ON THE DEFORMATION SPACE OF CLIFFORD-KLEIN FORMS OF HEISENBERG GROUPS

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ABSTRACT. Let H be an arbitrary closed connected subgroup of the connected, simply connected Heisenberg $G = H_{2n+1}$. We exhibit in this paper a complete description of the deformation space $\mathcal{T}(\Gamma, G, H)$ and the moduli space $\mathcal{M}(\Gamma, G, H)$ of a discontinuous abelian subgroup Γ of G for the homogeneous space G/H. The topological features of deformations, namely the topological stability, the rigidity and the local rigidity are also studied.

1. INTRODUCTION

This paper is the continuation of the papers [1] and [10] where attention is focused on the explicit determination of the deformation space and the moduli space of a discontinuous group acting on some nilpotent homogeneous spaces for which, the basis group in question is respectively exponential solvable and two-step nilpotent. The problem of describing explicitly deformations for Clifford-Klein forms in general settings was initiated by T. Kobayashi in [8] and was formalized as *Problem C* by the same author in [6]. The deformation space $\mathcal{T}(\Gamma, G, H)$ was first introduced in Kobayashi ([6], (5.3.1)) and the moduli space $\mathcal{M}(\Gamma, G, H) := Aut(\Gamma)\backslash\mathcal{T}(\Gamma, G, H)$ was in Kobayashi ([6], (5.3.2)) for general homogeneous space G/H and discontinuous groups Γ.

In [10], T. Kobayashi and S. Nasrin studied the setup of a properly discontinuous action of a discrete subgroup $\Gamma \simeq \mathbb{Z}^k$ which acts on $\mathbb{R}^{k+1} \simeq G/H$ through a certain nilpotent affine transformation group G of dimension $2k+1$ when the connected subgroup in question is $\mathbb{R}^k$. In these circumstances, the authors gave a complete description of the parameter space

$$(1) \qquad R(\Gamma, G, H) := \left\{ \varphi \in \mathrm{Hom}(\Gamma, G) \;\middle|\; \begin{array}{l} \varphi \text{ is injective, } \varphi(\Gamma) \text{ is discrete and} \\ \text{acts properly and fixed point freely} \\ \text{on } G/H \end{array} \right\}$$

which is introduced in [8] for general contexts.

On the basis of this description, they determine explicitly the deformation space $\mathcal{T}(\mathbb{Z}^k, G, \mathbb{R}^k)$ by building up an accurate cross-section of the adjoint orbits of the elements of $R(\Gamma, G, H)$. One of the principal aims of this paper (and also the papers

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[6, 10]) is to illustrate that Problem C cited above, would be very fruitful to be pursued in various different settings.

In [1], the two first authors tackled the case of exponential solvable Lie groups where the subgroup H is supposed to contain the first derivative group $[G, G]$. They exhibited then a complete description of the deformation space and the Moduli space. The policy drawn for such a study seems to be pretty different from that followed in the paper [10] but somehow tacitly generalizes it. This remark together with the above studies arouse our interest in the consideration of the deformation space and the moduli space in a more general context, especially when H is not predetermined. Among fundamental motivations for seeking an explicit determination of the deformation space, is its interest in understanding their local geometric structures which could be pretty complex when the Clifford-Klein forms in question are not necessarily compact. This paper appears therefore to be a new contribution of such a study in the context where the group in question is the Heisenberg group. This enables us to utterly understand the related topological features in this context, namely, the stability, the rigidity and the local rigidity.

Towards that purpose, we give a complete description of both the spaces $\mathcal{T}(\Gamma, G, H)$ and $\mathcal{M}(\Gamma, G, H)$ up to a homeomorphism without any restriction on the subgroups Γ and H. Our study makes use of Grassmannians and carries out an accurate description in terms of matrix-like forms. We show however that in the situation where the Clifford-Klein form is compact, these spaces are cutely obtained to be some classical product of set matrices.

Let us remark that one main fact to settle this setup is that the center of $G = \mathbb{R}^{2n+1}$ is one dimensional and that the Lie structure of the group in question utterly reposes on a non-degenerate alternative form on $\mathbb{R}^{2n}$ which intervenes at the level of the center, through the pointwise law of the group G. It comes out therefore that the passage through the quotient by the adjoint action only involves non-central coordinates. It is noteworthy to point out here that the situation turns out to be more complicated when for instance the center of G does not meet the direct product $H \cdot L$ where L designates the syndetic hull of the discrete group Γ. In such a setting, we are of course quite away from the setup of compact Clifford-Klein forms.

The paper is organized as follows. The next section is devoted to record some known results about proper actions and to recall the definitions of the spaces under study. In section 3, we recall the characterization of the parameter and deformation space in term of homomorphisms of Lie algebras when the basis group in question is exponential solvable. This will crucially be applied to our circumstances. Likewise, the description of the deformation and the moduli space in the case where the subgroup H contains the derivative group $[G, G]$ can also be regarded as one of the main genesis for the description for these spaces. We shall recall such a description as well. The fourth section deals with the main results and provides complete proofs of the spaces description. The last section is devoted to develop some examples. We hope that our results and methods could be extended to encompass new setups, specially when the group in question is of higher step.

2. Preliminaries

The material dealt with in this section is based on the papers [1, 4, 5, 6, 7, 8, 9, 10, 11], (particularly [6, chapter 5]) and on some references therein. The readers could consult these references for broader information about the subject.

2.1. **Generalities and natation.** Let X be a locally compact space and K a locally compact topological group. The action of the group K on X is said to be:

(1) Proper (in the sense of Palais [12]) if, for each compact subset $S \subset X$ the set $K_S = \{k \in K : \ k \cdot S \cap S \neq \emptyset\}$ is compact.

(2) Fixed point free (or merely free) if, for each $x \in X$, the isotropy group $K_x = \{k \in K : \ kx = x\}$ is trivial.

(3) Properly discontinuous if, K is discrete and for each compact subset $S \subset X$ the set K_S is finite.

In the case where $X = G/H$ is a homogeneous space and K a subgroup of G, then it is well known that the action of K on X is proper if $SHS^{-1} \cap K$ is relatively compact for any compact set S in G. Here, for two sets A and B of the locally compact topological group G, the product AB is the subset $\{ab : \ a \in A, b \in B\}$. Likewise the action of K on X is free if for every $g \in G$, $K \cap gHg^{-1} = \{e\}$. In such cases, we abusively say that the triple (G, H, K) is proper (respectively free). In this context, the subgroup K is said to be a discontinuous group for the homogeneous space X, if K is a discrete subgroup of G and K acts properly and fixed point freely on X.
For any given discontinuous subgroup Γ for the homogeneous space X, the quotient space $\Gamma\backslash G/H$ is said to be a *Clifford-Klein form* for the homogeneous space G/H. It is then well-known that any Clifford-Klein form is endowed through the action of Γ with a manifold structure for which the quotient canonical surjection

$$\pi : G/H \to \Gamma\backslash G/H \tag{2}$$

turns out to be an open covering and particularly a local diffeomorphism. On the other hand, any Clifford-Klein form $\Gamma\backslash G/H$ inherits any $G-$invariant geometric structure (e.g. complex structure, pseudo-Riemanian structure, conformal structure, symplectic structure,...) on the homogeneous space G/H through the covering map π defined as in equation (2) above. We designate by $\text{Hom}(\Gamma, G)$ the set of group homomorphisms from Γ to G endowed with the point wise convergence topology. The same topology is obtained by taking generators $\gamma_1, ..., \gamma_k$ of Γ, then using the injective map

$$\text{Hom}(\Gamma, G) \hookrightarrow G \times \cdots \times G, \ \varphi \mapsto (\varphi(\gamma_1), ..., \varphi(\gamma_k))$$

to equip $\text{Hom}(\Gamma, G)$ with the relative topology induced from the direct product $G \times \cdots \times G$. We consider then the parameter space $R(\Gamma, G, H)$ of $\text{Hom}(\Gamma, G)$ defined as in formula (1) above, this set plays an important role as we will see later. According to this definition and as earlier, for each $\varphi \in R(\Gamma, G, H)$, the space $\varphi(\Gamma)\backslash G/H$ is a Clifford-Klein form which is a Hausdorff topological space and even equipped with a structure of a manifold for which, the quotient canonical map is an open covering.

Let now $\varphi \in R(\Gamma, G, H)$ and $g \in G$, we consider the element $\varphi^g := g^{-1} \cdot \varphi \cdot g$ of $\mathrm{Hom}(\Gamma, G)$ defined by

$$\varphi^g(\gamma) = g^{-1}\varphi(\gamma)g, \gamma \in \Gamma.$$

It is then clear that the element $\varphi^g \in R(\Gamma, G, H)$ and that the map

$$\varphi(\Gamma)\backslash G/H \longrightarrow \varphi^g(\Gamma)\backslash G/H, \quad \varphi(\Gamma)xH \mapsto \varphi^g(\Gamma)g^{-1}xH$$

is a natural diffeomorphism. We consider then the orbits space

$$\mathcal{T}(\Gamma, G, H) = R(\Gamma, G, H)/G$$

instead of $R(\Gamma, G, H)$ in order to avoid the unessential part of deformations arising inner automorphisms and to be quite precise on parameters. We call the set $\mathcal{T}(\Gamma, G, H)$ as the space of the deformation of the action of Γ on the homogeneous space G/H.

On the other hand, let the group $\mathrm{Aut}(\Gamma)$ act on $\mathrm{Hom}(\Gamma, G)$ by

$$T \cdot \varphi(\gamma) := \varphi(T^{-1}(\gamma)), \ \varphi \in \mathrm{Hom}(\Gamma, G), \ T \in \mathrm{Aut}(\Gamma), \ \gamma \in \Gamma.$$

It is then easy to check that the group $\mathrm{Aut}(\Gamma)$ leaves the parameter space $R(\Gamma, G, H)$ invariant and its action on it is $G-$ equivariant. We define then (to avoid this unessential part too) the Moduli space as the double coset space

$$\mathcal{M}(\Gamma, G, H) := \mathrm{Aut}(\Gamma)\backslash R(\Gamma, G, H)/G.$$

For $\varphi \in R(\Gamma, G, H)$, the discontinuous subgroup $\varphi(\Gamma)$ for the homogeneous space G/H is said to be *locally rigid* as a discontinuous group of G/H in the sense of Kobayashi [8], if the orbit of φ through the inner conjugation is open in the set $R(\Gamma, G, H)$. This means equivalently that any point sufficiently close to φ should be conjugate to φ under an inner automorphism of G. So, the homomorphisms which are locally rigid are those which correspond to those which are isolated points in the deformation space $\mathcal{T}(\Gamma, G, H)$. When every point in $R(\Gamma, G, H)$ is locally rigid, the deformation space turns out to be discrete and then we say that the Clifford-Klein form $\Gamma\backslash G/H$ can not deform continuously through the deformation of Γ in G. If a given $\varphi \in R(\Gamma, G, H)$ is not locally rigid, we say that it admits a *continuous deformation* and that the related Clifford-Klein form is continuously deformable.

In the same context, one says that $\varphi \in R(\Gamma, G, H)$ is *rigid*, it its $G-$orbit $G \cdot \phi$ is open in $\mathrm{Hom}(\Gamma, G)$.

The homomorphism φ is said to be *topologically stable* or merely *stable* in the sense of Kobayashi-Nasrin [10], if there is an open set in $\mathrm{Hom}(\Gamma, G)$ which contains φ and is contained in $R(\Gamma, G, H)$. When the set $R(\Gamma, G, H)$ is an open subset of $\mathrm{Hom}(\Gamma, G)$, then obviously each of its elements is stable, which is the case for irreducible Riemannian symmetric spaces. Furthermore, we precise in this setting that the concept of stability may be one fundamental genesis to understand the local structure of the deformation space.

2.2. **Characterization of the parameter and deformation space.** We keep and remind our notations and settings. $\mathfrak{g}$ will denote a n-dimensional real exponential solvable Lie algebra, G will be the associated connected and simply connected exponential Lie group. The exponential map

$$\exp : \mathfrak{g} \to G$$

is a global C^∞−diffeomorphism from $\mathfrak{g}$ into G. Let log denote the inverse map of exp. The Lie algebra $\mathfrak{g}$ acts on $\mathfrak{g}$ by the adjoint representation $\mathrm{ad}_\mathfrak{g}$, that is:

$$\mathrm{ad}_\mathfrak{g}(T)(Y) = \mathrm{ad}(T)(Y) = [T, Y],\ T,\ Y \in \mathfrak{g}.$$

The group G acts on $\mathfrak{g}$ by the adjoint representation Ad_G, defined by $\mathrm{Ad}_G(g) = \mathrm{Ad}(g) = \exp(\mathrm{ad}(T)),\ g = \exp T \in G$.

The following upshot which generalizes the result of Nasrin [15], has been obtained separately in [2] and [18], is quite important and plays an important role in this paper.

Theorem 2.1. *Let G be a connected simply connected at most three step nilpotent Lie group, H and K be connected subgroups of G. Then the following assertions are equivalent:*

(i) *K acts properly on G/H.*
(ii) *The action of K on G/H is free, that is $K \cap gHg^{-1} = \{e\}$ for any $g \in G$.*
(iii) *$\mathfrak{k} \cap \mathrm{Ad}_g\mathfrak{h} = \{0\}$ for any $g \in G$. Here $\mathfrak{h}$ and $\mathfrak{k}$ are the Lie algebras of H and K respectively.*

Let $X = G/H$ be a homogeneous space, with G a connected simply connected exponential solvable Lie group and H a closed connected subgroup of G. Let Γ be a discrete subgroup of G of rank k, and define the parameter space $R(\Gamma, G, H)$ as given in (1). Let $\mathfrak{g}$, $\mathfrak{h}$ designate the Lie algebras of G and H respectively. In [1], we have obtained the following characterization of the parameter and the deformation space as follows. Let L be the syndetic hull of Γ (see Theorem (3.1) of [1]). Recall that the Lie subalgebra $\mathfrak{l}$ of L is the real span of the abelian lattice $\log\Gamma$, which is generated by $\log\gamma_1, \dots, \log\gamma_k$ where $\gamma_1, \dots, \gamma_k$ is a set of generators of Γ. Then the map

$$E_\mathfrak{l}\ :\ \mathrm{Hom}(\mathfrak{l}, \mathfrak{g}) \to \mathfrak{g}^k := \mathfrak{g} \times \cdots \times \mathfrak{g}\ (k \text{ times}), \quad \psi \mapsto (\psi(\log(\gamma_1)), \dots, \psi(\log(\gamma_k)))$$

is also injective. We consider the topology induced on $\mathrm{Hom}(\mathfrak{l}, \mathfrak{g})$ by the injection, which can also be defined as the point wise convergence topology, we identify here $\mathfrak{g}^k$ to $\mathcal{L}(\mathfrak{l}, \mathfrak{g})$, the set of linear maps from $\mathfrak{l}$ to $\mathfrak{g}$. Now, the group $Aut(\Gamma)$ can be identified to a subgroup of $Aut(\mathfrak{l})$ which leaves $\log\Gamma$ stable. Therefore $Aut(\Gamma)$ acts linearly on $\mathfrak{l}$ by,

$$T \cdot (\log\gamma_i) = \log(T(\gamma_i)),\ i = 1, \dots, k,\ T \in Aut(\Gamma) \tag{3}$$

and leaves the lattice $\log\Gamma$ stable. The induced action on $\mathrm{Hom}(\mathfrak{l}, \mathfrak{g})$ is given by

$$(T \cdot \psi)(\log\gamma_i) = \psi(T^{-1} \cdot (\log\gamma_i)),\ i = 1, \dots, k, T \in Aut(\Gamma), \psi \in \mathrm{Hom}(\mathfrak{l}, \mathfrak{g}). \tag{4}$$

The group G acts also on $\mathrm{Hom}(\mathfrak{l}, \mathfrak{g})$ by

$$\psi \cdot g = \mathrm{Ad}_{g^{-1}} \circ \psi. \tag{5}$$

The following useful result was obtained in [1].

Theorem 2.2. *Let G be an exponential solvable Lie group, $\Gamma \cong \mathbb{Z}^k$ a discrete subgroup of G and $L = \exp(\mathfrak{l})$ its syndetic hull. Then the parameter space is given by*

$$R(\Gamma, G, H) = \left\{ \psi \in \mathrm{Hom}(\mathfrak{l}, \mathfrak{g}) \;\middle|\; \begin{array}{l} \dim \psi(\mathfrak{l}) = \dim \mathfrak{l} \\ \exp \psi(\mathfrak{l}) \text{ acts properly on } G/H \end{array} \right\}.$$

The deformation space can equivalently be described as:

$$\mathcal{T}(\Gamma, G, H) = \left\{ \psi \in \mathrm{Hom}(\mathfrak{l}, \mathfrak{g}) \;\middle|\; \begin{array}{l} \dim \psi(\mathfrak{l}) = \dim \mathfrak{l} \\ \exp \psi(\mathfrak{l}) \text{ acts properly on } G/H \end{array} \right\} \Big/ \mathrm{Ad}(G),$$

where the action Ad *of G is given as in (5).*

2.3. **Deformation and moduli space for normal subgroups.** This subsection is devoted to describe the deformation and the moduli space when the subgroup H contains the derivative group $[G, G]$ and G is exponential. One of the important feature of this setting is as remarked in [1], that every discrete subgroup which acts properly on G/H is abelian, which allows us to use the above characterization. The results of this subsection will crucially be used to have the description of the spaces in the case of Heisenberg groups. We start then to recall some results on Grassmannians.

Let $M^{\circ}_{n,k}(\mathbb{R})$ be the set of the matrices of rank k in $M_{n,k}(\mathbb{R})$, we denote by $G_{n,k}(\mathbb{R})$ be the Grassmannian of k dimensional linear subspaces of $\mathbb{R}^n$ and

$$\begin{array}{rccc} \eta : & M^{\circ}_{n,k}(\mathbb{R}) & \longrightarrow & G_{n,k}(\mathbb{R}) \\ & M & \mapsto & M(\mathbb{R}^k) \end{array}$$

be the canonical surjection The linear group $GL_k(\mathbb{R})$ acts on $M^{\circ}_{n,k}(\mathbb{R})$ by right side multiplication and $\eta(M) = \eta(M')$ if and only if there exist $A \in GL_k(\mathbb{R})$ such that $M' = MA$. This means that the column vectors of an element of $M^{\circ}_{n,k}(\mathbb{R})$ generate an element of $G_{n,k}(\mathbb{R})$ and the column vectors of two elements of $M^{\circ}_{n,k}(\mathbb{R})$ generate the same element of $G_{n,k}(\mathbb{R})$ if and only if one of them is a multiple of the other by an element of $GL_k(\mathbb{R})$. It follows therefore that $G_{n,k}(\mathbb{R})$ is identified with $M^{\circ}_{n,k}(\mathbb{R})$ via the equivalence relation:

$$M \sim M' \text{ in } M^{\circ}_{n,k}(\mathbb{R}) \text{ if and only if } \; M' = MA \text{ for some } A \in GL_k(\mathbb{R}).$$

So, we regard the space $G_{n,k}(\mathbb{R})$ as the quotient space $M^{\circ}_{n,k}(\mathbb{R})/_{\sim} := M^{\circ}_{n,k}(\mathbb{R})/GL_k(\mathbb{R})$, endowed with the quotient topology. Let

$$I(n, k) = \{(i_1, \ldots, i_k) : \; 1 \le i_1 < \cdots < i_k \le n\}.$$

For $M \in M_{n,k}(\mathbb{R})$ and $\alpha = (i_1, \ldots, i_k) \in I(n, k)$, we denote by M_α its $k \times k$ relative minor and $U_\alpha = \{M \in M^{\circ}_{n,k}(\mathbb{R}) : \; M_\alpha = I_k\}$. It is not hard to check that the restriction η_α of η on U_α is a homeomorphism between U_α and the open set $\eta_\alpha(U_\alpha)$. Furthermore

$$G_{n,k}(\mathbb{R}) = \bigcup_{\alpha \in I(n,k)} \eta_\alpha(U_\alpha). \tag{6}$$

It is well know that $M^{\circ}_{n,k}(\mathbb{R})$ is a total space of a $GL_k(\mathbb{R})$-principal bundle, with the base space equal to the Grassmannian $G_{n,k}(\mathbb{R})$. The sets $\eta^{-1}(\eta_\alpha(U_\alpha))$, $\alpha \in I(n,k)$ constitute an open covering of $M^{\circ}_{n,k}(\mathbb{R})$ and for every α the map

$$\begin{array}{rcll} \chi_\alpha \ : \ GL_k(\mathbb{R}) \times \eta_\alpha(U_\alpha) & \to & \eta^{-1}(\eta_\alpha(U_\alpha)) \\ (A, W) & \mapsto & \eta_\alpha^{-1}(W)A \end{array} \tag{7}$$

is a local trivialization, its inverse is the map $M \mapsto \big(M_\alpha, \eta(M)\big)$ and for every $W \in \eta_\alpha(U_\alpha)$ the map $\chi_{\alpha,W} : \ GL_k(\mathbb{R}) \to F_W$ given by

$$\chi_{\alpha,W}(A) = \eta_\alpha^{-1}(W)A,$$

is a homeomorphism, where F_W is the orbit of $\eta_\alpha^{-1}(W)$, its inverse is the map defined by

$$\chi^{-1}_{\alpha,W}(M) = M_\alpha.$$

Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$ containing $[\mathfrak{g},\mathfrak{g}]$. We Fix a basis $X_1,\ldots,X_n$ of $\mathfrak{g}$ passing through $\mathfrak{h}$ and $[\mathfrak{g},\mathfrak{g}]$ and we identify $\mathfrak{g}$ to $\mathbb{R}^n$ through this basis. Let $s = \dim\mathfrak{h}$, $l = \dim\,[\mathfrak{g},\mathfrak{g}]$ and consider the bilinear forms $b_1,\ldots,b_l$ defined by

$$[X,Y] = \sum_{i=1}^{n} b_i(X,Y)X_i$$

If we identify the space of the linear maps $\mathcal{L}(\mathfrak{l},\mathfrak{g})$ to $M_{n,k}(\mathbb{R})$, then the set $\mathrm{Hom}(\mathfrak{l},\mathfrak{g})$ is identified to the $GL_k(\mathbb{R})-$stable set

$$\mathscr{V} = \{M \in M_{n,k}(\mathbb{R}) : \ {}^tMJ_{b_i}M = 0,\ i = 1,\ldots,l\}, \tag{8}$$

where $J_{b_1},\ldots J_{b_l}$ designate the matrices of $b_1,\ldots,b_l$ written through our basis. Let also for $\alpha \in I(n,k)$, $\mathscr{V}_\alpha = \mathscr{V} \cap U_\alpha$. The group G acts on $\mathrm{Hom}(\mathfrak{l},\mathfrak{g})$ by composition on the left. Furthermore for every

$$\alpha \in I_s(n,k) = \{(i_1,\ldots i_k) \in I(n,k),\ i_1 > s\}, \tag{9}$$

the set $\mathscr{V}_\alpha$ is G-stable and we have,

Theorem 2.3. *Let G be an exponential solvable Lie group of dimension n, H a connected subgroup of dimension s which contains $[G,G]$ and Γ a rank k discontinuous subgroup for G/H. Then*

$$\mathcal{T}(\Gamma,G,H) = \bigcup_{\alpha\in I_s(n,k)} \mathcal{T}_\alpha \quad \textit{and} \quad \mathcal{M}(\Gamma,G,H) = \bigcup_{\alpha\in I_s(n,k)} \mathcal{M}_\alpha,$$

where for every $\alpha \in I_s(n,k)$, the set $\mathcal{T}_\alpha$ is an open subset of $\mathcal{T}(\Gamma,G,H)$ homeomorphic to the product $GL_k(\mathbb{R}) \times (\mathscr{V}_\alpha/G)$ and $\mathcal{M}_\alpha$ is an open subset of $\mathcal{M}(\Gamma,G,H)$ homeomorphic to the product $GL_k(\mathbb{R})/GL_k(\mathbb{Z}) \times (\mathscr{V}_\alpha/G)$.

We point out here that this result stems from the fact that the parameter space is the total space of a topological $GL_k(\mathbb{R})$-principal bundle. A direct consequence of this result, is the following fact concerning the topological features of Clifford-Klein forms.

Theorem 2.4. *Let G be an exponential solvable Lie group, $H \subset G$ a normal connected subgroup and $\Gamma \simeq \mathbb{Z}^k$ a discrete subgroup of G. Then $R(\Gamma, G, H)$ is an open set in* $\mathrm{Hom}(\Gamma, G)$*. That is, every element of $R(\Gamma, G, H)$ is stable. If in addition H contains $[G, G]$, then every Clifford-Klein form $\Gamma\backslash G/H$ is continuously deformable. Actually the local rigidity propriety fails to hold for every element in $R(\Gamma, G, H)$.*

3. The case of Heisenberg groups

From now on, $\mathfrak{g} := \mathfrak{h}_{2n+1}$ will denote the Heisenberg Lie algebra of dimension $2n + 1$. It can be defined as a real vector space, with a skew-symmetric bilinear form b of rank $2n$ and a fixed generator Z of the kernel of b. The center $\mathfrak{z}$ of $\mathfrak{g}$ is then the kernel of b and it is the one dimensional subspace $[\mathfrak{g}, \mathfrak{g}]$, where for $X, Y \in \mathfrak{g}$, the Lie bracket is given by

$$[X, Y] = b(X, Y)Z.$$

3.1. **Some useful tools.** This subsection aims to prove some structure results concerning Heisenberg groups. Such results are quite important and will crucially be used in the sequel of the paper. Roughly speaking, it consists in building a symplectic basis of $\mathfrak{g}$ constructed from a given subalgebra. We begin by proving the following:

Proposition 3.1. *Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$. Then there exists a basis $\mathscr{B}_{\mathfrak{h}} = \{Z, X_1 \ldots, X_n, Y_1, \ldots, Y_n\}$ of $\mathfrak{g}$ with the Lie commutation relations*

$$[X_i, Y_j] = \delta_{i,j} Z,\ i, j = 1, ..., n$$

and satisfying:
1) If $\mathfrak{z} \subset \mathfrak{h}$, then there exist two integers p, $q \geq 0$ such that the family

$$\{Z, X_1, \ldots, X_{p+q}, Y_1, \ldots, Y_p\}$$

constitutes a basis of $\mathfrak{h}$.
2) If $\mathfrak{z} \not\subset \mathfrak{h}$, then $\dim \mathfrak{h} \leq n$ and $\mathfrak{h}$ is generated by $X_1, \ldots, X_s$, where $s = \dim \mathfrak{h}$. The symbol $\delta_{i,j}$ designates here the Kronecker index. The basis $\mathscr{B}_{\mathfrak{h}}$ is said to be a symplectic basis of $\mathfrak{g}$ adapted to $\mathfrak{h}$.

Proof. 1) Note that the assertion is obviously true if $\mathfrak{h} = \mathfrak{z}$. We can and do assume then that $\mathfrak{h} \supsetneq \mathfrak{z}$, the kernel of the restriction $b_{|\mathfrak{h}}$ is therefore non trivial, and there exists a subalgebra V_0 such that $\ker b_{|\mathfrak{h}} = \mathfrak{z} \oplus V_0$. For any complementary subspace V_1 of $\mathfrak{z} \oplus V_0$ in $\mathfrak{h}$, the bilinear form $b_{|V_1}$ is non degenerated. Let p, $q \geq 0$ such that dim $V_0 = q$ and dim $V_1 = 2p$. Let now

$$N := \{x \in \mathfrak{g} : b(x, V_1) = 0\}.$$

Remark that $\mathfrak{g} = N \oplus V_1$. Indeed, note first that $N \cap V_1 = \ker b_{|V_1} = \{0\}$. We now consider the map

$$\begin{array}{rcl} f : \mathfrak{g} & \longrightarrow & V_1^* \\ x & \mapsto & b(x, \cdot) \end{array}$$

which is surjective and verifies $\ker f = N$. We deduce therefore for dimension reasons that $\mathfrak{g} = V_1 \oplus N$. Let N_1 be any supplementary subspace of $\mathfrak{z}$ in N, that is $N = \mathfrak{z} \oplus N_1$.

For $x \in \ker b_{|N_1}$, we have $b(v,x) = 0$ for all $v \in \mathfrak{g}$ which means that x is central in $\mathfrak{g}$. But the intersection of N_1 and $\mathfrak{z}$ is trivial, then $\ker b_{|N_1} = \{0\}$ and finally the restriction $b_{|N_1}$ is non degenerate. Up to this step, one decomposes $\mathfrak{g} = \mathfrak{z} \oplus V_1 \oplus N_1$ as a sum of b-orthogonal subspaces and we can assume that V_0 is an isotropic subspace of N_1. It is therefore well known that any basis of V_0 can be extended to a symplectic basis of N_1 and the result follows by taking any symplectic basis of V_1, a symplectic basis of N_1 passing through V_0, and the generator of $\mathfrak{z}$.
2) Assume now that $\mathfrak{z} \not\subset \mathfrak{h}$. Let V be a complementary subspace to $\mathfrak{z}$ in $\mathfrak{g}$ containing $\mathfrak{h}$. Then $b_{|V}$ is non degenerate and $\mathfrak{h}$ is an isotropic subspace of V, in particular $\dim \mathfrak{h} \leq n$. Take any basis of $\mathfrak{h}$ and extend it to a symplectic basis of V by adding the central vector Z. We obtain a symplectic basis of $\mathfrak{g}$ adapted to $\mathfrak{h}$. □

We shall now make use of the upshot above to explicitly determine the deformation and the moduli spaces in our context. Remark first that the matrix J_b of b written in $\mathscr{B}_{\mathfrak{h}}$ is

$$J_b := \mathscr{M}(b, \mathscr{B}_{\mathfrak{h}}) = \begin{pmatrix} 0 & \cdots & & 0 \\ & (0) & & (-I_n) \\ \vdots & & & \\ 0 & (I_n) & & (0) \end{pmatrix}.$$

Using proposition 3.1 above, one can view G as the direct product of $\mathscr{D} = \mathbb{R}^{2n}$ and $\mathbb{R}$ with the following point wise multiplication

$$g_1 g_2 = (v + w, s + t + \frac{1}{2} b(v,w)),\ g_1 = (v,s),\ g_2 = (w,t)$$

where b is explicitly given on $\mathscr{D}$ by:

$$b(v,w) = \langle v_1, w_2 \rangle - \langle v_2, w_1 \rangle,\ v = (v_1, v_2),\ w = (w_1, w_2)$$

where v_1, w_1 designate the coordinates of v and w respectively through the basis vectors $(X_1, ..., X_n)$ and v_2, w_2 their coordinates through the vectors $(Y_1, ..., Y_n)$. We get now a nice characterization of the proper action for the Heisenberg setting using Theorem (2.1). We have the following:

Lemma 3.2. *Let $\mathfrak{h}$, $\mathfrak{l}$ be two subalgebras of $\mathfrak{g}$ and $H = \exp \mathfrak{h}$. Then $\exp \mathfrak{l}$ acts properly on G/H if and only if one of these two properties is satisfied:*
i) $\mathfrak{z} \subset \mathfrak{h}$ and $\mathfrak{l} \cap \mathfrak{h} = \{0\}$.
ii) $\mathfrak{z} \not\subset \mathfrak{h}$, $\mathfrak{l} \cap \mathfrak{h} = \{0\}$ and $\mathfrak{z} \cap (\mathfrak{h} \oplus \mathfrak{l}) = \mathfrak{l} \cap \mathfrak{z}$.

Proof. The Heisenberg Lie algebra is a two step nilpotent Lie algebra. Using Theorem (2.1), we get that the proper action is equivalent to the property $\mathrm{Ad}_g \mathfrak{h} \cap \mathfrak{l} = \{0\}$ for all $g \in G$, (or equivalently $\mathrm{Ad}_g \mathfrak{l} \cap \mathfrak{h} = \{0\}$ for all $g \in G$). If $\mathfrak{z} \subset \mathfrak{h}$, it is then clear that $\mathrm{Ad}_g \mathfrak{h} = \mathfrak{h}$ and that the proper action is equivalent to $\mathfrak{l} \cap \mathfrak{h} = \{0\}$. Assume that the action is proper and $\mathfrak{z} \not\subset \mathfrak{h}$, so obviously $\mathfrak{h} \cap \mathfrak{l} = \{0\}$. Towards the equality $\mathfrak{z} \cap (\mathfrak{h} \oplus \mathfrak{l}) = \mathfrak{l} \cap \mathfrak{z}$, it is sufficient to show that $\mathfrak{z} \cap (\mathfrak{l} \oplus \mathfrak{h}) \subset (\mathfrak{z} \cap \mathfrak{l})$. Let $x \in \mathfrak{z} \cap (\mathfrak{l} \oplus \mathfrak{h})$. There exist then $l \in \mathfrak{l}$ and

$h \in \mathfrak{h}$ such that $x = l + h$. We have to show that $x = l$. Suppose that $l \notin \mathfrak{z}$. Then there exist $X \in \mathfrak{g}$ such that $[X, l] = -x$ as the center is one dimensional. As such, the non trivial element $\mathrm{Ad}_{\exp X}\, l = l - x = -h$ belongs to the intersection $\mathrm{Ad}_{\exp X}\, \mathfrak{l} \cap \mathfrak{h}$, which is impossible. This leads to the fact that $x - l$ is a central element and belongs to $\mathfrak{h}$ which also means that it is trivial. Conversely, let $t \in \mathrm{Ad}_g\, \mathfrak{l} \cap \mathfrak{h}$. We have $t = l + x$ with $x \in \mathfrak{z}$ and $l \in \mathfrak{l}$. Then $x \in \mathfrak{z} \cap (\mathfrak{l} \oplus \mathfrak{h})$ which means that $x \in \mathfrak{l}$ and finally $t \in \mathfrak{h} \cap \mathfrak{l} = \{0\}$. This consideration shows conclusively that $\exp \mathfrak{l}$ acts properly on G/H. □

3.2. **The deformation and the moduli spaces when H contains the center.** We assume in this section that the subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ contains the center $\mathfrak{z} = [\mathfrak{g}, \mathfrak{g}]$ and that $\mathfrak{l}$ is a subalgebra of $\mathfrak{g}$ such that $\mathfrak{l} \cap \mathfrak{h} = \{0\}$. Then $\mathfrak{l}$ is an abelian subalgebra and if $\mathcal{L}(\mathfrak{l}, \mathfrak{g})$ designates the vector space of the linear maps from $\mathfrak{l}$ to $\mathfrak{g}$, the set $\mathrm{Hom}(\mathfrak{l}, \mathfrak{g})$ of Lie algebras homomorphisms can be regarded as the set

$$\mathrm{Hom}(\mathfrak{l}, \mathfrak{g}) := \{\psi \in \mathcal{L}(\mathfrak{l}, \mathfrak{g}),\ [\psi(x), \psi(y)] = 0 \text{ for all } x, y \in \mathfrak{l}\}.$$

We fix by the way a symplectic basis $\mathscr{B}_{\mathfrak{h}}$ of $\mathfrak{g}$ adapted to $\mathfrak{h}$ as provided by proposition (3.1). We identify $\mathfrak{g}$ to $\mathbb{R}^{2n+1}$, $\mathfrak{h}$ to a subspace of $\mathbb{R}^{2n+1}$ and $\mathcal{L}(\mathfrak{l}, \mathfrak{g})$ to a subset of real matrices $M_{2n+1,k}(\mathbb{R})$, where $k = \dim \mathfrak{l}$. Let as usual $s = \dim \mathfrak{h}$. For any $\alpha \in I_s(n, k)$, we consider the set

$$\mathscr{V}'_\alpha := \left\{ M = \begin{pmatrix} 0 \\ A \end{pmatrix},\ A \in M_{2n,k}(\mathbb{R}), \quad M_\alpha = \mathrm{I}_k \text{ and } {}^tMJ_bM = 0 \right\} \subset \mathscr{V}_\alpha, \tag{10}$$

where J_b is the matrix of b in $\mathscr{B}_{\mathfrak{h}}$. The following theorem provides a description of the deformation and the moduli space in this context.

Theorem 3.3. *Let G be the Heisenberg Lie group of dimension $2n + 1$, H a connected Lie subgroup of dimension s which contains the center of G and Γ a rank k discontinuous subgroup for G/H. Then*

$$\mathcal{T}(\Gamma, G, H) = \bigcup_{\alpha \in I_s(2n+1,k)} \mathcal{T}_\alpha \quad \textit{and} \quad \mathcal{M}(\Gamma, G, H) = \bigcup_{\alpha \in I_s(2n+1,k)} \mathcal{M}_\alpha,$$

where for every $\alpha \in I_s(2n+1, k)$, the set $\mathcal{T}_\alpha$ is an open subset of $\mathcal{T}(\Gamma, G, H)$ homeomorphic to the product $GL_k(\mathbb{R}) \times \mathscr{V}'_\alpha$ and $\mathcal{M}_\alpha$ is an open subset of $\mathcal{M}(\Gamma, G, H)$ homeomorphic to the product $GL_k(\mathbb{R})/GL_k(\mathbb{Z}) \times \mathscr{V}'_\alpha$.

Proof. We will make use of Theorem (2.3) as $\mathfrak{h}$ contains the first derivative group of $\mathfrak{g}$. As it stands there, we just have to prove that the quotient space $\mathscr{V}_\alpha/G$ is homeomorphic to $\mathscr{V}'_\alpha$ for any $\alpha \in I_s(2n + 1, k)$. We fix first of all a symplectic basis $\mathscr{B}_{\mathfrak{h}} = (Z, X_1, ..., X_n, Y_1, ..., Y_n)$ adapted to $\mathfrak{h}$. Take any $\alpha \in I_s(2n + 1, k)$ and $M \in \mathscr{V}_\alpha$, we can then write

$$M = \begin{pmatrix} a \\ A \end{pmatrix} \text{ with } a = (a_1, \ldots, a_k) \in \mathbb{R}^k \text{ and } A \in M_{2n,k}(\mathbb{R}).$$

Note that for $X \in \mathfrak{g}$ we have

$$\mathrm{Ad}_{\exp X} = \begin{pmatrix} 1 & b(X,X_1) & \cdots & b(X,Y_n) \\ . & & & \\ . & & & \\ . & & & \\ 0 & & (I_{2n}) & \end{pmatrix}.$$

Then the action of $\mathrm{Ad}_{\exp X}$ on M affects only the first line. More precisely, if we identify the columns c_i of A to a vector of $\mathfrak{g}$, then the first line of the product is

$$(a_1 + b(X,c_1), \ldots, a_k + b(X,c_k)).$$

The following result which has been proved in [1], will be used.

Lemma 3.4. (Lemma (4.3), [1]**)** *Let W_M denote the subspace of $\mathfrak{g}$ generated by the columns of M. If $M \in \mathscr{V}_\alpha$ for $\alpha \in I_s(2n+1,k)$ then $W_M \cap \mathfrak{h} = \{0\}$.*

It turns out as the center $\mathfrak{z}$ is not contained in W_M that the map $\mathfrak{g} \to W_M^*$, $x \mapsto b(x,.)$ is surjective and there exists therefore $X \in \mathfrak{g}$ such that

$$b(X,c_1) = -a_1, \ldots, b(X,c_k) = -a_k.$$

It follows then that any M in $\mathscr{V}_\alpha$ is G-equivalent to the matrix obtained from M by vanishing the first line of M. Conversely if the first lines of M and M' are zero, then $M = \mathrm{Ad}_{\exp X} M'$ only if $M = M'$.

Let π be the continuous map from $\mathscr{V}_\alpha$ to $\mathscr{V}'_\alpha$ which sends the matrix $M = \begin{pmatrix} a \\ A \end{pmatrix}$ to the matrix $\pi(M) = \begin{pmatrix} 0 \\ A \end{pmatrix}$, where we consider the trace topology on $\mathscr{V}'_\alpha$. Then the canonical surjection $p : \mathscr{V}_\alpha \to \mathscr{V}_\alpha/G$ factors through π to a continuous bijection f between $\mathscr{V}'_\alpha$ and $\mathscr{V}_\alpha/G$ defined by $p = f \circ \pi$. Now G acts continuously on $\mathscr{V}_\alpha$, then p is open and we can easily see that f^{-1} is continuous. This achieves the proof of the Theorem.

□

3.3. **The deformation and the moduli spaces when H does not contain the center.**

We now tackle the case where the center of $\mathfrak{g}$ does not meet $\mathfrak{h}$. In such a situation, H is an abelian subgroup of G. We still need some other results. The following lemma describes the structure of the parameter space in this case.

Lemma 3.5. *Let G be the Heisenberg Lie group, H a connected subgroup which does not contain the center, Γ a rank k abelian discontinuous subgroup for G/H and $L = \exp(\mathfrak{l})$ its syndetic hull in G. Then the parameters space $R(\Gamma,G,H)$ is the disjoint union of the two $G-$invariant sets*

$$R_1(\Gamma,G,H) = \left\{ \psi \in \mathrm{Hom}(\mathfrak{l},\mathfrak{g}) \;\middle|\; \begin{array}{c} \dim \psi(\mathfrak{l}) = k, \\ \mathfrak{h} \cap \psi(\mathfrak{l}) = \{0\} \\ \textit{and } \mathfrak{z} \subset \psi(\mathfrak{l}) \end{array} \right\}$$

and

$$R_2(\Gamma, G, H) = \left\{ \psi \in \mathrm{Hom}(\mathfrak{l}, \mathfrak{g}) \,\middle|\, \begin{array}{l} \dim \psi(\mathfrak{l}) = k \;\; \textit{and} \\ (\mathfrak{h} \oplus \mathfrak{z}) \cap \psi(\mathfrak{l}) = \{0\} \end{array} \right\}.$$

Proof. From Theorem (2.2) and lemma (3.2), we can easily see that $R(\Gamma, G, H)$ is the union of the following sets

$$R_1(\Gamma, G, H) = \left\{ \psi \in \mathrm{Hom}(\mathfrak{l}, \mathfrak{g}) \,\middle|\, \begin{array}{l} \dim \psi(\mathfrak{l}) = k, \\ \mathfrak{h} \cap \psi(\mathfrak{l}) = \{0\}, \\ \mathfrak{z} \cap (\mathfrak{h} \oplus \psi(\mathfrak{l})) = \psi(\mathfrak{l}) \cap \mathfrak{z} \\ \text{ and } \mathfrak{z} \subset \psi(\mathfrak{l}) \end{array} \right\}$$

and

$$R_2(\Gamma, G, H) = \left\{ \psi \in \mathrm{Hom}(\mathfrak{l}, \mathfrak{g}) \,\middle|\, \begin{array}{l} \dim \psi(\mathfrak{l}) = k, \\ \mathfrak{h} \cap \psi(\mathfrak{l}) = \{0\}, \\ \mathfrak{z} \cap (\mathfrak{h} \oplus \psi(\mathfrak{l})) = \psi(\mathfrak{l}) \cap \mathfrak{z} \\ \text{ and } \mathfrak{z} \not\subset \psi(\mathfrak{l}) \end{array} \right\}$$

$$= \left\{ \psi \in \mathrm{Hom}(\mathfrak{l}, \mathfrak{g}) \,\middle|\, \begin{array}{l} \dim \psi(\mathfrak{l}) = k, \\ \mathfrak{h} \cap \psi(\mathfrak{l}) = \{0\}, \\ \mathfrak{z} \cap (\mathfrak{h} \oplus \psi(\mathfrak{l})) = \{0\} \end{array} \right\}.$$

To conclude, note that the third condition $\mathfrak{z} \cap (\mathfrak{h} \oplus \psi(\mathfrak{l})) = \psi(\mathfrak{l}) \cap \mathfrak{z}$ involved in the set $R_1(\Gamma, G, H)$ is trivial as $\mathfrak{z} \subset \psi(\mathfrak{l})$. Likewise, it is easily seen that $\mathfrak{z} \cap (\mathfrak{h} \oplus \psi(\mathfrak{l})) = \{0\}$ if and only if $(\mathfrak{z} \oplus \mathfrak{h}) \cap \psi(\mathfrak{l}) = \{0\}$ and then the three last set equations of $R_2(\Gamma, G, H)$ together are equivalent to $(\mathfrak{h} \oplus \mathfrak{z}) \cap \psi(\mathfrak{l}) = \{0\}$. On the other hand, for any $g \in G$, $\mathfrak{z} \subset \mathrm{Ad}_{g^{-1}} \circ \psi(\mathfrak{l})$ if and only if $\mathfrak{z} \subset \psi(\mathfrak{l})$, which proves the $G-$invariance of $R_1(\Gamma, G, H)$. Furthermore, for any $\psi \in R_2(\Gamma, G, H)$ and any $g \in G$, one has

$$(\mathfrak{z} \oplus \mathfrak{h}) \cap \mathrm{Ad}_{g^{-1}}(\psi(\mathfrak{l})) = (\mathfrak{z} \oplus \mathfrak{h}) \cap (\psi(\mathfrak{l})) = \{0\},$$

which shows the $G-$ invariance of the set $R_2(\Gamma, G, H)$. □

We now fix a basis $\mathscr{B}_{\mathfrak{h}} = \{Z, X_1 \ldots, X_n, Y_1, \ldots, Y_n\}$ of $\mathfrak{g}$ adapted to $\mathfrak{h}$. We consider the decomposition

$$\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{h} \oplus \mathfrak{h}' \oplus \mathfrak{k} \oplus \mathfrak{k}'$$

where,

$$\mathfrak{h} = \langle X_1, \ldots, X_s \rangle, \; \mathfrak{h}' = \langle Y_1, \ldots, Y_s \rangle,$$
$$\mathfrak{k} = \langle X_{s+1}, \ldots, X_n \rangle \text{ and } \mathfrak{k}' = \langle Y_{s+1}, \ldots, Y_n \rangle.$$

We identify as previously $\mathfrak{g}$ to $\mathbb{R}^{2n+1} = \mathbb{R} \oplus \mathbb{R}^s \oplus \mathbb{R}^s \oplus \mathbb{R}^{n-s} \oplus \mathbb{R}^{n-s}$ and $\mathrm{Hom}(\mathfrak{l}, \mathfrak{g})$ to the set of matrices given in (8), with $l = 1$ and $b_1 = b$. Then with respect to this decomposition, any element of $\mathfrak{g}$

$$x = a_0 Z + \sum_{i=1}^{n} a_i X_i + \sum_{i=1}^{n} b_i Y_i,$$

is identified to the column vector

$${}^t(a_0 \, a_1 \, \cdots \, a_s \, b_1 \, \cdots \, b_s \, a_{s+1} \, \cdots \, a_n \, b_{s+1} \, \cdots \, b_n)$$

and every homomorphism $\psi \in \mathrm{Hom}(\mathfrak{l}, \mathfrak{g})$, can be written as a matrix

$$M = \begin{pmatrix} A_0 \\ A_1 \\ B_1 \\ A_2 \\ B_2 \end{pmatrix}$$

where $A_0 \in M_{1,k}(\mathbb{R})$, $A_1, B_1 \in M_{s,k}(\mathbb{R})$ and $A_2, B_2 \in M_{n-s,k}(\mathbb{R})$. Then from (8) and the lemma (3.5), we get,

$$R_1(\Gamma, G, H) = \left\{ M \in M_{2n+1,k}(\mathbb{R}) \;\middle|\; \begin{array}{l} \dim M(\mathbb{R}^k) = k, \\ \mathfrak{h} \cap M(\mathbb{R}^k) = \{0\}, \\ \mathfrak{z} \subset M(\mathbb{R}^k) \text{ and} \\ {}^t M J_b M = 0 \end{array} \right\}.$$

Up to this step, we consider the set

$$I_s^1(2n+1, k) = \{(i_1, \dots, i_k),\ i_1 = 1 \text{ and } i_2 > s+1\}. \tag{11}$$

Now, we can state the following:

Lemma 3.6. *The set $R_1(\Gamma, G, H)$ is open in* $\mathrm{Hom}(\mathfrak{l}, \mathfrak{g})$ *and the sets $\eta^{-1}\big(\eta_\alpha(\mathscr{V}_\alpha)\big)$, $\alpha \in I_s^1(2n+1, k)$ constitutes an open $G-$invariant covering of $R_1(\Gamma, G, H)$.*

Proof. The condition $\mathfrak{z} \subset M(\mathbb{R}^k)$ equivalent to the existence of a matrix

$$M' = \begin{pmatrix} 1 & 0 \\ 0 & A'_1 \\ 0 & B'_1 \\ 0 & A'_2 \\ 0 & B'_2 \end{pmatrix},$$

with $A'_1, B'_1 \in M_{s,k-1}(\mathbb{R})$, $A'_2, B'_2 \in M_{n-s,k-1}(\mathbb{R})$ and $M(\mathbb{R}^k) = M'(\mathbb{R}^k)$. The conditions $M(\mathbb{R}^k) \cap \mathfrak{h} = \{0\}$ and $\dim M(\mathbb{R}^k) = k$ are equivalent to

$$\mathrm{rank} \begin{pmatrix} B'_1 \\ A'_2 \\ B'_2 \end{pmatrix} = k - 1,$$

which is also equivalent to the existence of $\alpha \in I_s^1(2n+1, k)$ such that $M(\mathbb{R}^k) \in \eta_\alpha(U_\alpha)$. Now, if $M(\mathbb{R}^k) \in \eta_\alpha(U_\alpha)$ then ${}^t M J_b M = 0$ if and only if

$${}^t\{\eta_\alpha^{-1}(M(\mathbb{R}^k))\} J_b \{\eta_\alpha^{-1}(M(\mathbb{R}^k))\} = 0,$$

or equivalently $\eta_\alpha^{-1}(M(\mathbb{R}^k)) \in \mathscr{V}_\alpha$. Then $M \in R_1(\Gamma, G, H)$ if and only if

$$M(\mathbb{R}^k) \in \bigcup_{\alpha \in I_s^1(2n+1,k)} \eta_\alpha(\mathscr{V}_\alpha).$$

This means that

$$R_1(\Gamma, G, H) = \bigcup_{\alpha \in I_s^1(2n+1,k)} \eta^{-1}\big(\eta_\alpha(\mathscr{V}_\alpha)\big).$$

Furthermore,

$$\eta^{-1}\big(\eta_\alpha(\mathscr{V}_\alpha)\big) = \left\{ M \in M_{2n+1,k}(\mathbb{R}) \;\middle|\; \begin{array}{l} (\det M_\alpha) \neq 0 \\ \\ \text{and } {}^tMJ_bM = 0 \end{array} \right\},$$

which is an open set of $\mathscr{V}$ and then $R_1(\Gamma, G, H)$ is also open in $\mathrm{Hom}(\mathfrak{l}, \mathfrak{g})$. Let $X \in \mathfrak{g}$ and $M \in \eta^{-1}\big(\eta_\alpha(\mathscr{V}_\alpha)\big)$, then there exist $A \in GL_k(\mathbb{R})$ and $M' \in M_{2n+1,k}(\mathbb{R})$ such that

$$\mathrm{Ad}_{\exp X} = \begin{pmatrix} 1 & b(X, X_1) \;\cdots\; b(X, Y_n) \\ \cdot & \\ \cdot & \\ \cdot & \\ 0 & (I_{2n}) \end{pmatrix}$$

and $M = M'A$, with

$$M' = \begin{pmatrix} 1 & 0 \\ 0 & A_1' \\ 0 & B_1' \\ 0 & A_2' \\ 0 & B_2' \end{pmatrix}.$$

Therefore,

$$\mathrm{Ad}_{\exp X} M = M' \begin{pmatrix} 1 & b(X, c_2) \;\cdots\; b(X, c_n) \\ \cdot & \\ \cdot & \\ \cdot & \\ 0 & (I_{k-1}) \end{pmatrix} A,$$

where $c_2, \ldots, c_k$ are the $k-1$ last columns vectors of M' and we can see that $\eta(M) = \eta(\mathrm{Ad}_{\exp X} M)$, which proves the $G-$invariance of $\eta^{-1}\big(\eta_\alpha(\mathscr{V}_\alpha)\big)$ for any $\alpha \in I_s^1(2n+1, k)$. □

Now we are ready to state our main result in this section concerning the deformation and the moduli space in the case where $\mathfrak{h}$ does not meet the center of $\mathfrak{g}$. We have the following:

Theorem 3.7. *Let G be the Heisenberg Lie group, H a connected subgroup which does not meet the center of G, Γ a rank k abelian discontinuous subgroup for G/H and $L = \exp(\mathfrak{l})$ its syndetic hull in G. Then*

$$\mathcal{T}(\Gamma, G, H) = \bigcup_{\alpha \in I_{s+1}(2n+1,k)} \mathcal{T}_\alpha \bigcup_{\alpha \in I_s^1(2n+1,k)} \mathcal{T}_\alpha$$

and

$$\mathcal{M}(\Gamma, G, H) = \bigcup_{\alpha \in I_{s+1}(2n+1,k)} \mathcal{M}_\alpha \bigcup_{\alpha \in I_s^1(2n+1,k)} \mathcal{M}_\alpha,$$

where:

1) For every $\alpha \in I_{s+1}(2n+1, k)$, the set $\mathcal{T}_\alpha$ is open in $\mathcal{T}(\Gamma, G, H)$ and homeomorphic to the product $GL_k(\mathbb{R}) \times \mathscr{V}'_\alpha$ and the set $\mathcal{M}_\alpha$ is open in $\mathcal{M}(\Gamma, G, H)$ and homeomorphic to $GL_k(\mathbb{Z})\backslash GL_k(\mathbb{R}) \times \mathscr{V}'_\alpha$.

2) For every $\alpha \in I_s^1(2n+1, k)$, the set $\mathcal{T}_\alpha$ is open in $\mathcal{T}(\Gamma, G, H)$ and is homeomorphic to the product $O_k \times \mathbb{R}^k \times N_k \times \mathscr{V}_\alpha$, N_k designates here the set of upper triangular unipotent matrices. Likewise, the set $\mathcal{M}_\alpha$ is open in $\mathcal{M}(\Gamma, G, H)$ and homeomorphic to the product $(GL_k(\mathbb{Z})\backslash GL_k(\mathbb{R})/\mathbb{R}^{k-1}) \times \mathscr{V}_\alpha$.

Proof. We use Lemma (3.5) to write the following decomposition of the deformation space

$$R_1(\Gamma, G, H)/G \cup R_2(\Gamma, G, H)/G.$$

The set $R_2(\Gamma, G, H)$ can be identified to the parameter space $R(\Gamma, G, K)$, where $K = Z(G)H$ and $Z(G)$ is the center of G. Then by Theorem (3.3), we get the following description of the quotient set

$$R_2(\Gamma, G, H)/G = \bigcup_{\alpha \in I_{s+1}(2n+1,k)} \mathcal{T}_\alpha$$

where for any $\alpha \in I_{s+1}(2n+1, k)$, the set $\mathcal{T}_\alpha$ is open in $\mathcal{T}(\Gamma, G, H)$ and homeomorphic to the product $GL_k(\mathbb{R}) \times \mathscr{V}'_\alpha$. On the other hand, thanks to Lemma (3.6), one can write

$$R_1(\Gamma, G, H)/G = \bigcup_{\alpha \in I_s^1(2n+1,k)} \eta^{-1}\big(\eta_\alpha(\mathscr{V}_\alpha)\big)/G$$

as union of open sets. Let then $\mathcal{T}_\alpha = \eta^{-1}\big(\eta_\alpha(\mathscr{V}_\alpha)\big)/G$. Recall that the map χ_α is a homeomorphism between $GL_k(\mathbb{R}) \times \eta_\alpha(\mathscr{V}_\alpha)$ and $\eta^{-1}\big(\eta_\alpha(\mathscr{V}_\alpha)\big)$. Consider the $G-$action on $GL_k(\mathbb{R}) \times \eta_\alpha(\mathscr{V}_\alpha)$ given by

$$(A, W) \cdot g = (\chi_{\alpha,W}^{-1} \operatorname{Ad}_{g^{-1}} \chi_{\alpha,W} A, W).$$

Then the map χ_α is $G-$equivariant. Indeed,

$$\begin{aligned} \chi_\alpha((A, W) \cdot g) &= \chi_\alpha(\chi_{\alpha,W}^{-1} \circ \operatorname{Ad}_{g^{-1}} \circ \chi_{\alpha,W} \circ A, W) \\ &= \chi_{\alpha,W} \circ (\chi_{\alpha,W}^{-1} \circ \operatorname{Ad}_{g^{-1}} \circ \chi_{\alpha,W} \circ A) \\ &= \operatorname{Ad}_{g^{-1}} \circ \chi_\alpha(A, W) \\ &= \chi_\alpha(A, W) \cdot g. \end{aligned}$$

For every $\alpha \in I_s^1(2n+1,k)$ and $W \in \eta_\alpha(\mathscr{V}_\alpha)$, we can easily see that there is $A_1, B_1 \in M_{s,k}(\mathbb{R})$, $A_2, B_2 \in M_{n-s,k}(\mathbb{R})$ such that

$$\eta_\alpha^{-1}(W) = \begin{pmatrix} 1 & 0 \\ 0 & A_1 \\ 0 & B_1 \\ 0 & A_2 \\ 0 & B_2 \end{pmatrix}.$$

Then, for $g^{-1} = \exp X$ we have

$$\chi_{\alpha,W}^{-1} \operatorname{Ad}_{g^{-1}} \chi_{\alpha,W} A = \left(\operatorname{Ad}_{g^{-1}} \eta_\alpha^{-1}(W) A\right)_\alpha = \begin{pmatrix} 1 & b(X,c_2) \ \cdots \ b(X,c_n) \\ . & \\ . & \\ . & \\ 0 & (I_{k-1}) \end{pmatrix} A,$$

where $c_2, \ldots, c_k$ are the $k-1$ last columns of $\eta_\alpha^{-1}(W)$. We now consider the free action of $\mathbb{R}^{k-1}$ on $GL_k(\mathbb{R})$ defined by

$$(x_1, \ldots, x_{k-1}) \cdot A = \begin{pmatrix} 1 & x_1 \ \cdots \ x_{k-1} \\ . & \\ . & \\ . & \\ 0 & (I_{k-1}) \end{pmatrix} A.$$

The subspace W' of W generated by $c_2, \ldots, c_k$ is an abelian subalgebra of dimension $k-1$ which does not meet the center, which means that the map $\mathfrak{g} \to W'^*, X \mapsto b(X, \cdot)_{|W'}$ is surjective. It follows therefore that for any $(x_1, \ldots, x_{k-1}) \in \mathbb{R}^{k-1}$, there is $X \in \mathfrak{g}$ such that $b(X, c_i) = x_{i-1}$ for all $i = 2, \ldots, k$. Therefore the quotient map

$$\pi : GL_k(\mathbb{R}) \times \eta_\alpha(\mathscr{V}_\alpha) \longrightarrow \big(GL_k(\mathbb{R}) \times \eta_\alpha(\mathscr{V}_\alpha)\big)/G,$$

factors through the canonical surjection

$$p : GL_k(\mathbb{R}) \times \eta_\alpha(\mathscr{V}_\alpha) \longrightarrow \big(GL_k(\mathbb{R})/\mathbb{R}^{k-1}\big) \times \eta_\alpha(\mathscr{V}_\alpha)$$

to give a continuous surjective map

$$f : \big(GL_k(\mathbb{R})/\mathbb{R}^{k-1}\big) \times \eta_\alpha(\mathscr{V}_\alpha) \longrightarrow \big(GL_k(\mathbb{R}) \times \eta_\alpha(\mathscr{V}_\alpha)\big)/G$$

defined by $\pi = f \circ p$ and we can easily see that f is injective. Now, G acts continuously on $GL_k(\mathbb{R}) \times \eta_\alpha(\mathscr{V}_\alpha)$, which entails that π is open and that f is a homeomorphism. The following Lemma enables us to achieve the proof of the assertion concerning the deformation space.

Lemma 3.8. *Fix a positive integer p and regard $\mathbb{R}^p$ as a subgroup of $GL_{p+1}(\mathbb{R})$ through the writing*

$$R_p := \left\{ \begin{pmatrix} 1 & {}^t x \\ 0 & I_p \end{pmatrix} : x \in \mathbb{R}^p \right\}.$$

Then,

$$GL_{p+1}(\mathbb{R})/\mathbb{R}^p \simeq O_{p+1} \times \mathbb{R}^{p+1} \times N_p, \tag{12}$$

where N_p denotes the totality of upper triangular unipotent matrices.

Proof. Using the Iwasawa decomposition, we have

$$GL_{p+1}(\mathbb{R}) \simeq O_{p+1} \times A_{p+1} \times N_{p+1},$$

where $A_{p+1}(\simeq \mathbb{R}^{p+1})$ denotes the totality of diagonal matrices with positive entries. Thus we obtain (12) because of the decomposition $N_{p+1} \simeq N_p \times R_p$. □

As for the moduli space, recall that $\text{Aut}(\Gamma) = GL_k(\mathbb{Z})$ and if we consider the action of $\text{Aut}(\Gamma)$ on $GL_k(\mathbb{R}) \times \eta_\alpha(\mathscr{V}_\alpha)$ given by $T \cdot (A, W) = (AT^{-1}, W)$, then χ_α is $\text{Aut}(\Gamma)-$equivariant and the result follows immediately.

□

The following result provides another variant of description of the deformation space in our context.

Theorem 3.9. *Let G be the Heisenberg Lie group, H a connected subgroup which does not meet the center of G, Γ a rank k abelian discontinuous subgroup for G/H and $L = \exp(\mathfrak{l})$ its syndetic hull in G. Then*

$$\mathcal{T}(\Gamma, G, H) = \bigcup_{\alpha \in I_{s+1}(2n+1,k)} \mathcal{T}_\alpha \bigcup_{\alpha \in I_s^1(2n+1,k)} \bigcup_{j=1}^{k} \mathcal{T}_{\alpha,j}$$

where for every $\alpha \in I_{s+1}(2n+1, k)$, the set $\mathcal{T}_\alpha$ is open in $\mathcal{T}(\Gamma, G, H)$ and homeomorphic to the product $GL_k(\mathbb{R}) \times \mathscr{V}'_\alpha$. Furthermore, for any $\alpha \in I_s^1(2n+1, k)$ and $j \in \{1, ..., k\}$, the set $\mathcal{T}_{\alpha,j}$ is open in $\mathcal{T}(\Gamma, G, H)$ and is homeomorphic to the multiple direct product $\mathbb{R}^* \times \mathbb{R}^{k-1} \times GL_{k-1}(\mathbb{R}) \times \mathscr{V}_\alpha$.

Proof. In light of Theorem (3.7), we only need to show that

$$GL_k(\mathbb{R})/\mathbb{R}^{k-1} = \bigcup_{j=1}^{k} \mathscr{U}_i,$$

where for any $j = 1, ..., k$, $\mathscr{U}_j$ is homeomorphic to $\mathbb{R}^* \times \mathbb{R}^{k-1} \times GL_{k-1}(\mathbb{R})$. Indeed, let $A \in GL_k(\mathbb{R})$ and denote by A_i the matrix obtained from A by deleting the first line and the i^{th} column of A. Then the union of the open sets

$$U_i = \{A \in GL_k(\mathbb{R}), \det A_i \neq 0\}, \quad 1 \leq i \leq k$$

is equal to $GL_k(\mathbb{R})$ and each of them is $\mathbb{R}^{k-1}-$stable. Therefore

$$GL_k(\mathbb{R})/\mathbb{R}^{k-1} = \bigcup_{i=1}^{k} U_i/\mathbb{R}^{k-1}.$$

The following Lemma enables us to achieve the proof.

Lemma 3.10. *For $1 \leq i \leq k$ we have $U_i/\mathbb{R}^{k-1} \cong \mathbb{R}^* \times \mathbb{R}^{k-1} \times GL_{k-1}(\mathbb{R})$.*

Proof. Let $A \in U_i$, write

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1k} \\ (a_1) & \cdots & (a_k) \end{pmatrix},$$

where $a_{11}, \ldots, a_{1k} \in \mathbb{R}$ and $a_1, \ldots, a_k \in \mathbb{R}^{k-1}$. So for $x \in \mathbb{R}^{k-1}$ we have,

$$x \cdot A = \begin{pmatrix} a_{11} + \langle x, a_1 \rangle & \cdots & a_{1k} + \langle x, a_k \rangle \\ (a_1) & \cdots & (a_k) \end{pmatrix},$$

where $\langle ., . \rangle$ designates the natural scalar product on $\mathbb{R}^{k-1}$. For $x_0 = -b_i A_i^{-1}$, where $b_i \in \mathbb{R}^{k-1}$ obtained from $(a_{11}, \ldots, a_{1k})$ by eliminating of the $i - th$ coordinate, we have

$$a_{1j} + \langle x_0, a_j \rangle = 0, \text{ for all } j \neq i.$$

This means that A is equivalent (modulo $\mathbb{R}^{k-1}$) to a certain matrix in the set

$$K_i := \left\{ A = \begin{pmatrix} a_{11} & \cdots & a_{1k} \\ (a_1) & \cdots & (a_k) \end{pmatrix} \in GL_k(\mathbb{R}),\ a_{1j} = 0 \text{ for all } j \neq i \right\}.$$

Note that

$$K_i \cong \mathbb{R}^* \times \mathbb{R}^{k-1} \times GL_{k-1}(\mathbb{R}).$$

Let $\pi : U_i \longrightarrow U_i/\mathbb{R}^{k-1}$ be the canonical surjection and $p : U_i \longrightarrow K_i$ the continuous surjection defined by

$$p(A) = \begin{pmatrix} p_1(A) & \cdots & p_k(A) \\ (a_1) & \cdots & (a_k) \end{pmatrix},$$

where $p_j(A) = 0$, if $j \neq i$ and $p_i(A) = a_{1i} - \langle b_i A_i^{-1}, a_i \rangle \neq 0$. Then clearly the map $f : K_i \longrightarrow U_i/\mathbb{R}^{k-1}$ defined by $f\big(p(A)\big) = \pi(A)$ is surjective. For the injectivity, let $A,\ A' \in K_i$ such that $f(A) = f(A')$, which means that there is $x_0 \in \mathbb{R}^{k-1}$ such that $x_0 \cdot A = A'$. But $x \cdot A \in K_i$ only if $x = 0$. Thus $x_0 = 0$ and $A = A'$. Using the continuity of π and p with the fact that π is open, we obtain the bi-continuity of f. This achieves the proof of the lemma and also of the Theorem. □

3.4. **Case of compact Clifford-Klein forms.** We finally describe the deformation and the moduli space for compact Clifford-Klein forms. We have the following:

Theorem 3.11. *Let G be the Heisenberg Lie group of dimension $2n+1$, H a connected Lie subgroup of dimension s and Γ a rank k discontinuous subgroup for G/H. Assume in addition that the Clifford-Klein form $\Gamma\backslash G/H$ is compact. Then*

1) If H contains the center of G, then $k < s$ and

$$\mathcal{T}(\Gamma, G, H) = GL_k(\mathbb{R}) \times M_{p,q}(\mathbb{R})^2 \times Sym(\mathbb{R}^q) \times Sp(p, \mathbb{R})/Sp(p-r, \mathbb{R})$$

and equivalently

$$\mathcal{M}(\Gamma, G, H) = GL_k(\mathbb{R})/GL_k(\mathbb{Z}) \times M_{p,q}(\mathbb{R})^2 \times Sym(\mathbb{R}^q) \times Sp(p, \mathbb{R})/Sp(p-r, \mathbb{R})$$

where for $\mathfrak{h} = \log H$, $q+1 = \dim(\ker b_{|\mathfrak{h}})$, $2p+q+1 = \dim \mathfrak{h}$ and $p+q+r = n$.

2) If H does not contain the center of G then,

$$\mathcal{T}(\Gamma, G, H) = O_{n+1} \times \mathbb{R}^{n+1} \times N_n \times Sym(\mathbb{R}^n)$$

and

$$\mathcal{M}(\Gamma, G, H) = (GL_{n+1}(\mathbb{Z})\backslash GL_{n+1}(\mathbb{R})/\mathbb{R}^n) \times Sym(\mathbb{R}^n).$$

Proof. Note first of all that if Γ is a discontinuous group for G/H and H contains the center of G, then Γ is abelian and so is its syndetic hull L. By proposition (3.1), we get $k < n+1$ where k designates the rank of Γ. If $k = 2n+1-s$ and $k > s$ then obviously $k > n$, which means that either Γ is not a discontinuous group for G/H or Γ is not abelian. So, if $k > s$ and $2n+1-s = k$ then the parameters space is empty.
Assume now that $k < s$ and $2n+1-s = k$ then the set $I_s(2n+1, k)$ is reduced to the element $\alpha_0 = (s+1, \ldots, 2n+1)$. Using Theorem (3.3), we get:

$$\mathcal{T}(\Gamma, G, H) = GL_k(\mathbb{R}) \times \mathscr{V}'_{\alpha_0}.$$

To conclude we just have to prove that

$$\mathscr{V}'_{\alpha_0} = M_{p,q}(\mathbb{R})^2 \times \mathrm{Sym}(\mathbb{R}^q) \times Sp(p, \mathbb{R})/Sp(p-r, \mathbb{R}).$$

Having fixed an adapted basis $\mathscr{B}_{\mathfrak{h}} = \{Z, X_1 \ldots, X_n, Y_1, \ldots, Y_n\}$ of $\mathfrak{g}$ adapted to $\mathfrak{h}$, we consider the vector subspaces,

$$V_1^{'} = \langle X_1, \ldots, X_p \rangle, \quad V_1^{''} = \langle Y_1, \ldots, Y_p \rangle,$$

$$V_0 = \langle X_{p+1}, \ldots, X_{p+q} \rangle, \quad N_0 = \langle Y_{p+1}, \ldots, Y_{p+q} \rangle,$$

$$N_1^{'} = \langle X_{p+q+1}, \ldots, X_n \rangle \quad \text{and} \quad N_1^{''} = \langle Y_{p+q+1}, \ldots, Y_n \rangle.$$

So we have the following decompositions

$$\mathfrak{g} = \mathfrak{z} \oplus V_1^{'} \oplus V_1^{''} \oplus V_0 \oplus N_0 \oplus N_1^{'} \oplus N_1^{''} \quad \text{and} \quad \mathfrak{h} = \mathfrak{z} \oplus V_1^{'} \oplus V_1^{''} \oplus V_0. \tag{13}$$

Any matrix $M \in \mathscr{V}'_{\alpha}$ can be written as

$$M = \begin{pmatrix} 0 & 0 & 0 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \\ I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \begin{matrix} \mathfrak{z} \\ V_1' \\ V_1'' \\ V_0 \\ N_0 \\ N_1' \\ N_1'' \end{matrix}$$

where $A_1, B_1 \in M_{p,q}(\mathbb{R})$, $C_1 \in M_{q,q}(\mathbb{R})$, $A_2, A_3, B_2, B_3 \in M_{p,r}(\mathbb{R})$ and $C_2, C_3 \in M_{q,r}(\mathbb{R})$, for $r = n - p - q$. The matrix of b is

$$J_b = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -I & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -I \\ 0 & 0 & 0 & 0 & 0 & I & 0 \end{pmatrix}$$

and the condition ${}^tMJ_bM = 0$ is equivalent to the following system:

$$\begin{pmatrix} {}^tB_1A_1 - {}^tA_1B_1 + C_1 - {}^tC_1 & {}^tB_1A_2 - {}^tA_1B_2 + C_2 & {}^tB_1A_3 - {}^tA_1B_3 + C_3 \\ {}^tB_2A_1 - {}^tA_2B_1 - {}^tC_2 & {}^tB_2A_2 - {}^tA_2B_2 & {}^tB_2A_3 - {}^tA_2B_3 - I \\ {}^tB_3A_1 - {}^tA_3B_1 - {}^tC_3 & {}^tB_3A_2 - {}^tA_3B_2 + I & {}^tB_3A_3 - {}^tA_3B_3 \end{pmatrix} = 0.$$

This is in turn equivalent to

$$C_2 = {}^tA_1B_2 - {}^tB_1A_2,\ C_3 = {}^tA_1B_3 - {}^tB_1A_3,\ C_1 = \frac{1}{2}({}^tA_1B_1 - {}^tB_1A_1) + D$$

and

$$\begin{pmatrix} {}^tB_2A_2 - {}^tA_2B_2 & {}^tB_2A_3 - {}^tA_2B_3 \\ {}^tB_3A_2 - {}^tA_3B_2 & {}^tB_3A_3 - {}^tA_3B_3 \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \tag{14}$$

where $D \in \mathrm{Sym}(\mathbb{R}^q)$ and $A_1, B_1 \in M_{p,q}(\mathbb{R})$. Let

$$Y = \begin{pmatrix} B_2 & B_3 \\ A_2 & A_3 \end{pmatrix} \in M_{2p,2r}(\mathbb{R})$$

and

$$J_m = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix} \in M_{2m,2m}(\mathbb{R}).$$

Then, the condition (14) can be written as ${}^tYJ_pY = J_r$ and for

$$U = \{Y \in M_{2p,2r}(\mathbb{R}),\ {}^tYJ_pY = J_r\}$$

we easily see that

$$\mathscr{V}'_\alpha \cong M_{p,q}(\mathbb{R})^2 \times \mathrm{Sym}(\mathbb{R}^q) \times U.$$

To conclude, we finally prove the following lemma:

Lemma 3.12. *$U \cong Sp(p, \mathbb{R})/Sp(p-r, \mathbb{R})$.*

Proof. Note first that the symplectic group $\mathrm{Sp}(p, \mathbb{R})$ acts on U by multiplication on the left and its action is transitive. The matrix

$$Y = \begin{pmatrix} I_r & 0 \\ 0 & 0 \\ 0 & I_r \\ 0 & 0 \end{pmatrix}$$

belongs to U and with a direct verification, we get

$$\mathrm{Stab}(Y) = \left\{ P = \begin{pmatrix} I_r & 0 & 0 & 0 \\ 0 & A & 0 & B \\ 0 & 0 & I_r & 0 \\ 0 & C & 0 & D \end{pmatrix}, \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(p-r) \right\} \cong Sp(p-r).$$

□

We pay attention finally to the case where H does not meet $Z(G)$, the center of G. As we are dealing with compact Clifford-Klein forms, we are obviously submitted to write that $s + k = 2n + 1$, which entails that $I_{s+1}(2n+1, k)$ is empty and $I_s^1(2n+1, k)$ is merely reduced to the single element $\alpha_0 = (1, s+2, \ldots, 2n+1)$. As it stands here, the subgroups H and Γ are abelian, and we get by proposition (3.1) that $\dim \mathfrak{h} = n$ and $\operatorname{rank} \Gamma = n + 1$. Then Theorem (3.7) enables us to write that,

$$\mathcal{T}(\Gamma, G, H) = GL_{n+1}(\mathbb{R})/\mathbb{R}^n \times \mathscr{V}_{\alpha_0}.$$

Now every matrix M in $\mathscr{V}_{\alpha_0}$ can be written as

$$M = \begin{pmatrix} 1 & 0 \\ 0 & A \\ 0 & I_n \end{pmatrix}$$

for some $A \in M_n(\mathbb{R})$. The relation ${}^tMJ_bM = 0$ is then equivalent to ${}^tA - A = 0$ and the result follows from Lemma (3.8).

□

A straight consequence of the last theorem, is the following:

Corollary 3.13. *Let G be the Heisenberg Lie group, H a connected Lie subgroup of G and Γ an abelian discontinuous subgroup of G for G/H. Assume in addition that the Clifford-Klein form $\Gamma\backslash G/H$ is compact. Then the deformation space $\mathcal{T}(\Gamma, G, H)$ is endowed with a structure of a differential manifold.*

3.5. **The rigidity, local rigidity and stability.** We study now the topological features of the deformation space in our setting, namely the rigidity, local rigidity and the topological stability. We point out here that we will get as in [10] an example for which the stability holds globally but the rigidity fails to be true. More precisely, we prove the following:

Theorem 3.14. *Let G be the Heisenberg Lie group, H a connected Lie subgroup of G and Γ an abelian discontinuous subgroup of G for G/H. Then $R(\Gamma, G, H)$ is an open set in* $\mathrm{Hom}(\Gamma, G)$. *That is, every element of $R(\Gamma, G, H)$ is stable. In addition there is no isolated point in the deformation space. Actually the rigidity, the local rigidity properties fail to hold for every element in $R(\Gamma, G, H)$.*

Proof. Under the assumption that H contains the center of G which coincides with the first derivative group $[G, G]$, we get thanks to Theorem (2.4) that the parameter space $R(\Gamma, G, H)$ is open in $\mathrm{Hom}(\mathfrak{l}, \mathfrak{g})$ and there is no open isolated point inside the deformation space $\mathcal{T}(\Gamma, G, H)$. We now remove the assumption that H contains the center of G. In this context, by lemma (3.5) we can see that the parameter space

$$R(\Gamma, G, H) = R_1(\Gamma, G, H) \cup R_2(\Gamma, G, H).$$

By lemma (3.6) we see that $R_1(\Gamma, G, H)$ is an open set in $\mathrm{Hom}(\mathfrak{l}, \mathfrak{g})$. Furthermore $R_2(\Gamma, G, H) = R(\Gamma, G, K)$ where $K = HZ(G)$, which also open in $\mathrm{Hom}(\mathfrak{l}, \mathfrak{g})$. Let $[\psi]$ be an open point in $\mathcal{T}(\Gamma, G, H)$. By Theorem (3.7) there exists α in $I_s(2n + 1, k)$ or $\alpha \in I_s^1(2n+1, k)$ such that $[\psi] \in \mathcal{T}_\alpha$. In both cases, $\mathcal{T}_\alpha$ is homeomorphic to a topological space without isolated points. These arguments conclusively lead to the fact that the rigidity fails to hold globally as well. □

4. Examples

To end the paper, we present in this section some enriching examples for which we carry out explicit computations of some chosen layers $\mathcal{T}_\alpha$ and $\mathcal{M}_\alpha$ involved in the description of the deformation and moduli space as we did in the case of compact Clifford-Klein forms where only one single strate occurs. We precise that our computations take into account the precise basis of $\mathfrak{g}$ adapted to $\mathfrak{h}$ and utterly rely on the position of $\mathfrak{h}$ inside $\mathfrak{g}$. All the matrices considered in the following examples are written in a basis $\mathscr{B}_\mathfrak{h}$ of $\mathfrak{g}$ adapted to $\mathfrak{h}$.

Example 1. We Assume in this first example that $\mathfrak{h}$ does not contain the center of $\mathfrak{g}$ and we write accordingly to the notations of proposition (3.1) that $\dim \mathfrak{h} = s$. Take for instance $k = s + 1$ and $\alpha = (1, s + 2, \ldots, 2s + 1)$. Then any $M \in \mathscr{V}_\alpha$ can be written as

$$M = \begin{pmatrix} 1 & 0 \\ 0 & A \\ 0 & I_s \\ 0 & B \\ 0 & C \end{pmatrix}$$

where $A \in M_s(\mathbb{R})$, $B, C \in M_{n-s,s}(\mathbb{R})$. The matrix J_b of the bilinear form b is then given by

$$J_b = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -I_s & 0 & 0 \\ 0 & I_s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -I_{n-s} \\ 0 & 0 & 0 & I_{n-s} & 0 \end{pmatrix}.$$

By a routine computation, we can easily see that the condition ${}^tMJ_bM = 0$ gives rise to the following equation

$$-{}^tA + A - {}^tBC + {}^tCB = 0.$$

So $A = -\frac{1}{2}(-{}^tBC + {}^tCB) + D$, for some $D \in Sym(\mathbb{R}^s)$ and we finally get that

$$\mathcal{T}_\alpha = GL_k(\mathbb{R})/\mathbb{R}^{k-1} \times M^2_{n-s,s}(\mathbb{R}) \times Sym(\mathbb{R}^s)$$
$$\simeq O_k \times \mathbb{R}^k \times N_k \times M^2_{n-s,s}(\mathbb{R}) \times Sym(\mathbb{R}^s)$$

and

$$\mathcal{M}_\alpha = GL_k(\mathbb{Z})\backslash GL_k(\mathbb{R})/\mathbb{R}^{k-1} \times M^2_{n-s,s}(\mathbb{R}) \times Sym(\mathbb{R}^s).$$

We assume henceforth that $\mathfrak{h}$ contains the center of $\mathfrak{g}$ and therefore $\dim \mathfrak{h} = 1 + 2p + q$ according to the notations of proposition (3.1).

Example 2. Take for instance $p + q + k = n$ and $\alpha = (1 + 2p + 2q + k + 1, \ldots, 2n + 1)$. Let $M \in \mathscr{V}'_\alpha$. Then

$$M = \begin{pmatrix} 0 \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ I_k \end{pmatrix},$$

where $A_1,\ A_2 \in M_{p,k}(\mathbb{R}),\ A_3,\ A_4 \in M_{q,k}(\mathbb{R})$ and $A_5 \in M_k(\mathbb{R})$. The matrix of the bilinear form b is

$$J_b = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -I_p & 0 & 0 & 0 & 0 \\ 0 & I_p & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -I_q & 0 & 0 \\ 0 & 0 & 0 & I_q & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -I_k \\ 0 & 0 & 0 & 0 & 0 & I_k & 0 \end{pmatrix}.$$

So the condition ${}^tMJ_bM = 0$ is equivalent to the equation

$$-{}^tA_1A_2 + {}^tA_2A_1 - {}^tA_3A_4 + {}^tA_4A_3 + A_5 - {}^tA_5 = 0.$$

Therefore, for $A_1,\ A_2 \in M_{p,k}(\mathbb{R}),\ A_3,\ A_4 \in M_{q,k}(\mathbb{R})$ we can take

$$A_5 = -\frac{1}{2}(-A_1^t A_2 + {}^tA_2A_1 - {}^tA_3A_4 + {}^tA_4A_3) + D, \text{ with } D \in Sym(\mathbb{R}^k).$$

and then

$$\mathcal{T}_\alpha \cong GL_k(\mathbb{R}) \times M_{p,k}(\mathbb{R})^2 \times M_{q,k}(\mathbb{R})^2 \times Sym(\mathbb{R}^k).$$

Example 3. We still take $p+q+k=n$ and consider $\alpha = (2p+2q+2, \ldots, 2p+2q+k+1)$ and let $M \in \mathscr{V}'_\alpha$. Then

$$M = \begin{pmatrix} 0 \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ I_k \\ A_5 \end{pmatrix},$$

where $A_1,\ A_2 \in M_{p,k}(\mathbb{R}),\ A_3,\ A_4 \in M_{q,k}(\mathbb{R})$ and $A_5 \in M_k(\mathbb{R})$. Then the same calculation as in the first example gives

$$\mathcal{T}_\alpha \cong GL_k(\mathbb{R}) \times M_{p,k}(\mathbb{R})^2 \times M_{q,k}(\mathbb{R})^2 \times Sym(\mathbb{R}^k).$$

Example 4. Assume now that $k = q$ and take $\alpha = (2p+q+2, \ldots, 2p+2q+1)$ and let $M \in \mathscr{V}'_\alpha$. Then

$$M = \begin{pmatrix} 0 \\ A_1 \\ A_2 \\ A_3 \\ I_q \\ A_5 \\ A_6 \end{pmatrix},$$

where $A_1,\ A_2 \in M_{p,q}(\mathbb{R}),\ A_3 \in M_q(\mathbb{R})$ and $A_5, A_6 \in M_{r,q}(\mathbb{R})$ with $r = n-p-q$. So the condition ${}^tMJ_bM = 0$ is equivalent to the equation

$$-{}^tA_1A_2 + {}^tA_2A_1 - {}^tA_3 + A_3 - {}^tA_5A_6 + {}^tA_6A_5 = 0.$$

Then as above we have

$$A_3 = -\frac{1}{2}(-{}^tA_1A_2 + {}^tA_2A_1 - {}^tA_5A_6 + {}^tA_6A_5) + D, \text{ with } D \in Sym(\mathbb{R}^q).$$

We get then that

$$\mathcal{T}_\alpha \cong GL_q(\mathbb{R}) \times M_{p,q}(\mathbb{R})^2 \times M_{r,q}(\mathbb{R})^2 \times Sym(\mathbb{R}^q).$$

Example 5. Assume finally that $k = q+r$ where $p+q+r = n$ and take $\alpha = (2p+q+2, \ldots, 2p+2q+r+1)$. For $M \in \mathscr{V}'_\alpha$, we have

$$M = \begin{pmatrix} 0 & 0 \\ A_1 & B_1 \\ A_2 & B_2 \\ A_3 & B_3 \\ I_q & 0 \\ 0 & I_r \\ A_4 & B_4 \end{pmatrix},$$

where $A_1, A_2 \in M_{p,q}(\mathbb{R})$, $B_1, B_2 \in M_{p,r}(\mathbb{R})$, $A_3 \in M_q(\mathbb{R})$, $B_3 \in M_{q,r}(\mathbb{R})$, $A_4 \in M_{r,q}(\mathbb{R})$ and $B_4 \in M_r(\mathbb{R})$. Then

$${}^tMJ_bM = \begin{pmatrix} -{}^tA_1A_2 + {}^tA_2A_1 - {}^tA_3 + A_3 & -{}^tA_1B_2 + {}^tA_2B_1 + B_3 + {}^tA_4 \\ -{}^tB_1A_2 + {}^tB_2A_1 - {}^tB_3 - A_4 & -{}^tB_1B_2 + {}^tB_2B_1 + {}^tB_4 - B_4 \end{pmatrix}.$$

Then the condition ${}^tMJ_bM = 0$ is equivalent to

$$\begin{aligned} A_3 &= -\tfrac{1}{2}(-{}^tA_1A_2 + {}^tA_2A_1) + D, \quad D \in Sym(\mathbb{R}^q) \\ B_4 &= -\tfrac{1}{2}({}^tB_1B_2 - {}^tB_2B_1) + D', \quad D' \in Sym(\mathbb{R}^r) \\ -A_4 &= {}^tB_3 + {}^tB_1A_2 - {}^tB_2A_1. \end{aligned}$$

We obtain therefore that:

$$\mathcal{T}_\alpha = GL_k(\mathbb{R}) \times M_{p,q}(\mathbb{R})^2 \times M_{p,r}(\mathbb{R})^2 \times M_{q,r}(\mathbb{R}) \times Sym(\mathbb{R}^q) \times Sym(\mathbb{R}^r).$$

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