ON NON-ABELIAN DISCONTINUOUS SUBGROUPS ACTING ON EXPONENTIAL SOLVABLE HOMOGENEOUS SPACES

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Abstract. Let H be a closed connected subgroup of an exponential solvable Lie group G. We consider the deformation space $\mathscr{T}(\Gamma, G, H)$ of a discontinuous subgroup Γ of G for the homogeneous space G/H. When H contains $[G, G]$ or $[\Gamma, \Gamma]$ is uniform in $[G, G]$, we exhibit a description of the space $\mathscr{T}(\Gamma, G, H)$. This requires a precise algebraic description of the parameter and the deformation spaces, which is provided for more general contexts. When restricted to the setup of Heisenberg groups, we produce an accurate description of the space $\mathscr{T}(\Gamma, G, H)$ for arbitrary H and Γ, generalizing then an earlier works on the subject. The paper is also devoted to studying the topological features of the related deformations. Namely, we provide a necessary and sufficient condition for the rigidity of deformations and that the rigidity property globally fails to hold in the case of Heisenberg groups. $\mathscr{E}$

1. Introduction

This paper deals with the deformation space of the action of a *non-abelian* discontinuous subgroup of an exponential solvable Lie group G on a homogeneous space G/H, where H stands for an analytic subgroup of G. The problem of describing deformations was first advocated by T. Kobayashi for the general non-Riemannian setting in [12] where he formalized the study of the deformation of Clifford-Klein forms from a theoretic point of view. (See [11] for further perspectives and basic examples). Since [14], the deformation and the moduli spaces for abelian discontinuous subgroups have been recently found explicitly in a number of settings where the basis group G is exponential solvable and the discontinuous subgroup in question is abelian (see [1] and [3]). One substantial ingredient which has made such achievements possible, is that any abelian discrete subgroup of G admits a *syndetic hull* in G, a unique connected analytic subgroup of G containing it cocompactly. Such an upshot fails to hold beyond the setup of completely solvable Lie groups for non-abelian discrete subgroups of G (see [18]). In this paper, we shall provide a *new* proof of existence of the syndetic hull L of any closed subgroup Γ of G, assumed to be completely solvable. We provide in the same context a *new* positive answer to the rigidity problem of closed subgroups. It consists in showing that any continuous homomorphism from Γ to G uniquely extends to a continuous homomorphism

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from L to G and that the set of continuous homomorphisms $\mathrm{Hom}_c(L,G)$ and $\mathrm{Hom}_c(\Gamma,G)$ are homeomorphic when endowed with their natural topologies. As such, the restriction map $\xi:\mathrm{Hom}_c(L,G)\to\mathrm{Hom}_c(\Gamma,G)$ will be shown to be a homeomorphism. One distinguished consequent phenomenon is the useful algebraic interpretation of the parameter space $R(\Gamma,G,H)$ of $\mathrm{Hom}(\Gamma,G)$ defined by:

$$R(\Gamma,G,H):=\left\{\varphi\in\mathrm{Hom}(\Gamma,G)\;\middle|\;\begin{array}{l}\varphi(\Gamma)\text{ is isomorphic to }\Gamma\text{ and}\\ \text{acts properly and fixed point freely}\\ \text{on }G/H\end{array}\right\}, \tag{1}$$

which is shown (Theorem 4.1) to be homeomorphic to

$$R(\mathfrak{l},\mathfrak{g},\mathfrak{h})=\left\{\varphi\in\mathrm{Hom}(\mathfrak{l},\mathfrak{g})\;\middle|\;\begin{array}{l}\dim\varphi(\mathfrak{l})=\dim\mathfrak{l}\text{ and}\\ \exp\varphi(\mathfrak{l})\text{ acts properly on }G/H\end{array}\right\}. \tag{2}$$

Here, $H=\exp(\mathfrak{h})$ is the connected subgroup of G in question, $L=\exp(\mathfrak{l})$ and where isomorphism is meant as a topological isomorphism. This conclusively leads to a treatable expression of the deformation space as the group G acts on $R(\mathfrak{l},\mathfrak{g},\mathfrak{h})$ by adjoint action. We do believe that such an expression widely opens up many opportunities to tackle the problem of describing deformations for more general exponential contexts. We mention here that for abelian discontinuous subgroups, the idea to use a continuous analogue was first used in [14] in the nilpotent Lie group case, extended later in [1] to the exponential case.

In this paper, we cope with the case where G is an arbitrary completely solvable Lie group and under the hypothesis that $[G,G]=[L,L]$. One of the main achievements of the paper is to derive an explicit description of the deformation space in this context and later in the setting where G is the Heisenberg group and Γ is an arbitrary non-abelian discontinuous subgroup acting on a homogeneous space. The paper is also devoted to put the emphasis on the topological features of the deformation space(see section 6). The parameter space of the studied class is shown to be open and semi-algebraic so that the stability property globally holds. As for rigidity, we will provide a necessary and sufficient condition for a homomorphism to be rigid according to our context. In the setup of Heisenberg groups, the structure of Γ is shown to be irrelevant and we prove that the rigidity fails to hold everywhere on the parameter space. Our technique here is relevant from that singled out in [1] and makes use of Grassmannians to carry out a pretty expressible layering of the parameter space. If k designates the rank of Γ and $\mathrm{Aut}(\mathfrak{l})$ the closed subgroup of $GL_k(\mathbb{R})$ of all the Lie algebras automorphisms of $\mathfrak{l}$, we consider the canonical surjection $p:GL_k(\mathbb{R})\to GL_k(\mathbb{R})/\mathrm{Aut}(\mathfrak{l})$. The idea reduces to take an open covering of $GL_k(\mathbb{R})/\mathrm{Aut}(\mathfrak{l})$ and a family of local continuous sections which allows us to smoothly split the space $R(\mathfrak{l},\mathfrak{g},\mathfrak{h})$ into some open $G-$invariant components. This gives rise to a systematic covering of the deformation space for which any subsequent fiber involves the subgroup $\mathrm{Aut}(\mathfrak{l})$ as a direct component, when submitted to the action of G. It is worth noting that the aforementioned condition on Γ greatly intervenes to show that such a layering of the deformation space could be realized. When restricted to the Heisenberg case, the deformation space will be therefore precisely described as being sets of matrix-like form. The main result of the seventh section was announced in the paper [2].

2. Backgrounds

2.1. **Notation and basis facts.** We begin this section by reviewing some basic facts and notation for an exponential solvable Lie group. The next subsections are also devoted to that purpose. The material dealt with here is quite standard, we refer the reader to the references [4,8,9,11,12,13] for more complete details. Throughout the paper, $\mathfrak{g}$ will be a n-dimensional real exponential solvable Lie algebra, G will be the associated Lie group. *Exponential* means that the exponential mapping $\exp : \mathfrak{g} \to G$ is a $C^{+\infty}$ diffeomorphism. Therefore the group G is connected and simply connected. The Lie algebra $\mathfrak{g}$ acts on $\mathfrak{g}$ by the adjoint representation $ad_{\mathfrak{g}}$, that is:

$$ad_{\mathfrak{g}}(X)(Y) = ad(X)(Y) = [X,Y],\ X,\ Y \in \mathfrak{g}.$$

The group G acts on $\mathfrak{g}$ by the adjoint representation Ad_G, i. e.

$$Ad_G(g)(Y) = Ad(g)(Y) = \sum_{n\geq 0} \frac{(ad(X))^n}{n!} Y,\ g = \exp X \in G,\ X \text{ and } Y \in \mathfrak{g}.$$

2.2. **Proper and fixed point free actions.** Let X be a locally compact space and K a locally compact topological group. The action of the group K on X is said to be:

(1) Proper if, for each compact subset $S \subset X$ the set $K_S = \{k \in K : \ k \cdot S \cap S \neq \emptyset\}$ is compact.

(2) Fixed point free (or merely free) if, for each $x \in X$, the isotropy group $K_x = \{k \in K : \ k \cdot x = x\}$ is trivial.

(3) Properly discontinuous if, K is discrete and for each compact subset $S \subset X$ the set K_S is finite.

In the case where $X = G/H$ is a homogeneous space and K a subgroup of G, then it is well known that the action of K on X is proper if $SHS^{-1} \cap K$ is compact for any compact set S in G. Here, for two subsets A and B of the locally compact topological group G, the product AB is the subset $\{ab : \ a \in A, b \in B\}$. Likewise, the action of K on X is free if for every $g \in G$, $K \cap gHg^{-1} = \{e\}$. In such cases, we say that the triple (G,H,K) is proper (respectively free). To make this precise, the following lemma is obtained in [4].

Lemma 2.1. *Let G be an exponential solvable Lie group and let H and K be closed connected subgroups of G. Then the following assertions are equivalent:*

(i) *The triple (G,H,K) is free.*
(ii) $\mathfrak{k} \cap \mathrm{Ad}_g \mathfrak{h} = \{0\}$ *for any $g \in G$. Here $\mathfrak{h}$ and $\mathfrak{k}$ are the Lie algebras of H and K respectively.*

2.3. **Clifford-Klein forms.** Let now Γ be a discontinuous group for a homogeneous space G/H, which means that Γ is a discrete subgroup of G and acts properly and fixed point freely on G/H. In this case the quotient space $\Gamma\backslash G/H$ is said to be a *Clifford-Klein form* for the homogeneous space G/H. It is then well-known that any Clifford-Klein form is endowed through the action of Γ with a manifold structure for which the quotient canonical surjection

$$\pi : G/H \to \Gamma\backslash G/H \tag{3}$$

turns out to be an open covering and particularly a local diffeomorphism. On the other hand, any Clifford-Klein form $\Gamma\backslash G/H$ inherits any $G-$invariant geometric structure (e.g. complex structure, pseudo-Riemanian structure, conformal structure, symplectic structure,...) on the homogeneous space G/H through the covering map π defined as in equation (3) above.

3. Syndetic hulls and the rigidity problem of homomorphisms.

As remarked by T. Kobayashi, there is a strong relationship between the proper action of a cocompact discrete subgroup and the proper action of its syndetic hull on locally compact Hausdorff space. More precisely, one has the following:

Lemma 3.1. **([8], Lemma 2.3)** *Suppose a locally compact group L acts on a Hausdorff, locally compact space X. Let Γ be a cocompact discrete subgroup of L. Then the L-action on X is proper if and only if the Γ-action on X is properly discontinuous.*

This leads us to consider the following important object related to any closed subgroup of a Lie group.

Definition 3.2. Let G be a Lie group and Γ a closed subgroup of G. By a *syndetic hull* of Γ, we mean any connected Lie subgroup of G which contains Γ cocompactly.

In [18], Saito proved that any closed subgroup of a completely solvable Lie group admits a unique syndetic hull. Completely solvable means that any endomorphism $ad_X, X \in \mathfrak{g}$ has real eigenvalues. For the convenience of the readers and following some ideas from [21], we give here another proof of the result.

Theorem 3.3. *Let G be a completely solvable Lie group. Then any closed subgroup of G admits a unique syndetic hull.*

Proof. Let Γ be a closed subgroup of G, being completely solvable. We first show that the syndetic hull is unique provided its existence. Let L_1 and L_2 be two connected closed subgroups such that L_1/Γ and L_2/Γ are compact. We claim that $L_i/(L_1 \cap L_2),\ i = 1, 2$ are also compact. To see that, consider for $i = 1,\ 2$ the canonical surjection

$$\pi_i : L_i \longrightarrow L_i/(L_1 \cap L_2),$$

which factors through the canonical surjection $\rho_i : L_i \to L_i/\Gamma$ to a surjection $\tilde{\pi}_i : L_i/\Gamma \to L_i/(L_1 \cap L_2)$ such that $\pi_i = \tilde{\pi}_i \circ \rho_i$. The map $\tilde{\pi}_i$ is surjective and continuous and thus its image $L_i/(L_1 \cap L_2)$ is compact. Now, G is connected simply connected solvable Lie group, the quotient $L_i/(L_1\cap L_2)$ is diffeomorphic to $\mathbb{R}^d$, where $d=\dim L_i-\dim(L_1\cap L_2)$. It follows therefore that $d=0$ and that $L_1\cap L_2=L_1=L_2$, as it is to be shown.

We tackle now the proof of existence, we proceed by induction on the dimension of G. The case where Γ is abelian is already treated in [1]. Let now G_0 be a one co-dimensional closed normal subgroup of G. Provided that $\Gamma\subset G_0$, we are done using the induction hypothesis. We assume henceforth that $\Gamma\not\subset G_0$ and we consider $\Gamma_0=\Gamma\cap G_0$ which is a closed subgroup of G_0. Let $\tilde{G}=G/G_0\simeq\mathbb{R}$ and $\pi:G\to\tilde{G}$ the canonical projection. Provided that Γ_0 is trivial, the homomorphism $\tilde{\pi}=\pi|\Gamma:\Gamma\to\tilde{\Gamma}:=\tilde{\pi}(\Gamma)$ appears to be a groups isomorphism. This gives then that Γ is therefore abelian and the result follows. We assume from now on that Γ_0 is not trivial. There exists by the induction hypothesis, a closed connected subgroup L_0 of G_0 which contains Γ_0 cocompactly. As such, there exists a compact set C in G contained in G_0 and fulfills the following identity:

$$L_0=C\Gamma_0. \tag{4}$$

Assume for a while that L_0 is normal in G, then $L_0\Gamma$ is a subgroup of G and by equation (4), we get that

$$L_0\Gamma=C\Gamma_0\Gamma=C\Gamma \tag{5}$$

which is closed in G as Γ is. Let $G'=G/L_0$ and $\pi:G\to G'$ the associated canonical surjection. Then $\Gamma'=\pi(L_0\Gamma)$ is a closed subgroup of G'. Using the induction hypothesis, there exists a connected closed subgroup S' of G' such that S'/Γ' is compact, in particular there exists by Lemma 1 in [6] a compact set C' of G such that $S'=\overline{C'}\Gamma'$, where $\overline{C'}$ is the image of C' by π. Let S be the pre-image of S' by π , then S is closed subgroup of G which contains L_0. Moreover, we have

$$S=C'L_0\Gamma L_0=C'L_0\Gamma=C'C\Gamma$$

by (4) and (5), which merely entails that S/Γ is compact. On the other hand, we have $S'=S/L_0$ with S' and L_0 are connected. This shows that S is connected.
We finally treat the case where L_0 is not normal. Let $N_G(L_0)$ be the normalizer of L_0 in G. The subgroup G_0 being normal in G, we get that for any $\gamma\in\Gamma$, $\gamma\Gamma_0\gamma^{-1}\subset G_0\cap\Gamma=\Gamma_0$, and then Γ_0 is normal in Γ. Therefore for any $\gamma\in\Gamma$ the subgroup $\gamma L_0\gamma^{-1}$ is connected and closed in G_0 such that $\gamma L_0\gamma^{-1}/\Gamma_0$ is compact. By the uniqueness of L_0, we get that $\gamma L_0\gamma^{-1}=L_0$ and that $\Gamma\subset N_G(L_0)$. Recall that $N_G(L_0)$ is a connected closed subgroup of G and $\dim N_G(L_0)<\dim G$, the result follows using again the induction hypothesis.

□

A straight application of Theorem (3.3) is to establish a homeomorphism between the spaces $\mathrm{Hom}(\Gamma,G)$ and $\mathrm{Hom}(\mathfrak{l},\mathfrak{g})$ where $\mathfrak{l}$ stands for the Lie algebra of the syndetic hull L of Γ. Recall that in [18], it is proved that any homomorphism from Γ to G can be uniquely extended to a homomorphism from the syndetic hull of Γ to G. We now give a new proof of this upshot and even prove a more precise result (Proposition (3.6) below). We first have the following:

Theorem 3.4. *Let G be a completely solvable Lie group, Γ a closed subgroup of G and L its syndetic hull in G. Then any continuous homomorphism from Γ to G uniquely extends to a continuous homomorphism from L to G.*

Proof. We provide first a proof of existence. We start with a continuous homomorphism φ from Γ to G and we denote by $\hat{\Gamma} \subset L \times G$ its graph. Clearly $\hat{\Gamma}$ is a closed subgroup of $L \times G$. By Theorem (3.3), $\hat{\Gamma}$ has a unique syndetic hull K, say. Let

$$p : L \times G \to L, \quad q : L \times G \to G$$

be the natural homomorphisms projections. We now proceed to prove that $q_{|K} \circ p_{|K}^{-1}$ is a well defined continuous homomorphism from L to G and that its restriction to Γ coincides with φ. We first show that $p_{|K}$ realizes an isomorphism between K and L. Indeed, p is a continuous homomorphism and K is a connected exponential Lie subgroup of $L \times G$. Let ψ be the differential of $p_{|K}$ at e and $\mathfrak{k} = \log K$ the Lie algebra associated to K. Then

$$p_{|K} = \exp_L \circ \psi \circ \log_K \text{ and } p(K) = \exp_L\big(\psi(\mathfrak{k})\big) \tag{6}$$

where $\psi(\mathfrak{k})$ is a Lie subalgebra of $\mathfrak{l} = \log L$, in particular $p(K)$ is closed connected subgroup of L. Let π be the canonical surjection $L \to L/\Gamma$. The subgroup Γ is included in $p(K)$ being a saturated (with respect to π) closed set in L. This means that $\pi(p(K))$ is closed in L/Γ and then $p(K)/\Gamma$ is compact. From the uniqueness of the syndetic hull of Γ, we get that $p(K) = L$ and that $p_{|K}$ is a surjective homomorphism from K to L. To prove the injectivity we need the following lemma.

Lemma 3.5. *The maps $p_{|\hat{\Gamma}}$ and $p_{|K}$ are proper. That is, the inverse image of any compact set of L by $p_{|\hat{\Gamma}}$ and $p_{|K}$ is compact.*

Proof. Let C be a compact set in L, then

$$\begin{aligned} p^{-1}(C) \cap \hat{\Gamma} &= (C \times G) \cap \hat{\Gamma} \\ &= \{(x, \varphi(x)),\, x \in C \cap \Gamma\} \end{aligned}$$

which is compact as Γ is closed in G. This shows that $p_{|\hat{\Gamma}}$ is proper.

Let now C be a compact set of L and C_1 a compact set of K such that $K = C_1\hat{\Gamma}$. Clearly $p^{-1}(C) \cap K$ is closed in K and we have,

$$\begin{aligned} p^{-1}(C) \cap K = p^{-1}(C) \cap C_1\hat{\Gamma} &\subset C_1(C_1^{-1}p^{-1}(C) \cap \hat{\Gamma}) \\ &\subset C_1(p^{-1}\big(p(C_1^{-1})C\big) \cap \hat{\Gamma}) = C_1 p_{|\hat{\Gamma}}^{-1}\big(p(C_1^{-1})C\big). \end{aligned}$$

To conclude that $p^{-1}(C) \cap K$ is compact, it is thus sufficient to prove that $C_1 p_{|\hat{\Gamma}}^{-1}\big(p(C_1^{-1})C\big)$ is a compact in K. By the continuity of p, we see that $p(C_1^{-1})C$ is a compact set in L. Now, $p_{|\hat{\Gamma}}$ is proper. Then $p_{|\hat{\Gamma}}^{-1}\big(p(C_1^{-1})C\big)$ is compact in $\hat{\Gamma}$ and the result follows.

□

Now the map $p_{|K}$ is a continuous surjective homomorphism and from its properness, we see that its kernel is a compact subgroup of K. Up to this step, $p_{|K}$ is shown to be a continuous bijective homomorphism. Using the first equality in (6), we can see that

$$p_{|K}^{-1} = \exp_K \circ \psi^{-1} \circ \log_L,$$

which is also continuous. This conclusively shows that $p_{|K}$ is an isomorphism. Furthermore $q_{|K}$ is a continuous homomorphism, then $q_{|K} \circ p_{|K}^{-1}$ is a continuous homomorphism from L to G and for $x \in \Gamma$, we have

$$q_{|K} \circ p_{|K}^{-1}(x) = q_{|K}(x, \varphi(x)) = \varphi(x),$$

which entails that the restriction of $q_{|K} \circ p_{|K}^{-1}$ to Γ is φ. This achieves the proof of existence. As for the uniqueness, let L' be the subgroup of G generated by the family of the one parameter subgroups $(\gamma(t), t \in \mathbb{R})$ for $\gamma \in \Gamma$ defined by the derivative at $t = 0$, $\frac{d\gamma(t)}{dt}(0) = \log(\gamma)$. Then L' is a connected subgroup of G, contained in L and contains Γ, which entails that $L = \overline{L'}$. Let ψ be a continuous homomorphism from L to G and $\gamma \in \Gamma$. By the continuity of ψ, we get

$$\psi(\gamma(t)) = \exp\big(t \log(\psi(\gamma))\big), \ t \in \mathbb{R}. \tag{7}$$

Let ψ and ψ' be a two continuous homomorphisms from L to G. By continuity, $\psi = \psi'$ if and only if $\psi_{|L'} = \psi'_{|L'}$. But L' is generated by the subgroups $\gamma(t)$, $\gamma \in \Gamma$. This means therefore that $\psi = \psi'$ if and only if $\psi(\gamma(t)) = \psi'(\gamma(t))$ for all $\gamma \in \Gamma$, which is equivalent by means of (7) to $\psi_{|\Gamma} = \psi'_{|\Gamma}$. This achieves the proof of the Theorem. □

We finally prove the following result which will be of great interest in the sequel.

Proposition 3.6. *Let G be a completely solvable Lie group, Γ a closed subgroup of G and L its syndetic hull in G. Let $Hom_c(L, G)$ and $Hom_c(\Gamma, G)$ designate the sets of continuous homomorphisms from L to G and Γ to G respectively. Then the restriction natural map $R : Hom_c(L, G) \to Hom_c(\Gamma, G) : \psi \mapsto \psi_{|\Gamma}$ is a homeomorphism.*

Proof. We obviously get from Theorem (3.4) that R is bijective, and it is clear that R is continuous. To prove the continuity of its inverse, let $(\varphi_n)_n$ be a sequence in $\mathrm{Hom}(\Gamma, G)$ which converges to some element φ, say. We denote by $(\psi_n)_n$ (respectively ψ) the extensions of $(\varphi_n)_n$ (respectively φ). To prove that the sequence $(\psi_n)_n$ converges to ψ it is sufficient to show that $(\psi_n(\gamma(t)))_n$ converges to $\psi(\gamma(t))$ for every $\gamma \in \Gamma$. Finally by (7), we have for all $\gamma \in \Gamma$

$$\begin{aligned} \lim_{n\to\infty} \psi_n(\gamma(t)) &= \lim_{n\to\infty} \exp\big(t \log(\varphi_n(\gamma))\big) \\ &= \exp\big(t \log(\varphi(\gamma))\big) \\ &= \psi(\gamma(t)). \end{aligned}$$

□

4. First algebraic interpretation of the parameter, the deformation and the moduli spaces

4.1. **General theory of deformations.** The material dealt with in the following subsection basically comes from the paper [13].

Let G be a Lie group and Γ be a finitely generated discrete group. We designate by $\text{Hom}(\Gamma, G)$ the space of group homomorphisms from Γ to G endowed with the point wise convergence topology. The same topology is obtained by taking generators $\gamma_1, ..., \gamma_k$ of Γ, then using the injective map

$$\text{Hom}(\Gamma, G) \hookrightarrow G \times \cdots \times G, \ \varphi \mapsto (\varphi(\gamma_1), ..., \varphi(\gamma_k))$$

to equip $\text{Hom}(\Gamma, G)$ with the relative topology induced from the direct product $G \times \cdots \times G$. Let H be a closed subgroup of G. If H is not compact, then the discrete subgroup Γ does not necessarily act properly discontinuously on G/H. We consider then the parameters space $R(\Gamma, G, H)$ of $\text{Hom}(\Gamma, G)$ defined as in formula (1) above, this set plays an important role as we will see later. According to this definition, for each $\varphi \in R(\Gamma, G, H)$, the space $\varphi(\Gamma)\backslash G/H$ is a Clifford-Klein form which is a Hausdorff topological space and even equipped with a structure of a manifold for which, the quotient canonical map is an open covering.

Let now $\varphi \in R(\Gamma, G, H)$ and $g \in G$, we consider the element $\varphi^g := g \cdot \varphi \cdot g^{-1}$ of $\text{Hom}(\Gamma, G)$ defined by

$$\varphi^g(\gamma) = g\varphi(\gamma)g^{-1}, \gamma \in \Gamma.$$

It is then clear that the element $\varphi^g \in R(\Gamma, G, H)$ and that the map

$$\varphi(\Gamma)\backslash G/H \longrightarrow \varphi^g(\Gamma)\backslash G/H, \quad \varphi(\Gamma)xH \mapsto \varphi^g(\Gamma)gxH$$

is a natural diffeomorphism. We consider then the orbits space

$$\mathscr{T}(\Gamma, G, H) = R(\Gamma, G, H)/G$$

instead of $R(\Gamma, G, H)$ in order to avoid the unessential part of deformations arising inner automorphisms and to be quite precise on parameters. We call the set $\mathscr{T}(\Gamma, G, H)$ as the space of the deformation of the action of Γ on the homogeneous space G/H.

On the other hand, let the group $\text{Aut}(\Gamma)$ act on $\text{Hom}(\Gamma, G)$ by

$$T \cdot \varphi(\gamma) := \varphi(T^{-1}(\gamma)), \ \varphi \in \text{Hom}(\Gamma, G), \ T \in \text{Aut}(\Gamma), \ \gamma \in \Gamma.$$

It is then easy to check that the group $\text{Aut}(\Gamma)$ leaves the parameters space $R(\Gamma, G, H)$ invariant and its action on it is $G-$ equivariant. We define then (to avoid this unessential part too) the moduli space as the double coset space

$$\mathscr{M}(\Gamma, G, H) := \text{Aut}(\Gamma)\backslash R(\Gamma, G, H)/G.$$

4.2. **An algebraic interpretation.** Our first observation is that the parameter space defined above, only depends on the structure of the syndetic hull of Γ when the basis group G is completely solvable. Recall that any continuous homomorphism of a connected Lie groups is smooth and its derivative is a homomorphism of Lie algebras. We consider the smooth map $d : \text{Hom}_c(L, G) \longrightarrow \text{Hom}(\mathfrak{l}, \mathfrak{g}), \ \varphi \mapsto d\varphi_{|_e}$ where $\mathfrak{l}$ is the Lie algebras of L. In the case of exponential Lie groups $d\varphi_{|e}(X) = \log\circ\varphi\circ\exp(X)$ for any $X \in \mathfrak{g}$. The group G acts on the spaces $\mathrm{Hom}(\Gamma, G)$, $\mathrm{Hom}(L, G)$ and $\mathrm{Hom}(\mathfrak{l}, \mathfrak{g})$ respectively through the following laws:

$$(g\cdot\varphi)(\gamma) = g\varphi(\gamma)g^{-1},\ \gamma\in\Gamma(\text{or } L),\ \varphi\in\mathrm{Hom}(\Gamma, G)(\text{or }\mathrm{Hom}(L, G)), g\in G$$
$$g\cdot\psi = \mathrm{Ad}_g\circ\psi,\ \psi\in\mathrm{Hom}(\mathfrak{l},\mathfrak{g}), g\in G.$$

As L is exponential, the differential map d from $\mathrm{Aut}(L)$ to $\mathrm{Aut}(\mathfrak{l})$ is a topological isomorphism. Using the natural injection i of $\mathrm{Aut}(\Gamma)$ in $\mathrm{Aut}(L)$, we see that $\mathrm{Aut}(\Gamma)$ acts on $\mathrm{Hom}(L, G)$ and $\mathrm{Hom}(\mathfrak{l},\mathfrak{g})$ by

$$a\cdot\varphi = \varphi\circ i(a)^{-1} \text{ and } a\cdot\psi = \psi\circ d(i(a))_{|e}^{-1}, \tag{8}$$

for $\varphi\in\mathrm{Hom}(L, G)$, $\psi\in\mathrm{Hom}(\mathfrak{l},\mathfrak{g})$ and $a\in\mathrm{Aut}(\Gamma)$. Then for $R(\mathfrak{l},\mathfrak{g},\mathfrak{h})$ being the space defined as in (2), we get the preliminary algebraic interpretation of the parameter, deformation and moduli spaces:

Theorem 4.1. *Let G be a completely solvable Lie group, H a connected subgroup of G and Γ a discrete subgroup of G acting properly discontinuously on G/H. Then up to a homeomorphism,*

$$R(\Gamma, G, H) = R(\mathfrak{l},\mathfrak{g},\mathfrak{h}).$$

In particular if Γ and Γ' have the same syndetic hull, then $R(\Gamma, G, H)$ and $R(\Gamma', G, H)$ are homeomorphic. Furthermore, up to homeomorphism, the deformation space $\mathscr{T}(\Gamma, G, H)$ coincides with $R(\mathfrak{l},\mathfrak{g},\mathfrak{h})/G$ and the moduli space is identified to $Aut(\Gamma)\backslash R(\mathfrak{l},\mathfrak{g},\mathfrak{h})/G$.

Proof. The composition map $d\circ\xi$ is a homeomorphism from

$$\{\varphi\in\mathrm{Hom}(\Gamma, G),\ \varphi(\Gamma)\text{ isomorphic to }\Gamma\}$$

to its image, where ξ is the inverse of the restriction map as in proposition (3.6). We now show that this image is precisely the set

$$\{\varphi\in\mathrm{Hom}(\mathfrak{l},\mathfrak{g}),\ \dim\varphi(\mathfrak{l}) = \dim\mathfrak{l}\}.$$

Indeed, let $\varphi\in\mathrm{Hom}(\Gamma, G)$ such that Γ is isomorphic to $\Gamma' = \varphi(\Gamma)$. The homomorphism $\xi(\varphi)$ is continuous and L is connected, then $\xi(\varphi)(L)$ is a connected subgroup containing $\varphi(\Gamma)$. This means that $\xi(\varphi)(L)$ contains the syndetic hull of Γ' denoted by L'. In particular, $\dim L\geq\dim\xi(\varphi)(L)\geq\dim L'$. Let φ^{-1} be the inverse of $\varphi:\Gamma\to\Gamma'\subset G$. The composition of φ^{-1} and the natural injection of Γ in G, is a homomorphism from Γ' to G and Γ' is isomorphic to $\Gamma = \varphi^{-1}(\Gamma')$. Then for ξ' the extension map from $\mathrm{Hom}(\Gamma', G)$ to $\mathrm{Hom}(L', G)$, we have $\xi'(\varphi^{-1})(L')\supset L$, then $\dim L'\geq\dim\xi'(\varphi^{-1})(L')\geq\dim L$. This entails that $\dim L = \dim\xi(\varphi)(L)$, which is equivalent to $\dim\mathfrak{l} = \dim d\circ\xi(\varphi)(\mathfrak{l})$. Conversely, if ψ is a Lie algebras homomorphism, such that $\dim\mathfrak{l} = \dim\psi(\mathfrak{l})$, then $\exp\circ\psi\circ\log\in\mathrm{Hom}(L, G)$, is an isomorphism from L to its image. Its restriction to Γ denoted by φ say, is an isomorphism from Γ to its image and satisfies $d\circ\xi(\varphi) = \psi$. Now, from Lemma (3.1), the proper action of $\varphi(\Gamma)$ is equivalent to the proper action of its syndetic hull $L' = \xi(\varphi)(L) = \exp\circ d\circ\xi(\varphi)(\mathfrak{l})$. We conclude therefore that $d\circ\xi(R(\Gamma, G, H)) = R(\mathfrak{l},\mathfrak{g},\mathfrak{h})$. To achieve the proof of the theorem, we need the following:

Lemma 4.2. *With the same hypotheses as in Theorem (4.1). The maps d and ξ are G and $Aut(\Gamma)$-equivariant.*

Proof. For $g \in G$ and $\varphi \in \mathrm{Hom}(\Gamma, G)$ we have $g \cdot \xi(\varphi) = \xi(g \cdot \varphi)$ if and only if $(g \cdot \xi(\varphi))_{|\Gamma} = g \cdot \varphi$. Furthermore, for any $\gamma \in \Gamma$ we have, $(g \cdot \xi(\varphi))_{|\Gamma}(\gamma) = g\xi(\varphi)(\gamma)g^{-1} = (g \cdot \varphi)(\gamma)$. Let $\tau_g : G \to G$, $t \mapsto gtg^{-1}$ be the conjugation. For any $\varphi \in \mathrm{Hom}(L, G)$ and any $g \in G$, we have:

$$d(g \cdot \varphi)_{|e} = d(\tau_g \circ \varphi)_{|e} = d(\tau_g)_{|e} \circ d(\varphi)_{|e} = \mathrm{Ad}_g \circ d(\varphi)_{|e} = g \cdot d(\varphi)_{|e}.$$

Likewise, for $a \in \mathrm{Aut}(\Gamma)$ and $\varphi \in \mathrm{Hom}(\Gamma, G)$, we have $a \cdot \xi(\varphi) = \xi(a \cdot \varphi)$ if and only if $(a \cdot \xi(\varphi))_{|\Gamma} = a \cdot \varphi$, and as before we have $(a \cdot \xi(\varphi))_{|\Gamma}(\gamma) = \xi(\varphi) \circ i(a)^{-1}(\gamma) = \varphi \circ a^{-1}(\gamma) = (a \cdot \varphi)(\gamma)$. For $\varphi \in \mathrm{Hom}(L, G)$ we have $d(a \cdot \varphi)_{|e} = d(\varphi \circ i(a)^{-1})_{|e} = d(\varphi)_{|e} \circ d(i(a)^{-1})_{|e} = a \cdot d(\varphi)_{|e}$. □

Our Theorem is thus proved. □

Using Lemma (4.2) and Theorem (4.1), the following is immediate:

Corollary 4.3. *Retain the same hypotheses as in Theorem (4.1). If Γ is uniform in G, then H is trivial and we have, $R(\Gamma, G, H) = Aut(\mathfrak{g})$,*

$$\mathscr{T}(\Gamma, G, H) = Aut(\mathfrak{g}) / \mathrm{Ad}(G) \ \ \textit{and} \ \ \mathscr{M}(\Gamma, G, H) = Aut(\Gamma) \backslash Aut(\mathfrak{g}) / \mathrm{Ad}(G).$$

4.3. **Structure of a principal fiber bundle.** We retain all our hypotheses and notation. Let $k = \dim \mathfrak{l}$, $s = \dim \mathfrak{h}$ and $n = \dim \mathfrak{g}$. We fix a basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{g}$ passing through $\mathfrak{h}$. We identify the vector spaces $\mathfrak{g}$ to $\mathbb{R}^n$, $\mathfrak{l}$ to $\mathbb{R}^k$, $\mathfrak{h}$ to the s dimensional subspace $\mathbb{R}^s \times 0_{\mathbb{R}^{n-s}}$ of $\mathbb{R}^n$, $\mathscr{L}(\mathfrak{l}, \mathfrak{g})$ to $M_{n,k}(\mathbb{R})$ the real vector space of $n \times k$ matrices with real entries and $\mathrm{Hom}(\mathfrak{l}, \mathfrak{g})$ to a closed subset of $M_{n,k}(\mathbb{R})$. Let $M^\circ_{n,k}(\mathbb{R})$ be the open set of $M_{n,k}(\mathbb{R})$ consisting of rank k matrices in $M_{n,k}(\mathbb{R})$, which is also identified to the set $\{\varphi \in \mathscr{L}(\mathfrak{l}, \mathfrak{g}),\ \varphi \text{ injective}\}$. We define the set

$$I(n,k) = \{(i_1, \ldots, i_k) \in \mathbb{N}^k,\ 1 \le i_1 < \cdots < i_k \le n\}. \tag{9}$$

For

$$M = \begin{pmatrix} L_1 \\ \vdots \\ L_n \end{pmatrix} \in M_{n,k}(\mathbb{R}) \ \text{ and } \alpha = (i_1, \ldots, i_k) \in I(n,k),$$

we denote by M_α the $k \times k$ relative minor $\begin{pmatrix} L_{i_1} \\ \vdots \\ L_{i_k} \end{pmatrix}$. Let

$$U_\alpha = \{M \in M^\circ_{n,k}(\mathbb{R}) : \ M_\alpha = I_k\} \cong M_{n-k,k}(\mathbb{R})$$

and

$$\mathscr{U}_\alpha = \{M \in M^\circ_{n,k}(\mathbb{R}),\ \det M_\alpha \neq 0\} \cong M_{n-k,k}(\mathbb{R}) \times GL_k(\mathbb{R}),$$

where I_k designates the identity element of $M_k(\mathbb{R})$. Then clearly $M^{\circ}_{n,k}(\mathbb{R}) = \bigcup_{\alpha\in I(n,k)} \mathscr{U}_\alpha$. The group $GL_k(\mathbb{R})$ acts on $M_{n,k}(\mathbb{R})$ through the right multiplication and the Grassmannian $G_{n,k}(\mathbb{R})$ of the $k-$dimensional subspaces of $\mathbb{R}^n$ is identified to the quotient topological space $M^{\circ}_{n,k}(\mathbb{R})/GL_k(\mathbb{R})$. Let $\eta : M^{\circ}_{n,k}(\mathbb{R}) \to G_{n,k}(\mathbb{R})$, $M \mapsto M(\mathbb{R}^k)$ be the canonical surjection. It is easy to see that the restriction η_α of η to the set U_α is a homeomorphism between U_α and its image. The group $\mathrm{Aut}(\mathfrak{l})$ of automorphisms of $\mathfrak{l}$ is a closed subgroup of $GL_k(\mathbb{R})$, then the homogeneous space $GL_k(\mathbb{R})/\mathrm{Aut}(\mathfrak{l})$ is endowed with a manifold structure and the quotient map $p : GL_k(\mathbb{R}) \to GL_k(\mathbb{R})/\mathrm{Aut}(\mathfrak{l})$ admits local sections. Consider an open covering $\{V_\beta\}_{\beta\in I}$ of $GL_k(\mathbb{R})/\mathrm{Aut}(\mathfrak{l})$ such that for any $\beta \in I$, there is a section $s_\beta : V_\beta \to GL_k(\mathbb{R})$ satisfying $p \circ s_\beta = Id_{|V_\beta}$. For every $\alpha \in I(n,k)$ we consider the map

$$\begin{array}{rcl} \pi_\alpha : \mathscr{U}_\alpha & \longrightarrow & GL_k(\mathbb{R})/\mathrm{Aut}(\mathfrak{l}) \times \eta(U_\alpha) \\ M & \mapsto & \big(p(M_\alpha), \eta(M)\big) \end{array}$$

and let $\mathscr{U}_{\alpha\beta} = \pi_\alpha^{-1}\big(V_\beta \times \eta(U_\alpha)\big)$. Clearly π_α is continuous and surjective, $\mathscr{U}_{\alpha\beta}$ is open in $M_{n,k}(\mathbb{R})$ and the collection $(\mathscr{U}_{\alpha\beta})_{\beta\in I}$ constitutes an open covering of $\mathscr{U}_\alpha$. We finally consider the map

$$\begin{array}{rcl} \xi_{\alpha\beta} : \mathrm{Aut}(\mathfrak{l}) \times V_\beta \times \eta(U_\alpha) & \longrightarrow & \mathscr{U}_{\alpha\beta} \\ (A, x, W) & \mapsto & \eta_\alpha^{-1}(W) s_\beta(x) A. \end{array}$$

This map is well defined as for any $(A, x, W) \in \mathrm{Aut}(\mathfrak{l}) \times V_\beta \times \eta(U_\alpha)$, the conclusion $\pi_\alpha \circ \xi_{\alpha\beta}(A, x, W) = (x, W)$ holds. Indeed, it is clear that the matrix $M = \eta_\alpha^{-1}(W)s_\beta(x)$ is in $\pi_\alpha^{-1}(x, W)$ and the set $M\mathrm{Aut}(\mathfrak{l})$ is a subset of $\pi_\alpha^{-1}(x, W)$. We further have the following useful properties:

Lemma 4.4. *For every $\alpha \in I(n,k)$, β, $\beta' \in I$ and $(A, x, W) \in Aut(\mathfrak{l}) \times V_\beta \times \eta(U_\alpha)$ we have:*

1) $\xi_{\alpha\beta}$ is a homeomorphism.

2) $\pi_\alpha^{-1}(x, W) = \eta_\alpha^{-1}(W)s_\beta(x)Aut(\mathfrak{l})$.

3) The map $\xi_{\alpha\beta,x,W} : Aut(\mathfrak{l}) \to \pi_\alpha^{-1}(x, W)$ given by $\xi_{\alpha\beta,x,W}(A) = \eta_\alpha^{-1}(W)s_\beta(x)A$ is a homeomorphism.

4) The map $t_{\alpha\beta\beta'} : (V_\beta \cap V_{\beta'}) \times \eta(U_\alpha) \to Aut(\mathfrak{l})$ given by $t_{\alpha\beta\beta'}(x, W) = \xi^{-1}_{\alpha\beta,x,W} \circ \xi_{\alpha\beta',x,W}$ is continuous.

Proof. For the first statement, it is easy to see that the map $\xi_{\alpha\beta}$ is continuous. Note that for every $M \in \mathscr{U}_{\alpha\beta}$, we have $p(s_\beta(p(M_\alpha))) = p(M_\alpha)$ and then $s_\beta(p(M_\alpha))^{-1}M_\alpha \in \mathrm{Aut}(\mathfrak{l})$. Let

$$\begin{array}{rcl} \xi'_{\alpha\beta} : \mathscr{U}_{\alpha\beta} & \longrightarrow & \mathrm{Aut}(\mathfrak{l}) \times V_\beta \times \eta(U_\alpha) \\ M & \mapsto & \big(s_\beta(p(M_\alpha))^{-1}M_\alpha, p(M_\alpha), \eta(M)\big), \end{array} \tag{10}$$

then $\xi'_{\alpha\beta}$ is continuous and $\xi_{\alpha\beta} \circ \xi'_{\alpha\beta} = Id_{|\mathscr{U}_{\alpha\beta}}$. Concerning the second statement, Let $M \in \pi_\alpha^{-1}(x, W)$ then, $\eta(M) = W$ and $p(M_\alpha) = x$ which is equivalent to

$$M = \eta_\alpha^{-1}(W)M_\alpha \text{ and } s_\beta(x)^{-1}M_\alpha \in \mathrm{Aut}(\mathfrak{l}).$$

It follows that $M = \eta_\alpha^{-1}(W)s_\beta(x)B$ for some $B \in \mathrm{Aut}(\mathfrak{l})$. This proves the first direct inclusion and the converse is straight immediate. To prove (3), we consider the map

$\xi'_{\alpha\beta,x,W} : \pi_\alpha^{-1}(x,W) \to \mathrm{Aut}(\mathfrak{l})$ given by $\xi'_{\alpha\beta,x,W}(M) = s_\beta(x)^{-1}M_\alpha$. Then clearly $\xi_{\alpha\beta,x,W}$ and $\xi'_{\alpha\beta,x,W}$ are continuous and $\xi'_{\alpha\beta,x,W} \circ \xi_{\alpha\beta,x,W}$ is the identity map. For the last statement, it is pretty clear that

$$\xi^{-1}_{\alpha\beta,x,W} \circ \xi_{\alpha\beta',x,W}(A) = s_\beta(x)^{-1}s_{\beta'}(x)A.$$

The composition $\xi^{-1}_{\alpha\beta,x,W} \circ \xi_{\alpha\beta',x,W}$ is then the left translation identified to the element $s_\beta(x)^{-1}s_{\beta'}(x) \in \mathrm{Aut}(\mathfrak{l})$. Furthermore, the continuity of $t_{\alpha\beta\beta'}$ is a direct consequence of the continuity of s_β and $s_{\beta'}$. □

Let consider the set

$$\mathscr{V}_\alpha = \mathrm{Hom}(\mathfrak{l},\mathfrak{g}) \cap \mathscr{U}_\alpha. \tag{11}$$

Note that for any $\alpha \in I$, the set $\mathscr{V}_\alpha$ is closed and stable by the action of Aut $\mathfrak{l}$ on $\mathscr{U}_\alpha$, (regarded as a subgroup of $GL_k(\mathbb{R})$). Let define the sets

$$W_\alpha = \pi_\alpha(\mathscr{V}_\alpha),\ W_{\alpha\beta} = W_\alpha \cap \big(V_\beta \times \eta(U_\alpha)\big) \text{ and } \mathscr{W}_{\alpha\beta} = \pi_\alpha^{-1}(W_{\alpha\beta}) = \mathscr{V}_\alpha \cap \mathscr{U}_{\alpha\beta}. \tag{12}$$

Then obviously

$$\mathscr{V}_\alpha = \bigcup_{\beta\in I} \mathscr{W}_{\alpha\beta} \text{ and } W_\alpha = \bigcup_{\beta\in I} W_{\alpha\beta}. \tag{13}$$

The following Lemma is immediate:

Lemma 4.5. *The set W_α is closed in $(GL_k(\mathbb{R})/Aut(\mathfrak{l})) \times \eta(U_\alpha)$, $W_{\alpha\beta}$ is open in W_α and $\mathscr{W}_{\alpha\beta}$ is open in $\mathscr{V}_\alpha$.*

Proof. The set $\mathscr{V}_\alpha$ is a closed in $\mathscr{U}_\alpha$ and Aut($\mathfrak{l}$)-stable, then $\pi_\alpha(\mathscr{V}_\alpha) = W_\alpha$ is closed. The set $W_{\alpha\beta}$ (respectively $\mathscr{W}_{\alpha\beta}$) is an intersection of W_α (respectively $\mathscr{V}_\alpha$) with an open set, as it is to be shown. □

With the above in mind, we get the following:

Corollary 4.6. *For any $\alpha \in I(n,k)$ and any $\beta \in I$, the map $\xi_{\alpha\beta}$ realizes a homeomorphism between $Aut(\mathfrak{l}) \times W_{\alpha\beta}$ and $\mathscr{W}_{\alpha\beta}$.*

Proof. Towards the proof, it is sufficient to see that $\xi_{\alpha\beta}(\mathrm{Aut}(\mathfrak{l}) \times W_{\alpha\beta}) = \mathscr{W}_{\alpha\beta}$. Let $(A,x,W) \in (\mathrm{Aut}(\mathfrak{l}) \times W_{\alpha\beta})$, then $\xi_{\alpha\beta}(A,x,W) \in \pi_\alpha^{-1}(x,W) \subset \pi_\alpha^{-1}(W_{\alpha\beta}) = \mathscr{W}_{\alpha\beta}$. Conversely, let $M \in \mathscr{W}_{\alpha\beta}$, then $\xi'_{\alpha\beta}(M) \in \mathrm{Aut}(\mathfrak{l}) \times W_{\alpha\beta}$ and $M = \xi_{\alpha\beta}(\xi'_{\alpha\beta}(M))$. □

With the above in mind, we get the following:

Theorem 4.7. *The collection $\big(\pi_\alpha, \mathscr{U}_\alpha, (GL_k(\mathbb{R})/Aut(\mathfrak{l})) \times \eta(U_\alpha), Aut(\mathfrak{l})\big)$ defines a principal bundle, the maps $(\xi_{\alpha\beta})_{\beta\in I}$ are the local trivializations and the maps $(t_{\alpha\beta\beta'})_{\beta'\in I}$ are the transition functions. Furthermore, the data $(\pi_\alpha, \mathscr{V}_\alpha, W_\alpha, Aut(\mathfrak{l}))$ defines a principal bundle.*

We now make the following remark. For any α and β, the group $\mathrm{Aut}(\Gamma)$ acts on $\mathrm{Aut}(\mathfrak{l}) \times W_{\alpha\beta}$ by

$$a \cdot (A, x, W) = (A \circ d(i(a))^{-1}_{|e}, x, W) \tag{14}$$

and the map $\xi_{\alpha\beta}$ is $\mathrm{Aut}(\Gamma)$-equivariant. If the set $\mathscr{W}_{\alpha\beta}$ is G-stable, then we define a map $c_{W,\alpha}$ from G in $GL_k(\mathbb{R})$ given by

$$\mathrm{Ad}_g\, \eta_\alpha^{-1}(W) = \eta_\alpha^{-1}(g \cdot W) c_{W,\alpha}(g). \tag{15}$$

Using the identity $\mathrm{Ad}_{g'g}\, \eta_\alpha^{-1}(W) = \mathrm{Ad}_{g'}\, \mathrm{Ad}_g\, \eta_\alpha^{-1}(W)$ and the definition of $c_{W,\alpha}$, we get

$$c_{W,\alpha}(g'g) = c_{g\cdot W,\alpha}(g') c_{W,\alpha}(g). \tag{16}$$

As a direct consequence, we obtain:

Proposition 4.8. *For any $\alpha \in I(n,k)$ and any $\beta \in I$, if $\mathscr{W}_{\alpha\beta}$ is stable by the $G-$action, then G acts on $Aut(\mathfrak{l}) \times W_{\alpha\beta}$ through the following law:*

$$g \cdot (A, x, W) := \Big(s_\beta\big(p(c_{W,\alpha}(g) s_\beta(x))\big)^{-1} c_{W,\alpha}(g) s_\beta(x) A, p\big(c_{W,\alpha}(g) s_\beta(x)\big), g \cdot W \Big)$$

for any $(A, x, W) \in Aut(\mathfrak{l}) \times W_{\alpha\beta}$ and any $g \in G$. Moreover, this action commutes with the action of $Aut(\Gamma)$ and the trivialization map $\xi_{\alpha\beta}$ is G-equivariant.

Proof. Let $M = \eta_\alpha^{-1}(W) s_\beta(x) A \in \mathscr{W}_{\alpha\beta}$. As $\mathscr{W}_{\alpha\beta}$ is G-stable, $\mathrm{Ad}_g M \in \mathscr{W}_{\alpha\beta}$ and therefore $\xi'_{\alpha\beta}(\mathrm{Ad}_g M) \in \mathrm{Aut}(\mathfrak{l}) \times W_{\alpha\beta}$ for any $g \in G$. It is then easy to see that $\xi'_{\alpha\beta}(\mathrm{Ad}_g M) = g \cdot (A, x, W)$, where $\xi'_{\alpha\beta}$ is given as in (10), which proves that the action is well defined. Using (16), we directly get:

$$\begin{aligned} g' \cdot (g \cdot (A, x, W)) &= \Big(s_\beta\Big(p\big(c_{g\cdot W,\alpha}(g') s_\beta(p(c_{W,\alpha}(g) s_\beta(x)))\big)\Big)^{-1} c_{g\cdot W,\alpha}(g') s_\beta(p(c_{W,\alpha}(g) s_\beta(x))) \\ &\times\ s_\beta\big(p(c_{W,\alpha}(g) s_\beta(x))\big)^{-1} c_{W,\alpha}(g) s_\beta(x) A, \\ &\quad p\Big(c_{g\cdot W,\alpha}(g') s_\beta\big(p(c_{W,\alpha}(g) s_\beta(x))\big)\Big), g'g \cdot W \Big), \end{aligned}$$

for any $g, g' \in G$. Besides, $p(c_{g\cdot W,\alpha}(g') s_\beta(p(c_{W,\alpha}(g) s_\beta(x)))) = p(c_{g\cdot W,\alpha}(g') c_{W,\alpha}(g) s_\beta(x))$. This conclusively gives

$$\begin{aligned} g' \cdot (g \cdot (A, x, W)) &= (s_\beta(p(c_{g\cdot W,\alpha}(g') c_{W,\alpha}(g) s_\beta(x)))^{-1} c_{g\cdot W,\alpha}(g') c_{W,\alpha}(g) s_\beta(x) A, \\ &\quad p(c_{g\cdot W,\alpha}(g') c_{W,\alpha}(g) s_\beta(x)), g'g \cdot W) \\ &= (s_\beta(p(c_{W,\alpha}(g'g) s_\beta(x))^{-1} c_{W,\alpha}(g'g) s_\beta(x) A, p(c_{W,\alpha}(g'g) s_\beta(x)), g'g \cdot W) \\ &= g'g \cdot (A, x, W). \end{aligned}$$

The commutation properties is a consequence of (14). For the $G-$equivariance equation, let $g \in G$ and $(A, x, W) \in \mathrm{Aut}(\mathfrak{l}) \times W_{\alpha\beta}$. We have:

$$g \cdot \xi_{\alpha\beta}(A, x, W) = \mathrm{Ad}_g \eta_\alpha^{-1}(W) s_\beta(x) A = \eta_\alpha^{-1}(g \cdot W) c_{W,\alpha}(g) s_\beta(x) A = \xi_{\alpha\beta}(g \cdot (A, x, W)).$$

□

5. The deformation space of certain normal discontinuous subgroups

Throughout this section, H denotes a connected Lie subgroup of a completely solvable Lie group G and Γ a discontinuous subgroup for the homogeneous space G/H whose syndetic hull L fulfils $[L, L] = [G, G]$ (or equivalently $[\Gamma, \Gamma]$ is uniform in $[G, G]$). The aim here is to give a comprehensive description of the associated deformation space. It is worth mentioning that the techniques used here also allow us to treat the context where $[G, G] \subset H$ as it was the case in [1]. In such a situation and unlike the present setup, the discontinuous subgroup is easily shown to be abelian.

5.1. **The parameter space.** This subsection is devoted to produce a description of the subsequent parameter space $R(\Gamma, G, H)$. This is important to a major extent to see that such a space smoothly splits into some $G-$invariant constituents. We first see the following:

Proposition 5.1. *Retain the same hypotheses and notation and assume that H is normal or $[L, L] = [G, G]$. Then*

$$R(\Gamma, G, H) = \left\{ \varphi \in Hom(\mathfrak{l}, \mathfrak{g}) \;\middle|\; \begin{array}{l} \dim \varphi(\mathfrak{l}) = \dim \mathfrak{l} \\ \varphi(\mathfrak{l}) \cap \mathfrak{h} = \{0\} \end{array} \right\}.$$

In particular $R(\Gamma, G, H)$ is open in $Hom(\mathfrak{l}, \mathfrak{g})$ and semi-algebraic.

Proof. Our assumption says that $[\mathfrak{l}, \mathfrak{l}] = [\mathfrak{g}, \mathfrak{g}]$. Let $\varphi \in R(\mathfrak{l}, \mathfrak{g}, \mathfrak{h})$. Clearly $\varphi([\mathfrak{g}, \mathfrak{g}]) = [\varphi(\mathfrak{l}), \varphi(\mathfrak{l})] \subset [\mathfrak{g}, \mathfrak{g}]$. From the injectivity of φ, we deduce that $[\varphi(\mathfrak{l}), \varphi(\mathfrak{l})] = [\mathfrak{g}, \mathfrak{g}]$. In particular $\varphi(\mathfrak{l}) \supset [\mathfrak{g}, \mathfrak{g}]$, which means that $\varphi(\mathfrak{l})$ is an ideal of $\mathfrak{g}$. Using [4], the proper action of $\exp \varphi(\mathfrak{l})$ on G/H is equivalent to $\varphi(\mathfrak{l}) \cap \mathfrak{h} = \{0\}$, and the result follows. □

With the notation of subsection 4.3 in mind, we now consider the decomposition,

$$\{\varphi \in \mathrm{Hom}(\mathfrak{l}, \mathfrak{g}),\, \dim \varphi(\mathfrak{l}) = \dim \mathfrak{l}\} = \mathrm{Hom}(\mathfrak{l}, \mathfrak{g}) \cap M^{\circ}_{n,k}(\mathbb{R}) = \bigcup_{\alpha \in I(n,k)} \mathscr{V}_{\alpha}, \tag{17}$$

which is open in $\mathrm{Hom}(\mathfrak{l}, \mathfrak{g})$ and closed in $\mathscr{U}_{\alpha}$. The following Lemma is proved in ([1]).

Lemma 5.2. *For all $\varphi \in \mathscr{L}(\mathfrak{l}, \mathfrak{g})$ such that $\dim \varphi(\mathfrak{l}) = \dim \mathfrak{l}$ we have, $\varphi(\mathfrak{l}) \cap \mathfrak{h} = \{0\}$ if and only if there exists α in $I_s(n, k)$ such that $\varphi \in \mathscr{U}_{\alpha}$, where*

$$I_s(n, k) = \{(i_1, \ldots, i_k),\ s < i_1 < \cdots < i_k \leq n\}.$$

Now, from last lemma, (17) and the decomposition (13) of $\mathscr{V}_{\alpha}$, we get the following:

Corollary 5.3. *Suppose that H is normal or $[L, L] = [G, G]$. Then up to homeomorphism we have,*

$$R(\Gamma, G, H) = \bigcup_{\substack{\alpha \in I_s(n,k), \\ \beta \in I}} \mathscr{W}_{\alpha\beta}.$$

as a union of open sets in $Hom(\mathfrak{l}, \mathfrak{g})$.

5.2. **The deformation space.** Our main upshot in this section is the following:

Theorem 5.4. *Let G be a completely solvable Lie group, H a connected subgroup of G and Γ a discontinuous subgroup for G/H such that $[L,L]=[G,G]$. Then for any $\alpha\in I_s(n,k)$ and any $\beta\in I$, the set $\mathscr{W}_{\alpha\beta}$ is G-stable and the deformation and moduli spaces read*

$$\mathscr{T}(\Gamma,G,H)=\bigcup_{\substack{\alpha\in I_s(n,k),\\ \beta\in I}}\mathscr{W}_{\alpha\beta}/G,\quad \mathscr{M}(\Gamma,G,H)=\bigcup_{\substack{\alpha\in I_s(n,k),\\ \beta\in I}}Aut(\Gamma)\backslash\mathscr{W}_{\alpha\beta}/G,$$

as a union of open sets. Furthermore, G acts on $Aut(\mathfrak{l})\times W_{\alpha\beta}$, the set $\mathscr{W}_{\alpha\beta}/G$ is homeomorphic to $\big(Aut(\mathfrak{l})\times W_{\alpha\beta}\big)/G$ and the set $Aut(\Gamma)\backslash\mathscr{W}_{\alpha\beta}/G$ is homeomorphic to $Aut(\Gamma)\backslash\big(Aut(\mathfrak{l})\times W_{\alpha\beta}\big)/G$.

Proof. Note first from Corollary (5.3) that

$$R(\Gamma,G,H)=\bigcup_{\substack{\alpha\in I_s(n,k),\\ \beta\in I}}\mathscr{W}_{\alpha\beta}.$$

From corollary (4.6) and proposition (4.8) we are done, provided that for any $\alpha\in I_s(n,k)$ and $\beta\in I$, the set $\mathscr{W}_{\alpha\beta}$ is G-stable, which remains to be proved. Indeed, let $M\in\mathscr{W}_{\alpha\beta}$. For $W=\eta(M)$ and $x=p(M_\alpha)$, there is $A\in\mathrm{Aut}(\mathfrak{l})$ such that $M=\eta_\alpha^{-1}(W)s_\beta(x)A$. When $[\mathfrak{l},\mathfrak{l}]=[\mathfrak{g},\mathfrak{g}]$, for any $M\in R(\mathfrak{l},\mathfrak{g},\mathfrak{h})$, the subspace $M(\mathfrak{l})$ is an ideal of $\mathfrak{g}$. It follows therefore that $g\cdot W=W$ for any $g\in G$ and from (15) we obtain,

$$\mathrm{Ad}_g M=\eta_\alpha^{-1}(W)c_{W,\alpha}(g)s_\beta(x)A=Mc_{A,x,W,\alpha}(g), \tag{18}$$

where $c_{A,x,W,\alpha}(g)=A^{-1}s_\beta(x)^{-1}c_{W,\alpha}(g)s_\beta(x)A$. The set $\mathscr{W}_{\alpha\beta}$ is stable by the action of $\mathrm{Aut}(\mathfrak{l})$, then to prove that $\mathrm{Ad}_g M\in\mathscr{W}_{\alpha\beta}$, it is sufficient to show the following lemma:

Lemma 5.5. *The map $c_{A,x,W,\alpha}$ is a continuous homomorphism from G to $Aut(\mathfrak{l})$.*

Proof. The map $c_{A,x,W,\alpha}:G\to GL_k(\mathbb{R})$ is nothing but the composition of $c_{W,\alpha}$ and the conjugation by the element $A^{-1}s_\beta(x)^{-1}$. From the fact that $g\cdot W=W$ for any $g\in G$ and equation (16), we deduce that the map $c_{W,\alpha}$ is a homomorphism and so is $c_{A,x,W,\alpha}$. The continuity is also a direct consequence from the continuity of the action of G on $M_{n,k}(\mathbb{R})$ and the expression

$$c_{W,\alpha}(g)=\Big(g\cdot\eta_\alpha^{-1}(W)\Big)_\alpha,\ g\in G. \tag{19}$$

To conclude, we have to prove that $c_{A,x,W}(g)\in\mathrm{Aut}(\mathfrak{l})$. Let $X,Y\in\mathfrak{l}$, then

$$\begin{aligned} M\big([c_{A,x,W,\alpha}(g)(X),c_{A,x,W,\alpha}(g)(Y)]\big) &= [Mc_{A,x,W,\alpha}(g)(X),Mc_{A,x,W,\alpha}(g)(Y)]\\ &= \mathrm{Ad}_g M[X,Y]\\ &= Mc_{A,x,W,\alpha}(g)[X,Y]. \end{aligned}$$

This entails that $M\big([c_{A,x,W,\alpha}(g)(X),c_{A,x,W,\alpha}(g)(Y)]-c_{A,x,W,\alpha}(g)[X,Y]\big)=0$. As M is of maximal rank, we are done. □

□

We close this section by the following remark which will be of interest later on. Retain the same hypotheses and notation, specially G is completely solvable and that $[\mathfrak{l},\mathfrak{l}] = [\mathfrak{g},\mathfrak{g}]$. Then as in the previous Theorem, $\xi_{\alpha\beta}$ is a G-equivariant homeomorphism from $\mathscr{W}_{\alpha\beta}$ and $\mathrm{Aut}(\mathfrak{l}) \times W_{\alpha\beta}$ for any $\alpha \in I_s(n,k)$ and any $\beta \in I$. Under these circumstances, the action of G is defined as in Proposition (4.8) by

$$g \cdot (A, x, W) = (c_{x,W,\alpha}(g)A, x, W),\ (A, x, W) \in \mathrm{Aut}(\mathfrak{l}) \times W_{\alpha\beta},\ g \in G \tag{20}$$

where $c_{x,W,\alpha}(g) = s_\beta(x)^{-1}c_{W,\alpha}(g)s_\beta(x)$. Then $\mathrm{Aut}(\mathfrak{l}) \times W_{\alpha\beta}$ splits into a union of fibers of the natural projection on $W_{\alpha\beta}$ and each fiber is G-stable, or equivalently, the orbits of $\mathrm{Aut}(\mathfrak{l})$ in $\mathscr{W}_{\alpha\beta}$ are G-stable. From lemma (5.5), for $(x, W) \in W_{\alpha\beta}$ we can see that the map $c_{x,W,\alpha} : G \to \mathrm{Aut}(\mathfrak{l})$ is a group homomorphism. This entails in fact that $c_{x,W,\alpha}(G)$ is a Lie subgroup of $\mathrm{Aut}(\mathfrak{l})$. To get that, it is sufficient to see the following:

Lemma 5.6. *Let $c : G \to G'$ be a continuous Lie groups homomorphism. Suppose that G is connected, then the image of G is connected Lie subgroup of G'.*

Proof. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{k} = dc_{|e}(\mathfrak{g})$, then $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}'$, the Lie algebra of G'. Now G is connected, then it is generated by $\exp(\mathfrak{g})$ and

$$c \circ \exp_G(X) = \exp_{G'} \circ dc_{|e}(X),$$

for any $X \in \mathfrak{g}$. This conclusively shows that $c(G)$ is the connected Lie subgroup generated by $\exp_{G'}(\mathfrak{k})$. □

From (20) we can see that the natural projection of $\mathrm{Aut}(\mathfrak{l}) \times W_{\alpha\beta}$ on $W_{\alpha\beta}$ factors through the action of G to a continuous surjection from $\mathscr{W}_{\alpha\beta}/G$ to $W_{\alpha\beta}$ such that the fiber of (x, W) is homeomorphic to the homogeneous space $\mathrm{Aut}(\mathfrak{l})/c_{x,W,\alpha}(G)$.

6. Local geometrical proprieties of deformations

6.1. **The terminology.** We here consider the general setting and keep the same notation, G, H and Γ and so on. For any $\varphi \in R(\Gamma, G, H)$, the discontinuous subgroup $\varphi(\Gamma)$ for the homogeneous space G/H is said to be *locally rigid* (respectively *rigid*) as a discontinuous group of G/H if the orbit of φ through the inner conjugation is open in the set $R(\Gamma, G, H)$ (respectively in $\mathrm{Hom}(\Gamma, G)$). So, homomorphisms which are locally rigid are those which correspond to those which are open points in the deformation space $\mathscr{T}(\Gamma, G, H)$. When every point in $R(\Gamma, G, H)$ is locally rigid, the deformation space turns out to be discrete and then we say that the Clifford-Klein form $\Gamma\backslash G/H$ can not deform continuously through the deformation of Γ in G. If a given $\varphi \in R(\Gamma, G, H)$ is not locally rigid, we say that it admits a *continuous deformation* and that the related Clifford-Klein form is continuously deformable. The homomorphism φ is said to be *topologically stable* or merely *stable* in the sense of Kobayashi-Nasrin [14], if there is an open set in $\mathrm{Hom}(\Gamma, G)$ which contains φ and is contained in $R(\Gamma, G, H)$. When the set $R(\Gamma, G, H)$ is an open subset of $\mathrm{Hom}(\Gamma, G)$, then obviously each of its elements is stable.

6.2. **On rigidity.** Back now to our settings and notation, in particular G is completely solvable. Consider first the natural continuous action of $\mathrm{Aut}(\mathfrak{l})$ on $\mathrm{Hom}(\mathfrak{l},\mathfrak{g})$ which respects $R(\Gamma,G,H)$. Our upshot in this section is the following:

Theorem 6.1. *Let G be a completely solvable Lie group, H a connected subgroup of G and Γ a discontinuous subgroup for G/H such that $[L,L]=[G,G]$. Then $R(\Gamma,G,H)$ is an open set in Hom(Γ,G) and semi-algebraic. Moreover for $\varphi\in R(\Gamma,G,H)$ the following assertions are equivalent:*
ı) φ is rigid.
ıı) φ is locally rigid.
ııı) The orbit $\varphi Aut(\mathfrak{l})$ is open in Hom$(\mathfrak{l},\mathfrak{g})$ and

$$\dim Aut(\mathfrak{l})+\dim\varphi(\mathfrak{l})^{\perp}=\dim\mathfrak{g}, \tag{21}$$

where $\varphi(\mathfrak{l})^{\perp}=\{Y\in\mathfrak{g},\ [X,Y]=0\ for\ all\ X\in\varphi(\mathfrak{l})\}$.

Proof. Recall that $R(\Gamma,G,H)$ is open, by proposition (5.1), then clearly ($\imath$) and ($\imath\imath$) are equivalent. Let $\varphi\in R(\Gamma,G,H)$, M be the corresponding matrix to φ and $\xi'_{\alpha\beta}(M)=(A,x,W)$. Suppose that $G\cdot M$ is open. We have

$$G\cdot M=c_{A,x,W,\alpha}(G)\cdot M \text{ and } \mathrm{Aut}(\mathfrak{l})\cdot M=\bigcup_{a\in\mathrm{Aut}(\mathfrak{l})} ac_{A,x,W,\alpha}(G)\cdot M, \tag{22}$$

then $\mathrm{Aut}(\mathfrak{l})M$ is a union of open subsets. The $\mathrm{Aut}(\mathfrak{l})$-orbit of M is identified via $\xi_{\alpha\beta}$ with the set $\mathrm{Aut}(\mathfrak{l})\times\{(x,W)\}$. The restriction of the canonical surjection from $R(\Gamma,G,H)\to\mathscr{T}(\Gamma,G,H)$ to the $\mathrm{Aut}(\mathfrak{l})$-orbit of M, is a continuous map and from (20) its image is homeomorphic to the homogeneous space $\mathrm{Aut}(\mathfrak{l})/c_{x,W,\alpha}(G)$. Now the rigidity of φ implies that the image of φ in the homogeneous Hausdorff space $\mathrm{Aut}(\mathfrak{l})/c_{x,W,\alpha}(G)$ is open and closed. This means that this point is a connected component of $\mathrm{Aut}(\mathfrak{l})/c_{x,W,\alpha}(G)$, in particular $\dim(\mathrm{Aut}(\mathfrak{l})/c_{x,W,\alpha}(G))=0$. We now prove the following:

Lemma 6.2. $\dim c_{x,W,\alpha}(G)=\dim(\mathfrak{g})-\dim W^{\perp}$.

Proof. By the definition of $c_{W,\alpha}$ (15), we have

$$\begin{aligned}\ker(c_{W,\alpha}) &= \{g\in G,\ \mathrm{Ad}_g\,\eta_\alpha^{-1}(W)=\eta_\alpha^{-1}(W)\}\\ &= \{\exp(X),\ X\in\mathfrak{g} \text{ and } \mathrm{Ad}_{\exp(X)}(Y)=Y,\ \text{ for any } Y\in W\}\\ &= \{\exp(X),\ [X,Y]=0,\ \text{ for any } Y\in W\}=\exp(W^{\perp}).\end{aligned}$$

□

Using Lemma (6.2), we get $\dim\mathrm{Aut}(\mathfrak{l})+\dim\varphi(\mathfrak{l})^{\perp}=\dim\mathfrak{g}$. Suppose now that ($\imath\imath\imath$) holds. By the connectedness of G, we see that $c_{x,W,\alpha}(G)$ is the connected component of the identity of $\mathrm{Aut}(\mathfrak{l})$ and $c_{x,W,\alpha}(G)=c_{A,x,W,\alpha}(G)$. To prove that $G\cdot M$ is open, it is sufficient using (22) to see that $c_{A,x,W,\alpha}(G)\cdot M$ is open. Let $y\in c_{x,W,\alpha}(G)\cdot M$ and pick an open neighborhood V_0 of the identity in $\mathrm{Aut}(\mathfrak{l})$ included in $c_{x,W,\alpha}(G)$. Il follows therefore that $V_0\cdot y\subset c_{x,W,\alpha}(G)\cdot M$ and it is enough to show that $V_0\cdot y$ is open in $\mathrm{Hom}(\mathfrak{l},\mathfrak{g})$. Recall that the map $\mathrm{Aut}(\mathfrak{l})\to\mathrm{Aut}(\mathfrak{l})\cdot M$, $a\mapsto Ma^{-1}$ is a homeomorphism. Then $V_0\cdot y$ is open in $\mathrm{Aut}(\mathfrak{l})\cdot M$, and thus in $\mathrm{Hom}(\mathfrak{l},\mathfrak{g})$. □

The following important consequence is therefore immediate:

Corollary 6.3. *We keep the same hypotheses and notation. If the group* $Aut(\mathfrak{l})$ *is not solvable, then the local rigidity fails to hold on the parameter space.*

7. Case of Heisenberg groups

7.1. **The main result.** Throughout the present section, $\mathfrak{g} := \mathfrak{h}_{2n+1}$ designates the Heisenberg Lie algebra of dimension $2n+1$ and $G := H_{2n+1}$ the corresponding Lie group. $\mathfrak{g}$ can be defined as a real vector space endowed with a skew-symmetric bilinear form b of rank $2n$ and a fixed generator Z belonging to the kernel of b. The center $\mathfrak{z}$ of $\mathfrak{g}$ is then the kernel of b and it is the one dimensional subspace $[\mathfrak{g},\mathfrak{g}]$. For any $X, Y \in \mathfrak{g}$, the Lie bracket is given by

$$[X,Y] = b(X,Y)Z.$$

The following result is proved in ([3], Proposition 3.1) and provides a systematic way to construct a symplectic basis of $\mathfrak{g}$ starting from a given subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ and referred to be adapted to $\mathfrak{l}$.

Proposition 7.1. *Let* $\mathfrak{l}$ *be a Lie subalgebra of* $\mathfrak{g}$. *Then there exists a basis* $\mathscr{B}_\mathfrak{l} = \{Z, X_1 \ldots, X_n, Y_1, \ldots, Y_n\}$ *of* $\mathfrak{g}$ *with the Lie commutation relations*

$$[X_i, Y_j] = \delta_{i,j}Z,\ \ i,j = 1,...,n$$

and satisfying:
1) *If* $\mathfrak{z} \subset \mathfrak{l}$, *then there exist two integers* p, $q \geq 0$ *such that the family*

$$\{Z, X_1, \ldots, X_{p+q}, Y_1, \ldots, Y_p\}$$

constitutes a basis of $\mathfrak{l}$.
2) *If* $\mathfrak{z} \not\subset \mathfrak{l}$, *then* $\dim \mathfrak{l} \leq n$ *and* $\mathfrak{l}$ *is generated by* $X_1, \ldots, X_s$, *where* $s = \dim \mathfrak{l}$. *The symbol* $\delta_{i,j}$ *here designates the Kronecker symbol. The basis* $\mathscr{B}_\mathfrak{l}$ *is said to be a symplectic basis of* $\mathfrak{g}$ *adapted to* $\mathfrak{l}$.

Let $\mathfrak{l}$ be a non-abelian subalgebra and $\mathscr{B}_\mathfrak{l}$ a symplectic basis of $\mathfrak{g}$ adapted to $\mathfrak{l}$. We single out from $\mathscr{B}_\mathfrak{l}$ a new basis $\mathscr{B}'_\mathfrak{l}$ of $\mathfrak{l}$ as follows

$$\mathscr{B}'_\mathfrak{l} := \{Z, X_{p+1}, \ldots, X_{p+q}, X_1, \ldots, X_p, Y_1, \ldots, Y_p\}. \tag{23}$$

We also fix a basis of $\mathfrak{g}$ passing through $\mathfrak{h}$. Let us denote by $I_s^1(2n+1,k)$ the set of elements of the form $\alpha = (s+1, i_2, \ldots, i_k)$. The main result of this section consists in giving a comprehensive description of the deformation space $\mathscr{T}(\Gamma, G, H)$ of the action of a discontinuous non-abelian subgroup for a homogeneous space G/H, where H stands for an arbitrary connected closed subgroup of G. Pick first a finite open covering $\{V_\beta\}_{\beta\in I}$ of $GL_k(\mathbb{R})/\mathrm{Aut}(\mathfrak{l})$ and its local sections $s_\beta : V_\beta \to GL_k(\mathbb{R}), \beta \in I$. For $\alpha \in I_s^1(2n+1,k)$ and $\beta \in I$, we consider the set

$$\mathscr{A}_{\alpha,\beta} := \{(A, M) \in s_\beta(V_\beta) \times U_\alpha :\ {}^tA{}^tMJ_\mathfrak{g}MA = J_\mathfrak{l}\},$$

where $J_\mathfrak{g}$ and $J_\mathfrak{l}$ designates the matrices of b and of $b_{|\mathfrak{l}}$ respectively, written in the basis $\mathscr{B}_\mathfrak{l}$. Our main result in this section is the following:

Theorem 7.2. *Let G be the $2n+1-$dimensional Heisenberg group, H a connected Lie subgroup of G and Γ a non-abelian discontinuous subgroup of G for G/H. Let $L=\exp(\mathfrak{l})$ be the syndetic hull of Γ. There exists a finite set of local sections $(s_\beta)_{\beta\in I}$ for the canonical surjection $GL_k(\mathbb{R})\to GL_k(\mathbb{R})/Aut(\mathfrak{l})$ such that, the deformation space of Γ acting on G/H reads*

$$\mathscr{T}(\Gamma,G,H)=\bigcup_{\substack{\beta\in I\\ \alpha\in I_s^1(2n+1,k)}}\mathscr{T}_{\alpha\beta},$$

where for $\beta\in I$ and $\alpha\in I_s^1(2n+1,k)$, the set $\mathscr{T}_{\alpha\beta}$ is open in $\mathscr{T}(\Gamma,G,H)$ and homeomorphic to the set $\mathbb{R}^*\times\mathbb{R}^{2pq}\times Sp(2p)\times GL_q(\mathbb{R})\times\mathscr{A}_{\alpha,\beta}$. Here $1+2p+q=k$ and $q+1=\dim\mathfrak{z}(\mathfrak{l})$, where $\mathfrak{z}(\mathfrak{l})$ is the center of $\mathfrak{l}$.

7.2. **Proof of the main result.** We now proceed to the proof of Theorem (7.2). We first start by proving a series of useful results on which the proof of the main upshot rests. Let f be an automorphism of $\mathfrak{l}$, then clearly $f(Z)\in\mathfrak{z}$ and $f(\mathfrak{z}(\mathfrak{l}))=\mathfrak{z}(\mathfrak{l})$. Then with respect to the basis $\mathscr{B}'_{\mathfrak{l}}$, the matrix M of f is of the form,

$$M=\begin{pmatrix} a & c & d\\ 0 & G & F\\ 0 & 0 & E\end{pmatrix},\quad \begin{array}{c}\text{with } a\in\mathbb{R}^*,\ c\in M_{1,q}(\mathbb{R}),\ d\in M_{1,2p}(\mathbb{R}),\\ G\in GL_q(\mathbb{R}),\ F\in M_{q,2p}(\mathbb{R}),\ E\in GL_{2p}(\mathbb{R}).\end{array}\tag{24}$$

Let us denote by a_M, the coefficient a of the matrix M. As f is a homomorphism, it comes out that

$${}^tMJ_{\mathfrak{l}}M=a_MJ_{\mathfrak{l}}\text{ where }J_{\mathfrak{l}}=\begin{pmatrix}0_{\mathbb{R}} & 0 & 0\\ 0 & 0_{\mathbb{R}^q} & 0\\ 0 & 0 & J_{2p}\end{pmatrix}\in M_k(\mathbb{R}),\text{ with }J_{2p}=\begin{pmatrix}0 & I_p\\ -I_p & 0\end{pmatrix}.$$

By a direct calculation, this condition is equivalent to ${}^tEJ_{2p}E=a_MJ_{2p}$.

Lemma 7.3. *Let $\mathfrak{l}$ be a non-abelian Lie subalgebra of $\mathfrak{g}$,*

$$K=\begin{pmatrix}1 & M_{1,q}(\mathbb{R}) & M_{1,2p}(\mathbb{R})\\ 0 & GL_q(\mathbb{R}) & M_{q,2p}(\mathbb{R})\\ 0 & 0 & Sp(2p)\end{pmatrix}\ \textit{and}\ Q=\left\{\begin{pmatrix}a & 0 & 0\\ 0 & I_{q+p} & 0\\ 0 & 0 & aI_p\end{pmatrix},\ a\in\mathbb{R}^*\right\}.$$

Then $Aut(\mathfrak{l})=KQ$. Here p and q are given as in Proposition (7.1).

Proof. Clearly any matrix in K or Q is an automorphism of $\mathfrak{l}$. Conversely, let f be an element of $\mathrm{Aut}(\mathfrak{l})$ and M its matrix written as in (24). Let us write

$$E=\begin{pmatrix}D_1 & D_2\\ D_3 & D_4\end{pmatrix}\text{ and }E_M=\begin{pmatrix}D_1 & a^{-1}D_2\\ D_3 & a^{-1}D_4\end{pmatrix},$$

then ${}^tEJ_{2p}E=aJ_{2p}$ if and only if $E_M\in Sp(2p)$ and

$$M=\begin{pmatrix}1 & c & d'\\ 0 & G & F'\\ 0 & 0 & E_M\end{pmatrix}\begin{pmatrix}a & 0 & 0\\ 0 & I_{p+q} & 0\\ 0 & 0 & aI_p\end{pmatrix},$$

where for $d=(d_1,\dots,d_{2p})$, $d'=(d_1,\dots,d_p,a^{-1}d_{p+1},\dots,a^{-1}d_{2p})$ and for $F=(F_1,F_2)$ with F_1,F_2 in $M_{q,p}(\mathbb{R})$, $F'=(F_1,a^{-1}F_2)$. □

The condition $[\mathfrak{l},\mathfrak{l}]=[\mathfrak{g},\mathfrak{g}]$ implies that any homomorphism from $\mathfrak{l}$ to $\mathfrak{g}$ stabilizes the center of $\mathfrak{g}$. The following lemma then is immediate,

Lemma 7.4. *If $\mathfrak{l}$ is a non-abelian Lie subalgebra of $\mathfrak{g}$. Then $\mathfrak{l}$ contains the center $\mathfrak{z}$, and any homomorphism from $\mathfrak{l}$ to $\mathfrak{g}$ stabilizes $\mathfrak{z}$.*

We consider a basis $\mathscr{B}_{\mathfrak{h}}=\{Z,X'_1\dots,X'_n,Y'_1,\dots,Y'_n\}$ of $\mathfrak{g}$ adapted to $\mathfrak{h}$ as in Proposition (7.1) with the following ordering

$$X'_1\dots,X'_s,Z,Y'_1,\dots,Y'_s,X'_{s+1},\dots,X'_n,Y'_{s+1},\dots,Y'_n.$$

and for $\mathfrak{l}$ we keep fixing the basis (23). As a direct consequence, the set $\mathrm{Hom}(\mathfrak{l},\mathfrak{g})$ is identified with a subset of matrices in $M_{2n+1,k}(\mathbb{R})$ written through the bases $\mathscr{B}'_{\mathfrak{l}}$ and $\mathscr{B}_{\mathfrak{h}}$ as

$$M=\begin{pmatrix}0 & M_1\\ a & d\\ 0 & M_2\end{pmatrix}, \tag{25}$$

where $a\in\mathbb{R}$, $d\in\mathbb{R}^{k-1}$, $M_1\in M_{s,k-1}(\mathbb{R})$, $M_2\in M_{2n-s,k-1}(\mathbb{R})$. As a direct consequence of (25), we have the following:

Lemma 7.5. *For any $\alpha\in I_s(2n+1,k)$, the set $\mathscr{V}_\alpha$ is empty whenever $\alpha\notin I_s^1(2n+1,k)$.*

Proof. From (25), for any M in $\mathrm{Hom}(\mathfrak{l},\mathfrak{g})$ and $\alpha\notin I_s^1(2n+1,k)$, we get $\det(M_\alpha)=0$. Then our result is a direct consequence of (11). □

Lemma 7.6. *For $W=\eta_\alpha(M)$ and $\alpha\in I_s^1(2n+1,k)$, the image of the group homomorphism $c_{W,\alpha}$ is independent from the choice of W and α. More precisely, $c_{W,\alpha}(G)$ equals the matrix group*

$$c(G):=\begin{pmatrix}1 & M_{1,k-1}(\mathbb{R})\\ 0 & I_{k-1}\end{pmatrix}.$$

Proof. As a direct consequence of (19), we get for $X\in\mathfrak{g}$:

$$c_{W,\alpha}(e^X)=\begin{pmatrix}1 & b(X,c_2)\ \cdots\ b(X,c_k)\\ \cdot & \\ \cdot & \\ \cdot & \\ 0 & (I_{2p+q})\end{pmatrix}.$$

where the c_i are the column of $\eta_\alpha^{-1}(W)$. Furthermore, the column vectors $c_2,\dots c_k$ are linearly independent and the center of $\mathfrak{g}$ is not contained in the space generated by this vectors. This means that the linear forms on $\mathfrak{g}$, $b(\cdot,c_2),\dots,b(\cdot,c_k)$ are independent, then the system $b(X,c_2)=x_1,\dots,b(X,c_k)=x_{k-1}$ has a solution for any $(x_1,...,x_{k-1})\in\mathbb{R}^{k-1}$ and the result follows. □

The following result is immediate.

Lemma 7.7. *The normalizer of $c(G)$ in $GL_k(\mathbb{R})$ is the parabolic subgroup P of matrices of the form* $\begin{pmatrix} a & b \\ 0 & B \end{pmatrix}$, *where $a \in \mathbb{R}^*$, $b \in \mathbb{R}^{k-1}$ and $B \in GL_{k-1}(\mathbb{R})$.*

Lemma 7.8. *Let $\alpha \in I_s(2n+1,k)$ and $\beta \in I$. For any $(x,W) \in V_\beta \times \eta(U_\alpha)$ we have, $\eta_\alpha^{-1}(W)s_\beta(x) \in \mathscr{V}_\alpha$ only if $s_\beta(x) \in P$. In other words, $c_{x,W,\alpha}(G) = c(G)$. In particular $c(G)$ is normal in Aut$(\mathfrak{l})$.*

Proof. By Lemma (7.7), the parabolic subgroup P coincides with the normalizer $N(c(G))$ and it is clear that $\mathrm{Aut}(\mathfrak{l})$ is a subgroup of P. Now $s_\beta(x) = \big(\eta_\alpha^{-1}(W)s_\beta(x)\big)_\alpha$ and $\eta_\alpha^{-1}(W)s_\beta(x) \in \mathscr{V}_\alpha$ only if $\eta_\alpha^{-1}(W)s_\beta(x)$ is written as in (25). Then $s_\beta(x) \in P$. □

Lemma 7.9. *The topological group Aut$(\mathfrak{l})/c(G)$ is isomorphic to $K'Q$, where*

$$K' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & GL_q(\mathbb{R}) & M_{q,2p}(\mathbb{R}) \\ 0 & 0 & Sp(2p) \end{pmatrix}.$$

In particular, Aut$(\mathfrak{l})/c(G)$ is homeomorphic to $\mathbb{R}^* \times \mathbb{R}^{2pq} \times Sp(2p) \times GL_q(\mathbb{R})$.

Proof. Consider the projection $p : \mathrm{Aut}(\mathfrak{l}) \to K'Q$, where $p(A)$ is the matrix obtained from A by vanishing the $k-1$ last coefficients of the first line of A. Then p is a surjective homomorphism and its kernel is $c(G)$. Now K' is normal in $K'Q$ and $K' \cap Q$ is trivial, then $K'Q$ is isomorphic to the semi-direct product $K \times Q$. The question of the related topology is straight clear. □

Back now to Theorem (7.2). Take a family of continuous local sections $(s_\beta)_{\beta \in I}$ of the canonical surjection $GL_k(\mathbb{R}) \to GL_k(\mathbb{R})/\mathrm{Aut}(\mathfrak{l})$. From Theorem (5.4), the set $\mathscr{T}_{\alpha\beta}$ is homeomorphic to the quotient space $\big(\mathrm{Aut}(\mathfrak{l}) \times W_{\alpha\beta}\big)/G$. By Lemmas (7.6) and (7.8), we have $c_{x,W,\alpha}(G) = c(G)$ and then the map

$$\begin{array}{rcl} i \, : \mathrm{Aut}(\mathfrak{l})/c(G) \times W_{\alpha\beta} & \to & \big(\mathrm{Aut}(\mathfrak{l}) \times W_{\alpha\beta}\big)/G \\ (\overline{A}, x, W) & \mapsto & \overline{(A,x,W)} \end{array}$$

is a well defined bijection and the canonical surjection $p_1 : \mathrm{Aut}(\mathfrak{l}) \times W_{\alpha\beta} \to \big(\mathrm{Aut}(\mathfrak{l}) \times W_{\alpha\beta}\big)/G$ factors through the canonical surjection $p_2 : \mathrm{Aut}(\mathfrak{l}) \times W_{\alpha\beta} \to \mathrm{Aut}(\mathfrak{l})/c(G) \times W_{\alpha\beta}$ and i. Now p_1 and p_2 are continuous and open, then i is bicontinuous. We now focus attention on the set $W_{\alpha\beta}$. By (12) and Lemma (4.4) we can write,

$$W_{\alpha\beta} \;=\; \{(x,W) \in V_\beta \times \eta(U_\alpha) :\; \eta_\alpha^{-1}(W)s_\beta(x) \in \mathscr{V}_\alpha\}, \tag{26}$$

Using Lemma (7.5), we can see that $W_{\alpha\beta}$ is empty for $\alpha \notin I_s^1(2n+1,k)$. Let us denote by $a_{s_\beta(x)}$ the coefficient of the first line and column in the matrix $s_\beta(x)$. Then $\eta_\alpha^{-1}(W)s_\beta(x)(Z) = a_{s_\beta(x)}Z$ and we can state that,

$$W_{\alpha\beta} \;=\; \left\{(x,W) \in V_\beta \times \eta(U_\alpha) \,\middle|\, {}^t s_\beta(x) {}^t\eta_\alpha^{-1}(W) J_{\mathfrak{g}} \eta_\alpha^{-1}(W) s_\beta(x) = a_{s_\beta(x)} J_{\mathfrak{l}} \right\}. \tag{27}$$

To finish the proof, we have to replace the sections $(s_\beta)_\beta$ by new sections satisfying $W_{\alpha\beta}$ is empty or $a_{s_\beta(x)} = 1$ for all $(x,W) \in W_{\alpha\beta}$. For $i = 1, \dots, k$, consider the open covering of $GL_k(\mathbb{R})$

$$G_i = \{A = (a_{ij}) \in GL_k(\mathbb{R}),\, a_{i1} \neq 0\}.$$

Then all the G_i's are $\mathrm{Aut}(\mathfrak{l})$-stable and we can actually replace the sections $(s_\beta)_\beta$ and the covering $(V_\beta)_\beta$, by the covering $(V_{\beta,i})_{\beta,i} = (V_\beta \cap G_i)_{\beta,i}$ and the sections $(s_{\beta,i})_{\beta,i} = (s_{\beta|V_{\beta,i}})_{\beta,i}$. For $i \neq 1$, $s_{\beta,i}(x) \notin P$ for all $x \in V_{\beta,i}$ and then $W_{\alpha,\beta,i}$ is empty by Lemma (7.8). For $i = 1$, let

$$G_1 = \{A = (a_{ij}) \in GL_k(\mathbb{R}),\, a_{11} = 1\}$$

and consider the map

$$\begin{array}{rcl} \delta_\beta \,:\, s_\beta(V_{\beta,1}) & \rightarrow & G_1 \\ s_\beta(x) & \mapsto & s_\beta(x)Q(a_{s_\beta(x)}) \end{array}$$

where $Q(a_{s_\beta(x)}) = \begin{pmatrix} a^{-1}_{s_\beta(x)} & 0 & 0 \\ 0 & I_{q+p} & 0 \\ 0 & 0 & a^{-1}_{s_\beta(x)} I_p \end{pmatrix}$.

Lemma 7.10. *The map δ_β is a homeomorphism from $s_\beta(V_{\beta,1})$ on its image.*

Proof. Clearly δ_β is continuous. Note that $s_\beta(x)$ and its image have the same coset class modulo $\mathrm{Aut}(\mathfrak{l})$. Then δ_β is injective and the map $\delta'_\beta : \delta_\beta\big(s_\beta(V_{\beta,1})\big) \rightarrow s_\beta(V_{\beta,1})$ given by $\delta'_\beta(M) = s_\beta(p(M))$ is well defined, continuous and turns out to be the inverse of δ_β. □

We finally end up with new family of continuous local sections $(s'_{\beta,i})_{\beta,i}$ given by:

$$s'_{\beta,i} = \begin{cases} \delta_\beta \circ s_{\beta,1} & \text{if} \quad i = 1 \\ s_{\beta,i} & \text{if} \quad i \neq 1 \end{cases}$$

defined on the open sets $V_{\beta,i}$ such that $a_{s'_{\beta,i}}(x) = 1$. This finishes the proof of the Theorem. □

Corollary 7.11. *Let G be the Heisenberg Lie group, H a connected Lie subgroup of G and Γ a discontinuous subgroup of G for G/H. Then $R(\Gamma, G, H)$ is an open set in $Hom(\Gamma, G)$. That is, every element of $R(\Gamma, G, H)$ is stable. In addition the rigidity, the local rigidity properties fail to hold globally on the parameter space.*

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