection $\hat{\mathfrak{z}} \to \mathbb{C}^r$. We equip $\hat{\mathfrak{z}}$ with the topology of $\mathbb{C}^r$. This topology does not depend on the choice of the generators $z_1, \dots, z_r$.

Let $\Gamma \subset G$ be a discrete cocompact subgroup. Let $\hat{\mathfrak{z}}_\Gamma$ be the set of all $\chi \in \hat{\mathfrak{z}}$ such that the generalized eigenspace $C^\infty(\Gamma\backslash G)(\chi)$ is non-zero. The $\hat{\mathfrak{z}}_\Gamma$ is discrete in $\hat{\mathfrak{z}}$, more sharply there exists $\varepsilon_\Gamma > 0$ such that for any two $\chi \neq \chi'$ in $\hat{\mathfrak{z}}_\Gamma$ there is $j \in \{1, \dots, r\}$ such that $|\chi(z_j) - \chi'(z_j)| > \varepsilon_\Gamma$.

Proposition 3.6 *Let* $* \in \{\emptyset, \mathrm{mg}, \mathrm{umg}\}$. *For every* $q \geq 0$ *and every* $\chi \in \hat{\mathfrak{z}}$ *the space* $V_q(\chi) = H_q^0(\Gamma, \Sigma, C_*^\infty(G))(\chi)$ *coincides with*

$$\bigcap_{z \in \mathfrak{z}} \ker(z - \chi(z))^{2^q},$$

and is therefore a closed subspace of V_q. *The representation of* G *on* $V_q(\chi)$ *is of finite length.*

The space $V_q(\chi)$ *is non-zero only if* $\chi \in \hat{\mathfrak{z}}_\Gamma$. *One has a decomposition*

$$H_q^0(\Gamma, \Sigma, C_*^\infty(G)) = \overline{\bigoplus_{\chi \in \hat{\mathfrak{z}}_\Gamma}} H_q^0(\Gamma, \Sigma, C_*^\infty(G))(\chi).$$

The exact sequence of Proposition 3.4 induces an exact sequence

$$0 \to V_{q-1}(\chi) \to V_q(\chi) \to \bigoplus_{\pi \in \hat{G}_\chi} m_\Gamma(\pi) N_{\Gamma,\Sigma}(q) \pi \to 0.$$

Proof: All assertions are clear for $q = 0$. We proceed by induction. Fix $\chi \in \hat{\mathfrak{z}}_\Gamma$. The exact sequence

$$0 \to V_{q-1} \to V_q \xrightarrow{\psi} V_0^{N_{\Gamma,\Sigma(q)}} \to 0.$$

induces an exact sequence

$$0 \to V_{q-1}(\chi) \to V_q(\chi) \xrightarrow{\psi_\chi} V_0(\chi)^{N_{\Gamma,\Sigma(q)}}.$$

Let $v \in V_0(\chi)^{N_{\Gamma,\Sigma(q)}}$. As ψ is surjective, one finds $u \in V_q$ with $\psi(u) = v$. We have to show that one can choose u to lie in $V_q(\chi)$. We have $(z - \chi(z))v = 0$ for every $z \in \mathfrak{z}$. Therefore $(z - \chi(z))u \in V_{q-1}$. Inductively we assume the decomposition to holds for V_{q-1}, so we can write

$$(z_j - \chi(z_j))u = \sum_{\chi' \in \hat{\mathfrak{z}}_\Gamma} u_{j,\chi'},$$

for $1 \leq j \leq r$ and $u_{j,\chi'} \in \ker(z - \chi'(z))^{2^{q-1}}$ for every $z \in \mathfrak{z}$. For every $\chi' \in \hat{\mathfrak{z}}_\Gamma \smallsetminus \{\chi\}$ we fix some index $1 \leq j(\chi') \leq r$ with $|\chi(z_{j(\chi')}) - \chi'(z_{j(\chi')})| > \varepsilon_\Gamma$. On the space

$$\overline{\bigoplus_{\chi' : j(\chi') = j} V_q(\chi')}$$

the operator $z_j - \chi(z_j)$ is invertible and the inverse $(z_j - \chi(z_j))^{-1}$ is continuous. We can replace u with

$$u - \sum_{\chi' \in \hat{\mathfrak{z}}_\Gamma \smallsetminus \{\chi\}} (z_{j(\chi')} - \chi(z_{j(\chi')}))^{-1} u_{j(\chi'),\chi'}.$$

We end up with u satisfying $\psi(u) = v$ and

$$(z_1 - \chi(z_1)) \cdots (z_r - \chi(z_r))u \ \in \ V_{q-1}(\chi) = \bigcap_{z \in \mathfrak{z}} \ker(z - \chi(z))^{2^{q-1}}.$$

So for every $z \in \mathfrak{z}$ one has

$$0 = (z_1 - \chi(z_1)) \cdots (z_r - \chi(z_r))(z - \chi(z))^{2^{q-1}} u,$$

which implies

$$(z - \chi(z))^{2^{q-1}} u \in \ker\left((z_1 - \chi(z_1)) \cdots (z_r - \chi(z_r))\right).$$

As the set $\mathfrak{z}_\Gamma$ is countable, one can, depending on χ, choose the generators $z_1, \dots, z_r$ in a way that $\chi(z_j) \neq \chi'(z_j)$ holds for every j and every $\chi' \in \hat{\mathfrak{z}}_\Gamma \smallsetminus \{\chi\}$. Therefore the operator $(z_1 - \chi(z_1)) \cdots (z_r - \chi(z_r))$ is invertible on $V_{q-1}(\chi')$ for every $\chi' \in \hat{\mathfrak{z}}_\Gamma \smallsetminus \{\chi\}$ and it follows $(z - \chi(z))^{2^{q-1}} u \in V_{q-1}(\chi) \subset \ker(z - \chi(z))^{2^{q-1}}$ and therefore $u \in \ker((z - \chi(z))^{2^q})$. Since this holds for every z it follows $u \in V_q(\chi)$ and hence ψ_χ is indeed surjective. One has an exact sequence

$$0 \to V_{q-1}(\chi) \to V_q(\chi) \to V_0(\chi)^{N_{\Gamma,\Sigma}(q)} \to 0.$$

Taking the sum over all $\chi \in \mathfrak{z}_\Gamma$ we arrive at an exact sequence

$$0 \to V_{q-1} \to \overline{\bigoplus_{\chi \in \hat{\mathfrak{z}}_\Gamma}} V_q(\chi) \to V_0^{N_{\Gamma,\Sigma}(q)} \to 0.$$

Hence we get a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & V_{q-1} & \longrightarrow & \overline{\bigoplus_{\chi \in \hat{\mathfrak{z}}_\Gamma}} V_q(\chi) & \longrightarrow & V_0^{N_{\Gamma,\Sigma}(q)} & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle =} & & \Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle =} & & \\
0 & \longrightarrow & V_{q-1} & \longrightarrow & V_q & \longrightarrow & V_0^{N_{\Gamma,\Sigma}(q)} & \longrightarrow & 0,
\end{array}$$

where i is the inclusion. By the 5-Lemma, i must be a bijection. The proposition follows. $\square$

We now finish the proof of the theorem. We keep the notation V_q for the space $\mathbb{H}_q^0(\Gamma, \Sigma, C_*^\infty(G))$. For a given $\chi \in \hat{\mathfrak{z}}_\Gamma$ the G-representation $V_q(\chi)$ is of finite length, so the K-isotypical decomposition

$$V_q(\chi) = \overline{\bigoplus_{\tau \in \hat{K}}} V_q(\chi)(\tau)$$

has finite dimensional isotypes, i.e., $\dim V_q(\chi)(\tau) < \infty$. Let $U(\mathfrak{g})^K$ be the algebra of all $D \in U(\mathfrak{g})$ such that $\mathrm{Ad}(k)D = D$ for every $k \in K$. Then the action of $D \in U(\mathfrak{g})$ commutes with the action of each $k \in K$, and so $K \times U(\mathfrak{g})^K$ acts on every smooth G-module. For $\pi \in \hat{G}$ the $K \times U(\mathfrak{g})^K$-module $V_\pi(\tau)$ is irreducible and $V_\pi(\tau) \cong V_{\pi'}(\tau')$ as a $K \times U(\mathfrak{g})^K$-module implies $\pi = \pi'$ and $\tau = \tau'$, see [4], Proposition 3.5.4. As $V_q(\chi)(\tau)$ is finite dimensional. one gets

$$V_q(\chi)(\tau) = \bigoplus_{\substack{\pi \in \hat{G} \\ \chi_\pi = \chi}} V_q(\chi)(\tau)(\pi),$$

where $V_q(\chi)(\tau)(\pi)$ is the largest $K \times U(\mathfrak{g})^K$-submodule of $V_q(\chi)(\tau)$ with the property that every irreducible subquotient is isomorphic to $V_\pi(\tau)$. Let

$$V_q(\pi) = \overline{\bigoplus_{\tau \in \hat{K}}} V_q(\chi_\pi)(\tau)(\pi).$$

The claims of the theorem follow from the proposition. □

4 The higher order Borel conjecture

Let (σ, E) be a finite dimensional representation of G. In [1], A. Borel has shown that the inclusions $C^\infty_{\text{umg}}(G) \hookrightarrow C^\infty_{\text{mg}}(G) \hookrightarrow C^\infty(G)$ induce isomorphisms in cohomology:

$$H^p_{\mathfrak{g},K}(H^0(\Gamma, C^\infty_{\text{umg}}(G)) \otimes E) \xrightarrow{\cong} H^p_{\mathfrak{g},K}(H^0(\Gamma, C^\infty_{\text{mg}}(G)) \otimes E) \xrightarrow{\cong} H^p_{\mathfrak{g},K}(H^0(\Gamma, C^\infty(G)) \otimes E).$$

In [3], J. Franke proved a conjecture of Borel stating that the inclusion $\mathcal{A}(G) \hookrightarrow C^\infty(G)$ induces an isomorphism

$$H^p_{\mathfrak{g},K}(H^0(\Gamma, \mathcal{A}(G)) \otimes E) \xrightarrow{\cong} H^p_{\mathfrak{g},K}(H^0(\Gamma, C^\infty(G)) \otimes E).$$

Conjecture 4.1 (Higher order Borel conjecture) *For every $q \geq 0$, the inclusion $\mathcal{A}(G) \hookrightarrow C^\infty(G)$ induces an isomorphism*

$$H^p_{\mathfrak{g},K}(H^0_q(\Gamma, \Sigma, \mathcal{A}(G)) \otimes E) \xrightarrow{\cong} H^p_{\mathfrak{g},K}(H^0_q(\Gamma, \Sigma, C^\infty(G)) \otimes E).$$

We can prove the higher order version of Borel's result.

Theorem 4.2 *For each $q \geq 0$, the inclusions $C^\infty_{\rm umg}(G) \hookrightarrow C^\infty_{\rm mg}(G) \hookrightarrow C^\infty(G)$ induce isomorphisms in cohomology:*

$$H^p_{\mathfrak{g},K}(H^0_q(\Gamma,\Sigma,C^\infty_{\rm umg}(G))\otimes E) \stackrel{\cong}{\longrightarrow} H^p_{\mathfrak{g},K}(H^0_q(\Gamma,\Sigma,C^\infty_{\rm mg}(G))\otimes E)$$
$$\stackrel{\cong}{\longrightarrow} H^p_{\mathfrak{g},K}(H^0_q(\Gamma,\Sigma,C^\infty(G))\otimes E).$$

Proof: Let Ω be one of the spaces $\mathbb{C}^\infty_{\rm umg}(G)$ or $\mathbb{C}^\infty_{\rm mg}(G)$.

We will now leave Σ out of the notation. By Proposition 3.4 we get an exact sequence

$$0 \to H^0_{q-1}(\Gamma,\Omega) \to H^0_q(\Gamma,\Omega) \to H^0(\Gamma,\Omega)^{N_{\Gamma,\Sigma}(q)} \to 0,$$

and the corresponding long exact sequences in $(\mathfrak{g},K)$-cohomology. For each $p \geq 0$ we get a commutative diagram with exact rows

$$\begin{array}{ccccc}
H^p_{\mathfrak{g},K}(H^0_{q-1}(\Gamma,\Omega)\otimes E) & \longrightarrow & H^p_{\mathfrak{g},K}(H^0_q(\Gamma,\Omega)\otimes E) & \longrightarrow & H^p_{\mathfrak{g},K}(H^0(\Gamma,\Omega)\otimes E)^{N_{\Gamma,\Sigma}(q)} \\
{\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & {\scriptstyle\gamma}\downarrow \\
H^p_{\mathfrak{g},K}(H^0_{q-1}(\Gamma,C^\infty(G))\otimes E) & \longrightarrow & H^p_{\mathfrak{g},K}(H^0_q(\Gamma,C^\infty(G))\otimes E) & \to & H^p_{\mathfrak{g},K}(H^0(\Gamma,C^\infty(G))\otimes E)^{N_{\Gamma,\Sigma}(q)}.
\end{array}$$

Borel has shown that γ is an isomorphism and that α is an isomorphism for $q = 0$. We prove that β is an isomorphism by induction on q. For the induction step we can assume that α is an isomorphism. Since the diagram continues to the left and right with copies of itself where p is replaced by $p-1$ or $p+1$, we can deduce that β is an isomorphism by the 5-Lemma. □

By Proposition 3.2 (b) this proof cannot be applied to $\mathcal{A}(G)$.

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