STABILITY ON KÄHLER-RICCI FLOW ON A COMPACT KÄHLER MANIFOLD WITH POSITIVE FIRST CHERN CLASS; I

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ABSTRACT. In this paper, we prove that Kähler-Ricci flow converges to a Kähler-Einstein metric (or a Kähler-Ricci soliton) in the sense of Cheeger-Gromov as long as an initial Kähler metric is very closed to g_{KE} (or g_{KS}) if a compact Kähler manifold with $c_1(M) > 0$ admits a Kähler Einstein metric g_{KE} (or a Kähler-Ricci soliton g_{KS}). The result improves Main Theorem in [TZ3] in the sense of stability of Kähler-Ricci flow.

0. INTRODUCTION

The Ricci flow was first introduced by R. Hamilton in [Ha]. If the underlying manifold M is Kähler with positive first Chern class $c_1(M) > 0$, it is more natural to study the following Kähler-Ricci flow (normalized),

$$\frac{\partial g(t,\cdot)}{\partial t} = -\mathrm{Ric}(g(t,\cdot)) + g(t,\cdot),$$
$$g(0,\cdot) = g, \tag{0.1}$$

where g is an initial Kähler metric with its Kähler form $\omega_g \in 2\pi c_1(M) > 0$. It can be shown that (0.1) preserves the Kähler class. Moreover, (0.1) has a global solution $g_t = g(t,\cdot)$ for any $t > 0$ ([Ca]). So, the main interest and difficulty of (0.1) is to study the limiting behavior of g_t as t tends to ∞ (cf. [CT1], [CT2], [TZ3], etc.).

In this paper, we study a stability problem of Kähler-Ricci flow (0.1), namely, we assume that M admits a Kähler-Einstein metric or a Kähler-Ricci soliton, and then we analysis the behavior of evolved Kähler metrics g_t of (0.1). We shall prove

Theorem 0.1 (Main Theorem). *Let M be a compact Kähler manifold with $c_1(M) > 0$ which admits a Kähler Einstein metric g_{KE} (or a Kähler-Ricci soliton (g_{KS,X_0}) with respect some holomorphic vector field X_0 on M) with*

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its Kähler form in $2\pi c_1(M)$. Let ψ be a Kähler potential of an initial metric g of (0.1) and $\varphi=\varphi_t$ be a family of Kähler potentials of evolved metrics g_t of (0.1), i.e., $\omega_g=\omega_{KE}+\sqrt{-1}\partial\bar\partial\psi$ (or $\omega_g=\omega_{KS}+\sqrt{-1}\partial\bar\partial\psi$) and $\omega_\varphi=\omega_{KE}+\sqrt{-1}\partial\bar\partial\varphi$ (or $\omega_g=\omega_{KS}+\sqrt{-1}\partial\bar\partial\varphi$), where ω_g, $\omega_\varphi=\omega_{g_t}$ and ω_{KE} (or ω_{KS}) denote Kähler forms of g, g_t and g_{KE} (or g_{KS}), respectively. Then there exists a small ϵ such that if

$$\|\psi-\underline{\psi}\|_{C^{2,\alpha}}\le\epsilon,$$

where $\underline{\psi}=\frac{1}{\int_M\omega_{KE}^n}\int_M\psi\omega_{KE}^n$ (or $\underline{\psi}=\frac{1}{\int_M\omega_{KS}^n}\int_M\psi\omega_{KS}^n$), then there exist a family of holomorphisms $\sigma=\sigma_t$ on M such that Kähler potentials $(\varphi_\sigma-\underline{\varphi_\sigma})$ are C^k-norm uniformly bounded, where $\varphi_\sigma=\sigma^*\varphi+\rho$ and $\rho=\rho_t$ are Kähler potentials defined by $\rho^*(\omega_{KE})=\omega_{KE}+\sqrt{-1}\partial\overline{\partial}\rho$ and $\int_M e^{-\rho}\omega_{KE}^n=\int_M\omega_{KE}^n$ (or $\rho^*(\omega_{KS})=\omega_{KS}+\sqrt{-1}\partial\overline{\partial}\rho$ and $\int_M e^{-\rho-X_0(\rho)}\omega_{KS}^n=\int_M\omega_{KS}^n$). As a consequence, g_t converge to g_{KE} (or g_{KS}) smoothly in the sense of Cheeger-Gromov.

The main step in the proof of Theorem 0.1 is to obtain a decay estimate for $\dot\varphi$ and φ both when one studies the convergence of Kähler-Ricci flow as in [CT2], [PS], [TZ3] etc. In case that M admits a Kähler Einstein metric or M admits a Kähler-Ricci soliton and an initial potential ψ is K_{X_0}-invariant, we can obtain an exponential decay estimate for both $\dot\varphi$ and φ, so we can improve that Kähler potentials $(\sigma^*\varphi+\rho)$ in the theorem exponentially converge to zero as long as $\|\psi-\underline{\psi}\|_{C^{2,\alpha}}$ is small (cf. Theorem 1.1 and Theorem 2.1). Here K_{X_0} denotes an one-parameter compact subgroup generated by the imaginary part X' of X_0 ([TZ1],[TZ2]). This result is also obtained in [TZ3] where a crucial step is to use the monotonicity and the properness of the Mabuchi's K-energy on a Kähler-Einstein manifold with $c_1(M)>0$ (or the monotonicity and the properness of the generalized K-energy on a compact Kähler manifold which admits a Kähler-Ricci soliton , cf. [CTZ]). But at the present paper, we avoid to use these energies in our case of the stability problem. This advantage allows us to remove the K_{X_0}-invariant condition for the initial potential ψ in case of Kähler-Ricci soliton in Theorem 0.1, although we need more careful computations than the case of Kähler-Einstein metric. Basically, we shall use the generalized Futaki-invariant and the Gauge Transformation induced by the reductive subgroup $\mathrm{Aut}_r(M)$ of holomorphisms transformation group $\mathrm{Aut}(M)$ on M to control the modified Kähler potentials $(\sigma^*\varphi+\rho)$ along the Kähler-Ricci flow. We note that the definition of generalized Futaki-invariant is independent of the choice of Kähler metric, which needs no K_{X_0}-invariant condition ([TZ2]). Unfortunately, we could not improve the convergence of $(\sigma^*\varphi+\rho)$ exponentially without the assumption of K_{X_0}-invariant condition. But we believe that it is still true if one can extend the Gauge Transformation $\mathrm{Aut}_r(M)$ to $\mathrm{Aut}(M)$ (cf. Proposition 2.10).

Theorem 0.1 will be proved in Section 1 and Section 2 while in Section 1 we consider the case of Kähler-Einstein metric and, in Section 2 we consider the case of Kähler-Ricci soliton. The rest of paper is as follows: In Section 3, we prove a uniqueness result for the limit of Kähler-Ricci flow as an application of Theorem 0.1; Section 4 and Section 5 are two appendixes, one is a lemma about a $W^{k,2}$-estimate for $\dot\varphi_t$ and another is a lemma about the existence of almost orthonormality of a Kähler potential to the space of first eigenvalue-functions of operator (P, ω_{KS}) defined in Lemma 2.2 in Section 2.

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1. In case of Kähler-Einstein metric

In this section, we assume that M admits a Kähler Einstein metric g_{KE} with its Kähler form $\omega_{KE} \in 2\pi c_1(M)$. For simplicity, we set a class of Kähler potentials by

$$\mathcal{M}(\omega_{KE}) = \{\phi \in C^\infty(M, R) |\ \omega_{KE} + \sqrt{-1}\partial\bar\partial\phi > 0\}.$$

Then for any Kähler metric g with its Kähler form $\omega_g \in 2\pi c_1(M)$, we have $\omega_g = \omega_{KE} + \sqrt{-1}\partial\bar\partial\psi$ for some $\psi \in \mathcal{M}(\omega_{KE})$ and Kähler-Ricci flow (0.1) is equivalent to a parabolic equation of complex Monge-Ampère type for Kähler potentials $\varphi_t = \varphi(t, \cdot)$ with $\omega_{g_t} = \omega_{KE} + \sqrt{-1}\partial\bar\partial\varphi_t$,

$$\frac{\partial\varphi}{\partial t} = \log\frac{\omega_\varphi^n}{\omega_{KE}^n} + \varphi,$$

$$\varphi(0) = \psi - \underline{\psi}, \tag{1.1}$$

where $\underline{\psi} = \frac{1}{V}\int_M \psi\omega_{KE}^n$ and $V = \int_M \omega_{KE}^n$.

Set a Hölder space by

$$\mathcal{K}(\epsilon_0) = \{\phi \in \mathcal{M}(\omega_{KE}) |\ \|\phi - \underline{\phi}\|_{C^{2,\alpha}} \le \epsilon_0\}.$$

Let $\mathrm{Aut}_0(M)$ be the connected component of holomorphisms transformation group of M which contains the identity map of M. Then we shall prove

Theorem 1.1. *There exists a small ϵ such that for any initial data $\psi \in \mathcal{K}(\epsilon)$ in equation (1.1), $\|\varphi - \underline{\varphi}\|_{C^{2,\alpha}}$ are uniformly bounded, where $\varphi = \varphi_t = \varphi(t, \cdot)$ are evolved Kähler potentials of (1.1). Moreover, there exist a family of $\sigma = \sigma_t \in Aut_0(M)$ such that Kähler potentials $(\varphi_\sigma - \underline{\varphi_\sigma})$ converge exponentially* *to 0 as $t \to \infty$, where $\varphi_\sigma = (\sigma^*\varphi + \rho)$, and $\rho = \rho_t$ are Kähler potentials defined by*

$$\sigma^*(\omega_{KE}) = \omega_{KE} + \sqrt{-1}\partial\overline{\partial}\rho,$$
$$\int_M e^{-\rho}\omega_{KE}^n = \int_M \omega_{KE}^n. \tag{1.2}$$

As a consequence, Kähler metrics $\sigma^*(\omega_\varphi)$ converge exponentially to ω_{KE}.

We need several lemmas to prove Theorem 1.1. Let $\Lambda_1(M, \omega_{KE})$ be a finite dimensional linear space of the first eigenvalue-functions of Lapalace operator $\triangle_{\omega_{KE}}$ associated to the metric ω_{KE}. Then by using the Bochner formula, it is well-known that the first non-zero eigenvalue is 1 and $\Lambda_1(M, \omega_{KE}) = \mathrm{span}\{\theta_X |\ X \in \eta(M)\}$, where $\eta(M)$ is a linear space consisting of holomorphic vector fields on M which is isomorphic to the Lie algebra of $\mathrm{Aut}_0(M)$ and θ_X is a potential of X defined by

$$\sqrt{-1}\overline{\partial}\theta_X = i_X(\omega_{KE}),$$
$$\int_M \theta_X \omega_{KE}^n = 0. \tag{1.3}$$

By using the continuity of eigenvalues of Lapalcian operators, one sees

Lemma 1.2. *Let $\lambda_1(\omega_\phi)$ and $\lambda_2(\omega_\phi)$ be the first and the second eigenvalues of Lapalcian operator associated to Kähler metric ω_ϕ, respectively. Then there exists a δ_0 such that for any $\phi \in \mathcal{K}(\epsilon_0)$, we have*

$$\lambda_1(\omega_\phi) \geq 1 + \delta_0, \ \text{if } \eta(M) = 0,$$
$$\lambda_2(\omega_\phi) \geq 1 + \delta_0, \ \text{if } \eta(M) \neq 0,$$

where ϵ_0 is a small positive number.

Fix a large number T and N, we can choose a sufficient small ϵ depends on T, ϵ_0 and N such that for any $t \leq T$, evolved Kähler potentials φ_t of (1.1) lie in $\mathcal{K}(\frac{\epsilon_0}{2})$ and satisfy

$$|\dot\varphi_t - c(t)|_{C^0} \leq (\frac{\epsilon_0}{2N})^2, \text{ and } osc(\varphi_t) \leq \frac{\epsilon_0}{4N}, \tag{1.4}$$

whenever $\|\psi - \underline{\psi}\|_{C^{2,\alpha}} \leq \epsilon$. Here $c(t) = \frac{1}{V}\int_M \dot\varphi_t \omega_{\varphi_t}^n$. Choose a maximal $\delta(T)$ such that $\varphi_t \in \mathcal{K}(\epsilon_0)$ for any $t < T + \delta(T)$. We shall show that $\delta(T)$ must be the infinity whenever T and N are large enough. First we prove

Lemma 1.3. *Let $H(t) = \frac{1}{V}\int_M |\dot\varphi_t - c(t)|^2 \omega_{\varphi_t}^n$. Then for any $t \in [0, T + \delta(T))$, there exists a $\theta > 0$ such that*

$$H(t) \leq H(0)e^{-\theta t}. \tag{1.5}$$

Proof. For simplicity, we let $\varphi = \varphi_t$. By (1.1), one sees

$$|\dot{\varphi}| \leq 3\epsilon_0, \ \forall t \in [0, T + \delta(T)).$$

Since φ satisfies,

$$\ddot{\varphi} = \triangle\dot{\varphi} + \dot{\varphi}, \tag{1.6}$$

then by a direct computation, we have

$$\begin{aligned}
&\frac{d}{dt}H_0(t)\\
&= 2\frac{1}{V}\int_M (\dot{\varphi} - c(t))(\ddot{\varphi} - \dot{c}(t))\omega_\varphi^n + \frac{1}{V}\int_M (\dot{\varphi} - c(t))^2 \triangle_\varphi \dot{\varphi}\omega_\varphi^n\\
&= 2\frac{1}{V}\int_M (\dot{\varphi}_t - c(t))(\triangle_\varphi \dot{\varphi} + \dot{\varphi})\omega_\varphi^n + \frac{1}{V}\int_M (\dot{\varphi} - c(t))^2 \triangle_\varphi \dot{\varphi}\omega_\varphi^n\\
&= -2\frac{1}{V}\int_M |\nabla(\dot{\varphi} - c(t))|^2\omega_\varphi^n + 2\frac{1}{V}\int_M (\dot{\varphi} - c(t))^2\omega_\varphi^n\\
&- 2\frac{1}{V}\int_M (\dot{\varphi} - c(t))\|\nabla(\dot{\varphi} - c(t))\|^2\omega_\varphi^n\\
&= 2H_0(t) - 2\frac{1}{V}\int_M (1 + \dot{\varphi} - c(t))\|\nabla(\dot{\varphi} - c(t))\|^2\omega_\varphi^n\\
&\leq 2H_0(t) - 2(1 - 6\epsilon_0)\frac{1}{V}\int_M \|\nabla(\dot{\varphi} - c(t))\|^2\omega_\varphi^n. \qquad (1.7)
\end{aligned}$$

Case 1, $\eta(M) = 0$. Then by (1.7) and Lemma 1.1, we have

$$\frac{d}{dt}H_0(t) \leq -[-2 + 2(1 - 6\epsilon_0)(1 + \delta_0)]H_0(t).$$

By choosing $\theta = -2 + 2(1 - 4\epsilon_0)(1 + \delta_0) \geq \delta_0$, we will get

$$H_0(t) \leq H_0(0)e^{-\theta t}. \tag{1.8}$$

Case 2, $\eta(M) \neq 0$. Since the Futaki-invariant vanishes, for any $X \in \eta(M)$, we have

$$\int_M \triangle(\theta_X + X(\varphi))(\dot{\varphi} - c(t))\omega_\varphi^n = 0,$$

where θ_X is the potential of X defined by (1.3) and $X(\phi)$ is the derivative of ϕ along X. It follows

$$|\int_M \theta_X(\dot{\varphi} - c(t))\omega_\varphi^n| \leq C\epsilon_0 \int_M |\dot{\varphi} - c(t)|\omega_\varphi^n,$$

for any $X \in \eta(M)$ with satisfying $\int_M \|X\|^2_{\omega_{KE}}\omega_{KE}^n = 1$. Here we used an estimate

$$\|\varphi - \underline{\varphi}\|_{C^3} = O(\epsilon_0)$$

by the regularity of Kähler potentials $\varphi \in \mathcal{K}(\epsilon_0)$, which can be obtained by the Implicity Functional Theorem for equation (1.22) (cf. an argument at the last paragraph of this section) with the help of $W^{k,2}$-estimate

$$\|\dot{\varphi} - c(t)\|_{W^{k,2}} = O(\epsilon_0)$$

for $\dot{\varphi}$ (cf. the argument in Appendix 1). Thus by using the continuity of the eigenvalue functions, one sees

$$|\int_M \psi^i(\dot{\varphi} - c(t))\omega_\varphi^n| \le C'\epsilon_0 \int_M |\dot{\varphi} - c(t)|\omega_\varphi^n, \tag{1.9}$$

where ψ^i are the first eigenvalue functions of the Lapalacian operator associated to the metric ω_φ, which satisfy $\int_M |\psi^i|^2 \omega_\varphi^n = 1$.

Let $\Lambda_1(M, \omega_\varphi)$ be a linear space spanned by a basis $\{\psi^i\}$ and $\Lambda_1^\perp(M, \omega_\varphi)$ be a subspace of L^2-integral functions which are orthogonal to $\Lambda_1(M, \omega_\varphi) \cup \mathbb{R}$. Then we can decompose $(\dot{\varphi}_t - c(t))$ as

$$\dot{\varphi}_t - c(t) = \phi + \phi^\perp,$$

with $\phi \in \Lambda_1(M, \omega_\varphi)$ and $\phi^\perp \in \Lambda_1^\perp(M, \omega_\varphi)$. Thus by (1.9), we get

$$\int_M |\phi|^2 \omega_\varphi^n \le A\epsilon_0^2 \int_M (\dot{\varphi}_t - c(t))^2 \omega_\varphi^n,$$

for some uniform constant A. It follows

$$\int_M |\phi^\perp|^2 \omega_\varphi^2 \ge (1 - A\epsilon_0^2) \int_M (\dot{\varphi}_t - c(t))^2 \omega_\varphi^2. \tag{1.10}$$

Hence by Lemma 1.1, we get

$$\begin{aligned} \int_M \|\nabla(\dot{\varphi}_t - c(t)\|^2 \omega_\varphi^n &\ge \int_M \|\nabla \psi^\perp\|^2 \omega_\varphi^n \\ &\ge (1+\sigma_0)(1 - A\epsilon_0^2) \int_M (\dot{\varphi}_t - c(t))^2 \omega_\varphi^n. \end{aligned} \tag{1.11}$$

By choosing $\theta = 2(1 - 6\epsilon_0)(1+\sigma_0)(1 - C\epsilon_0^2) - 2 \ge \sigma_0$, we obtain from (1.7),

$$\frac{dH_0(t)}{dt} \le -\theta H_0(t).$$

As a consequence, we have

$$H_0(t) \le H_0(0)e^{-\theta t}.$$

□

Next we want to use Perelman's deep estimates for the gradient of $\dot{\varphi}_t$ and the non-collapsing result for metric ω_{φ_t} to get a C^0-estimate of $\dot{\varphi}_t$ with help of Lemma 1.3. Let's state the Perelman's result (a detailed proof can be found in [ST]).

Lemma 1.4. *(Perelman) Let g_t be the evolved Kähler metrics of (0.1) and $\varphi = \varphi_t$ be Kähler potentials of g_t. Then there exists a uniform constant C independent of t (just depending on the initial metric g) such that the following two facts hold,*

i) $\|\nabla\dot{\varphi}\|_{\omega_\varphi} \le C$;

ii) for $x \in M$ and $0 < r \le diam(M, g(t))$, $\int_{B_r(x)} \omega^n > C^{-1} r^{2n}$, where $diam(M, g(t))$ denote the diameters of $(M, g(t))$ which are uniformly bounded.

Lemma 1.5. *For any $t \in [0, T + \delta(T))$, we have*

$$|\dot{\varphi}_t - c(t)| \le \min\{(\frac{\epsilon_0}{2N})^2, Ce^{-\frac{\theta}{2(n+1)}t}\} \tag{1.12}$$

and

$$||\dot{\varphi}_t - c(t)||_{C^\alpha} \le C\{\min\{(\frac{\epsilon_0}{2N})^2, Ce^{-\frac{\theta}{2(n+1)}t}\}\}^{\frac{1}{2}}. \tag{1.13}$$

Here $\alpha \le \frac{1}{4}$ and C depends only on the constant in Lemma 1.4.

Proof. We suffice to consider the case of $\alpha = \frac{1}{4}$. Let x_0 be the point where $|\tilde{h}| = |\tilde{h}_t| = |\dot{\varphi}_t - c(t)|$ achieves its maximum. Choose a small ball $B_r(x_0)$ for $r = \min\{\mathrm{diam}(M, g(t)), e^{-\frac{\theta}{2(n+1)}t}\}$. So we have for any $x \in B_r(x_0)$,

$$0 \le |\tilde{h}(x_0)| \le |\tilde{h}(x)| + \|\nabla\tilde{h}\| r = |\tilde{h}(x)| + \|\nabla h\| r. \tag{1.14}$$

Case 1), $e^{-\frac{\theta}{2(n+1)}t} \ge \mathrm{diam}(M, g(t))$. Then by Lemma 1.4, one sees

$$\begin{aligned}\int_M |\tilde{h}(x_0)|^2 \omega_\varphi^n &\le 2\int_M |\tilde{h}(x)|^2 \omega_\varphi^n + 2V \mathrm{diam}(M, g(t))^2 \|\nabla\tilde{h}\|^2 \\ &\le Ce^{-\frac{\theta}{n+1}t}.\end{aligned}$$

Thus

$$|\tilde{h}(x_0)| \le Ce^{-\frac{\theta}{2(n+1)}t}. \tag{1.15}$$

Case 2), $e^{-\frac{\theta}{2(n+1)}t} \le \mathrm{diam}(M, g(t))$. Then

$$\begin{aligned}\frac{1}{V(B_r(x_0))}\int_{B_r(x_0)} |\tilde{h}(x_0)|^2 \omega_\varphi^n &\le \frac{2}{V(B_r(x_0))}\int_{B_r(x_0)} |\tilde{h}(x)|^2 \omega_\varphi^n \\ &\quad + \frac{2}{V(B_r(x_0))}\int_{B_r(x_0)} \|\nabla\tilde{h}\|^2 r^2 \omega_\varphi^n.\end{aligned}$$

Thus by Lemma 1.4, we get

$$\begin{aligned}|\tilde{h}(x_0)|^2 &\le Ce^{\frac{2n\theta}{2(n+1)}t}\int_M |\tilde{h}(x)|^2 \omega_\varphi^n + Cr^2 \\ &\le Ce^{-\frac{\theta}{n+1}t}.\end{aligned}$$

It follows

$$|\tilde{h}(x_0)| \le C' e^{-\frac{\theta}{2(n+1)}t}. \tag{1.16}$$

Therefore, both (1.15) and (1.16) give the estimate (1.12).

For any $x, y \in M$, by (1.12), we have: if $\mathrm{dist}(x,y) = \|x-y\|_{\omega_\varphi} \geq e^{-\frac{\theta}{2(n+1)}t}$,

$$\begin{aligned}
\frac{|\tilde{h}(x)-\tilde{h}(y)|}{\|x-y\|_{\omega_{KE}}^{\frac{1}{4}}} &\leq 2\frac{|\tilde{h}(x)-\tilde{h}(y)|}{\|x-y\|_{\omega_\varphi}^{\frac{1}{4}}} \\
&\leq 2|\tilde{h}(x)-\tilde{h}(y)|^{\frac{1}{2}}\frac{|\tilde{h}(x)-\tilde{h}(y)|_{\omega_\varphi}^{\frac{1}{4}}}{\|x-y\|_{\omega_\varphi}^{\frac{1}{4}}} \\
&\leq C\{\min\{(\frac{\epsilon_0}{2N})^2, Ce^{-\frac{\theta}{2(n+1)}t}\}\}^{\frac{1}{2}}; \qquad (1.17)
\end{aligned}$$

if $\mathrm{dist}(x,y) \leq e^{-\frac{\theta}{2(n+1)}t}$,

$$\begin{aligned}
\frac{|\tilde{h}(x)-\tilde{h}(y)|}{\|x-y\|_{\omega_{KE}}^{\frac{1}{4}}} &\leq 2\frac{|\tilde{h}(x)-\tilde{h}(y)|}{\|x-y\|_{\omega_\varphi}^{\frac{1}{4}}} \\
&= 2\frac{|\tilde{h}(x)-\tilde{h}(y)|^{\frac{1}{2}}|\tilde{h}(x)-\tilde{h}(y)|^{\frac{1}{2}}}{\|x-y\|_{\omega_\varphi}^{\frac{1}{2}}}\|x-y\|_{\omega_\varphi}^{\frac{1}{4}} \\
&\leq C|\tilde{h}|_{C^0}^{\frac{1}{2}}(\mathrm{diam}(M,g(t)))^{\frac{1}{4}} \\
&\leq C'\{\min\{(\frac{\epsilon_0}{2N})^2, Ce^{-\frac{\theta}{2(n+1)}t}\}\}^{\frac{1}{2}}. \qquad (1.18)
\end{aligned}$$

Here we used Perelman's estimates again. (1.17) and (1.18) give the estimate (1.13). □

Remark 1.6. *We can avoid to use Perelman's estimates to prove Lemma 1.5 by replacing to estimate the* $W^{k,2}$*-norm of* $\dot{\varphi}_t$*. See Appendix 1 in this paper.*

Proposition 1.7. *Choose some large* T *such that*

$$C\frac{4(n+1)}{\theta}e^{-\frac{\theta}{2(n+1)}T} \leq \frac{\epsilon_0}{4N},$$

where C *is the constant chosen in Lemma 1.5. Then*

$$|\tilde{\varphi}| \leq \frac{3\epsilon_0}{4N}, \ \forall\ t \in [0, T+\delta(T)), \qquad (1.19)$$

where $\tilde{\varphi} = \tilde{\varphi}_t = \varphi(t) - \frac{1}{V}\int_M \varphi\omega_\varphi^n$.

Proof. Notice that

$$\frac{d}{dt}\tilde{\varphi} = \tilde{h} - \frac{1}{V}\int_M \tilde{h}\triangle_\varphi\varphi\omega_\varphi^n.$$

Then by Lemma 1.5, we have

$$\begin{aligned}\tilde{\varphi} &= \tilde{\varphi}_T + \int_T^{T+\delta(T)} \tilde{h} dt - \int_T^{T+\delta(T)} \frac{1}{V}\int_M \tilde{h}\triangle_\varphi \varphi\omega_\varphi^n dt \\ &\le \frac{\epsilon_0}{2N} + C\int_T^{T+\delta(T)} e^{-\frac{\theta}{2(n+1)}t}dt + 2C\epsilon_0\int_T^{T+\delta(T)} e^{-\frac{\theta}{2(n+1)}t}dt \\ &\le \frac{\epsilon_0}{2N} + 2C\frac{2(n+1)}{\theta}e^{-\frac{\theta}{2(n+1)}T}.\end{aligned}$$

□

Proof of Theorem 1.1. First we want to show that $\varphi_t \in \mathcal{K}(\epsilon_0)$ for any $t > 0$. By the contradiction, we may assume that there exists a number $\delta(T) < \infty$ such that $\varphi_t \in \mathcal{K}(\epsilon_0)$ for any $t < T + \delta(T)$ and there exists a sequence of $t_i \to T + \delta(T)$ such that

$$\|\overline{\varphi_{t_i}}\|_{C^{2,\alpha}} = \|\varphi_{t_i} - \underline{\varphi_{t_i}}\|_{C^{2,\alpha}} \to \epsilon_0. \tag{1.20}$$

Let b_t be a constant so that $\overline{\varphi} = \tilde{\varphi} + b_t$. Then by Proposition 1.7, it is easy to see $b_t \le \frac{2\epsilon_0}{N}$. Decompose $\overline{\varphi}$ by $\overline{\varphi} = \phi + \phi^\perp$, where $\phi \in \Lambda_1(M, \omega_{KE})$ and $\phi^\perp \in \Lambda_1^\perp(M, \omega_{KE})$, where $\Lambda_1^\perp(M, \omega_{KE})$ is a subspace of L^2-integral functions which are orthogonal to $\Lambda_1(M, \omega_{KE}) \cup \mathbb{R}$. Thus $\phi = \sum_i a_i\theta_i$ for some constants a_i, where θ_i is a basis of the space $\Lambda_1(M, \omega_{KE})$. As a consequence, by Proposition 1.7, we have $|a_i| \le \frac{2\epsilon_0}{N}$, so

$$\|\phi\|_{C^{2,\alpha}} \le \frac{A_0\epsilon_0}{N}, \tag{1.21}$$

for some uniform constant A_0.

By equation (1.1), we have

$$\omega_\varphi^n = \omega_{KE}^n e^{\tilde{h}+\overline{\varphi}+a}, \tag{1.22}$$

where $\tilde{h} = \dot{\phi}_t - c_t$ and $a = a_t$ are constants. By Lemma 1.5 and Proposition 1.7, it is easy to see that $|a| \le \frac{4A_0\epsilon_0}{N}$. Let P be a projection from Banach space $H^{2,\alpha}(M)$ to Banach space $H^\alpha(M) \cap \Lambda_1^\perp(M, \omega_{KE})$. Then $\phi^\perp$ is a solution of equation

$$P[\log(\frac{[\omega_{\phi+\phi^\perp}]^n}{\omega_{KE}^n})] - \phi^\perp = P(\tilde{h} + a),$$

where ϕ and $\tilde{h}+a$ are regarded as two perturbation functions. On the other hand, by Lemma 1.5, we have

$$\|P(\tilde{h}+a)\|_{C^\alpha} = \|P(\tilde{h})\|_{C^\alpha} \le C\min\{(\frac{2\epsilon_0}{N})^2, Ce^{-\frac{\theta}{2(n+1)}t}\}^{\frac{1}{2}}.$$

Thus by using the Implicity Functional Theorem, we get

$$\|\phi^\perp\|_{C^{2,\alpha}} \le c = O(\frac{\epsilon_0}{N}), \tag{1.23}$$

where constant c is independent of t and ϵ_0 and goes to zero as $N \to \infty$. Consequently, $c \leq \frac{\epsilon_0}{4}$ by choosing a large number N. Hence by combining (1.21) and (1.23), we obtain

$$\|\overline{\varphi}\|_{C^{2,\alpha}} \leq \frac{\epsilon_0}{2}. \forall\ t \in [T, T+\delta(T)). \tag{1.24}$$

But this is impossible according to (1.20). Therefore we prove that $\varphi_t \in \mathcal{K}(\epsilon_0)$ for any $t > 0$.

By the above argument and lemma 1.5 and Proposition 1.7, we conclude that there exists an ϵ such that if $\|\psi - \underline{\psi}\|_{C^{2,\alpha}} \leq \epsilon$, then for any $t > 0$, we have

$$\text{a) } \varphi_t \in \mathcal{K}(\epsilon_0), \tag{1.25}$$

$$\text{b) } |\tilde{\varphi}| \leq \frac{3\epsilon_0}{4N}, \tag{1.26}$$

$$\text{c) } \|\tilde{h}\|_{C^\alpha} \leq C\{\min\{(\frac{\epsilon_0}{2N})^2, Ce^{-\frac{\theta}{2(n+1)}t}\}\}^{\frac{1}{2}}. \tag{1.27}$$

On the other hand, according to [BM], one can choose an element $\sigma_t \in \mathrm{Aut}_0(M)$ for each φ such that potential $(\varphi_\sigma - \underline{\varphi_\sigma})$ lies in $\Lambda_1^\perp(M, \omega_{KE})$, where $\varphi_\sigma = \varphi_{\sigma_t} = \varphi_t(\sigma_t(.)) + \rho_t(.)$ and ρ_t is Kähler potential defined by (1.2). Furthermore, by the fact $\varphi \in \mathcal{K}(\epsilon_0)$, one can prove easily

$$\mathrm{dist}(\sigma, Id) \leq 1.$$

Consequently, by (1.27), we have

$$\|\tilde{h}(\sigma_t(.))\|_{C^\alpha} \leq Ce^{-\frac{\theta}{(n+1)}t}.$$

Thus by applying the Implicity Functional Theorem to the modified equation of (1.22),

$$\omega_{\varphi_\sigma}^n = \omega_{KE}^n e^{\tilde{h}(\sigma_t(.)) - \varphi_\sigma + a},$$

we have

$$\|\varphi_\sigma - \underline{\varphi_\sigma}\|_{C^{2,\alpha}} \leq C(\|\tilde{h}(\sigma_t(.))\|_{C^\alpha}).$$

Furthermore, one can get an explicit estimate

$$\|\varphi_\sigma - \underline{\varphi_\sigma}\|_{C^{2,\alpha}} \leq 2\|\tilde{h}(\sigma_t(.))\|_{C^\alpha} \leq C'e^{-\frac{\theta}{(n+1)}t}.$$

To get higher-order estimates for the modified Kähler potentials φ_ρ, one can use Lemma 4.1 in Appendix 1 and the embedding theory of Sobolev spaces to obtain

$$\|\tilde{\dot{\varphi}}\|_{C^{k,\alpha}} \leq C_k e^{-\frac{\theta}{n+1}t},\ \forall\ t > 0,$$

where constants C_k depends only on k, ϵ_0 and higher-order derivatives of the initial Kähler potential ψ (we may assume that ψ is smooth since we can replace it by an evolved Kähler metric $\varphi_{t=1}$). Then by the Implicity Functional Theorem as the above, we derive

$$\|\varphi_\sigma - \underline{\varphi_\sigma}\|_{C^{k+2,\alpha}} \leq 2\|\tilde{\dot{\varphi}}\|_{C^{k,\alpha}} \leq C'_k e^{-\frac{\theta}{n+1}t}.$$

Therefore we prove that Kähler metrics $\sigma^*(\omega_\varphi)$ converge exponentially to ω_{KE}.

□

2. In case of Kähler-Ricci soliton

In this section, we assume that M admits a Kähler Ricci soliton (ω_{KS}, X_0) with some holomorphic vector field X_0 on M, i.e., (ω_{KS}, X_0) satisfies equation,

$$\mathrm{Ric}(\omega_{KS}) - \omega_{KS} = L_{X_0}\omega_{KS},$$

where L_{X_0} is a Lie derivative along the vector field X_0. By the Hodge theorem, one can define a real-valued potential θ_X of X_0 by

$$L_{X_0}\omega_{KS} = \sqrt{-1}\partial\overline{\partial}\theta_X,$$
$$\int_M e^{\theta_X}\omega_{KS}^n = \int_M \omega_{KS}^n.$$

So if we let X and X' be a real part and imaginary part of X_0, respectively, then for any $\phi \in \mathcal{M}(\omega_{KS})$, we have

$$L_{X_0}\omega_\phi = \sqrt{-1}\partial\overline{\partial}(\theta_X + X_0(\phi)), \tag{2.1}$$

and so

$$L_X\omega_\phi = \sqrt{-1}\partial\overline{\partial}(\theta_X + X(\phi))$$

and

$$L_{X'}\omega_\phi = \sqrt{-1}\partial\overline{\partial}(X'(\phi)).$$

(2.1) also implies that for any $\psi \in C^\infty(M)$ it holds

$$< \overline{\partial}(\theta_X + X_0(\phi)), \overline{\partial}\psi >_{\omega_\phi} = X_0(\psi) = X(\psi) + \sqrt{-1}X'(\psi).$$

Thus

$$| < \nabla(\theta_X + X(\phi)), \nabla\psi >_{\omega_\phi} - X(\psi)| \leq |X'(\phi)|\|\nabla\psi\|_{\omega_\phi}. \tag{2.2}$$

We now consider a modified equation of (0.1),

$$\frac{\partial g(t,\cdot)}{\partial t} = -\mathrm{Ric}(g) + g + L_X g,$$
$$g(0) = g. \tag{2.3}$$

Then (2.3) is equivalent to a parabolic equation of complex Monge-Ampère type,

$$\frac{\partial\varphi}{\partial t} = \log\frac{\omega_\varphi^n}{\omega_{KS}^n} + \varphi + X(\varphi),$$
$$\varphi(0) = \psi - \underline{\psi}, \tag{2.4}$$

where $\underline{\psi} = \frac{1}{V}\int_M \psi\omega_{KS}^n$, $V = \int_M \omega_{KE}^n$, and $\varphi = \varphi_t$ are potentials of evolved Kähler metrics g_t of (2.3). Let K_{X_0} be an one-parameter compact subgroup of $\mathrm{Aut}_0(M)$ generated by the imaginary part X' of X_0. By choosing a reductive subgroup $\mathrm{Aut}_r(M)$ of $\mathrm{Aut}_0(M)$ such that $\mathrm{Aut}_r(M)$ contains K_{X_0}, we can prove

Theorem 2.1. *Let M be a compact Kähler manifold M with $c_1(M) > 0$ which admits a Kähler-Ricci soliton ω_{KS}. Then there exists a small ϵ such that for any initial data, potential $\psi \in \mathcal{M}(\omega_{KS})$ in equation (2.4) with $\psi \in \mathcal{K}(\epsilon)$, there exist a family of $\sigma = \sigma_t \in Aut_r(M)$ for evolved Kähler potentials $\varphi = \varphi_t$ of (2.4) at t such that Kähler potentials $(\varphi_\sigma - \underline{\varphi_\sigma})$ are C^k-norm uniformly bounded, where $\varphi_\sigma = \sigma^*\varphi + \rho$ and $\rho = \rho_t$ are Kähler potentials defined by*

$$\rho^*(\omega_{KS}) = \omega_{KS} + \sqrt{-1}\partial\overline{\partial}\rho,$$
$$\int_M e^{-\rho - X_0(\rho)}\omega_{KS}^n = \int_M \omega_{KS}^n. \tag{2.5}$$

As a consequence, evolved Kähler metrics g_t of (2.3) converge to g_{KS} smoothly in the sense of Cheeger-Gromov. Furthermore, if in addition that ψ is K_{X_0}-invariant, then there exist a family of $\sigma = \sigma_t \in Aut_r(M)$ such that $(\varphi_\sigma - \underline{\varphi_\sigma})$ converge exponentially to 0 as $t \to \infty$, and consequently Kähler metrics $\sigma^*(\omega_\varphi)$ converge exponentially to ω_{KS}.

As in Section 1, to prove Theorem 2.1, we need to estimate $\dot\varphi$ of Kähler potentials $\varphi = \varphi_t$ of (2.4). We introduce a modified functional of $H_0(t)$ by

$$\tilde{H}_0(t) = \frac{1}{V}\int_M (\dot\varphi - c(t))^2 e^{\tilde{h}}\omega_\varphi^n,$$

where $c(t) = \int_M \dot\varphi e^{\tilde h}\omega_\varphi^n$ is a constant, $\tilde h = \tilde h_t = \theta_X + X(\varphi) - \dot\varphi$ and $V = \int_M \omega_{KS}^n$. By a direct computation, one shows

$$\begin{aligned}
\frac{d}{dt}\tilde H_0(t) &= 2\frac{1}{V}\int_M (\dot\varphi - c(t))(\ddot\varphi - \dot c(t))e^{\tilde h}\omega_\varphi^n\\
&+ \frac{1}{V}\int_M (\dot\varphi - c(t))^2(\triangle_\varphi\dot\varphi + X(\dot\varphi) - \ddot\varphi)e^{\tilde h}\omega_\varphi^n\\
&= 2\frac{1}{V}\int_M (\dot\varphi - c(t))(\triangle_\varphi\dot\varphi + \dot\varphi + X(\dot\varphi) - c(t))e^{\tilde h}\omega_\varphi^n\\
&+ \frac{1}{V}\int_M (\dot\varphi - c(t))^2(\triangle_\varphi\dot\varphi + X(\dot\varphi) - \ddot\varphi)e^{\tilde h}\omega_\varphi^n\\
&= 2\frac{1}{V}\int_M [\dot\varphi - c(t)](\triangle_\varphi\dot\varphi + X(\dot\varphi))e^{\tilde h}\omega_\varphi^n\\
&+ 2\frac{1}{V}\int_M (\dot\varphi - c(t))^2(1 + \dot\varphi)e^{\tilde h}\omega_\varphi^n. \qquad (2.6)
\end{aligned}$$

On the other hand, by (2.2), we see

$$\begin{aligned}
&\int_M [\dot\varphi - c(t)](\triangle_\varphi\dot\varphi + X(\dot\varphi))e^{\tilde h}\omega_\varphi^n\\
&= -\int_M \|\nabla(\dot\varphi - c(t))\|^2 e^{\tilde h}\omega_\varphi^n\\
&+ \int_M [\dot\varphi - c(t)][X(\dot\varphi) - <\nabla(\theta_X + X(\phi) - \dot\varphi), \nabla\dot\varphi>]e^{\tilde h}\omega_\varphi^n\\
&\le -\int_M \|\nabla(\dot\varphi - c(t))\|^2 e^{\tilde h}\omega_\varphi^n + V|\dot\varphi - c(t)||X'(\dot\varphi)|\|\nabla\dot\varphi\|.
\end{aligned}$$

Thus inserting the above inequality into (2.6), we get

$$\begin{aligned}
\frac{d}{dt}\tilde H_0(t) &\le 2\frac{1}{V}\int_M (\dot\varphi - c(t))^2(1 + \dot\varphi)e^{\tilde h}\omega_\varphi^n\\
&- 2\frac{1}{V}\int_M \|\nabla(\dot\varphi - c(t))\|^2 e^{\tilde h}\omega_\varphi^n\\
&+ 2|\dot\varphi - c(t)||X'(\dot\varphi)|\|\nabla\dot\varphi\|. \qquad (2.7)
\end{aligned}$$

We shall estimate the L^2-integral of $\nabla\dot\varphi$ and need the following lemma,

Lemma 2.2. *Let $P = (P, \omega_\phi)$ be an elliptic operator on $C^{k,\alpha}(M)$ defined by*

$$P\psi = \triangle\psi + \psi + Re < \overline\partial h, \overline\partial\psi >_{\omega_\phi},$$

where $\triangle$ is the Lapalace operator with respect to a Kähler metric ω_ϕ and h is a Ricci potential of ω_ϕ. Then $ker(P, \omega_\phi) \subset \eta_r(M)$, where $\eta_r(M)$ is a reductive part of Lie algebraic $\eta(M)$ consisting of all holomorphic vector fields on M. Moreover, if $\omega_\phi = \omega_{KS}$, then $ker(P, \omega_{KS}) \cong \eta_r(M)$.

Proof. Let $L\psi = \triangle\psi + \psi + <\overline{\partial}h, \overline{\partial}\psi>_{\omega_\phi}$ and $\overline{L}\psi = \triangle\psi + \psi + \overline{<\overline{\partial}h, \overline{\partial}\psi>_{\omega_\phi}}$, where h is a Ricci potential of the metric ω_ϕ. Then by the Bochner formula, one can show (cf. Lemma 3.1 in [TZ3]),

$$\int_M -(L\psi)\psi e^h \omega_\phi^n = \int_M (\|\nabla\psi\|^2 - \psi^2) e^h \omega_\phi^n \geq 0, \tag{2.8}$$

and

$$\int_M -(\overline{L}\psi)\psi e^h \omega_\phi^n = \int_M (\|\nabla\psi\|^2 - \psi^2) e^h \omega_\phi^n \geq 0. \tag{2.9}$$

Moreover, the equality (2.8) or (2.9) holds if and only if the corresponding vector field of $(0,1)$-form $\overline{\partial}\phi$ is holomorphic. Thus

$$\begin{aligned} &-2\int_M (P\psi)\psi e^h \omega_\phi^n = -\int_M (L\psi + \overline{L}\psi)\psi e^h \omega_\phi^n \\ &= 2\int_M (\|\nabla\psi\|^2 - \psi^2) e^h \omega_\phi^n \geq 0, \end{aligned} \tag{2.10}$$

and the equality holds if and only if the corresponding vector field of $(0,1)$-form $\overline{\partial}\psi$ is holomorphic. Since ψ is a real-valued function the corresponding vector field must lie in $\eta_r(M)$. Furthermore, if one defines a potential θ'_Y by

$$L_Y \omega_{KS} = \sqrt{-1}\partial\overline{\partial}\theta'_Y \text{ and } \int_M \theta'_Y e^{\theta_X} \omega_{KS}^n = 0,$$

for an element Y in $\eta_r(M)$, then in case of $\omega_\phi = \omega_{KS}$, by using the fact that X_0 is an element of center of $\eta_r(M)$ [TZ1] and $h = \theta_{X_0}$, one can show θ'_Y must be in $\ker(P, \omega_{KS})$. □

Set a Banach space by

$$\overline{\mathcal{K}}(\epsilon_0) = \{\phi \in \mathcal{M}(\omega_{KS}) |\ \|\phi - \underline{\phi}\|_{C^{2,\alpha}} \leq \epsilon_0\}.$$

Then we have

Lemma 2.3. *Let $\varphi = \varphi_t$ be an evolved Kähler potential of (2.4) at t and $\theta'_Y \in ker(P, \omega_\varphi)$ be a potential of $Y \in \eta_r(M)$ with $\int_M \|Y\|^2 \omega_{KS}^n = 1$. If $\varphi \in \overline{\mathcal{K}}(\epsilon_0)$, then there exist two uniform constants C_1 and C_2 such that*

$$\begin{aligned} &|\int_M \theta'_Y(\dot{\varphi}_t - c(t)) e^{\theta_X + X(\varphi)} \omega_\varphi^n| \\ &\leq C_1\epsilon_0 \int_M |\dot{\varphi}_t - c(t)| e^{\theta_X + X(\varphi)} \omega_\varphi^n + C_2\epsilon_0^2. \end{aligned} \tag{2.11}$$

Proof. Recall a generalized Futaki-invariant defined in [TZ2] by

$$F_{X_0}(Y) = \int_M Y[h_{\omega_\phi} - (\theta_X + X_0(\phi))] e^{\theta_X + X_0(\phi))} \omega_\phi^n,\ \forall\ Y \in \eta(M).$$

It was proved that the invariant is independent of the choice of Kähler metric ω_ϕ on M and the invariant vanishes if M admits the Kähler-Ricci soliton (ω_{KS}, X_0). So we have

$$F_{X_0}(Y) \equiv 0, \ \forall\ Y \in \eta(M).$$

By applying the metrics ω_φ to the above identity, one sees

$$\int_M Y[\dot\varphi - c(t) - \sqrt{-1}X'(\varphi)]e^{\theta_X + X_0(\varphi))}\omega_\varphi^n = 0, \ \forall\ Y \in \eta_r(M).$$

It follows

$$\begin{aligned}&|\mathrm{Re}(\int_M Y[\dot\varphi - c(t) - \sqrt{-1}X'(\varphi)]e^{\theta_X + X(\varphi) + \cos(X'(\varphi))}\omega_\varphi^n)|\\ &\le A_0\|\varphi\|_{C^2}|X'(\varphi)|.\end{aligned} \tag{2.12}$$

On the other hand, by using the Stoke's formula, we have

$$\begin{aligned}&\int_M Y[\dot\varphi - c(t) - \sqrt{-1}X'(\varphi)]e^{\theta_X + X(\varphi) + \ln\cos(X'(\varphi))}\omega_\varphi^n\\ &= -\int_M (\dot\varphi - c(t)) - \sqrt{-1}X'(\varphi))\\ &\times [\triangle(\theta'_Y + Y(\varphi)) + < \overline\partial(\theta'_Y + Y(\varphi)), \overline\partial(\theta_X + X(\varphi) + \ln\cos(X'(\varphi))) >]\\ &\times e^{\theta_X + X(\varphi) + \ln\cos(X'(\varphi))}\omega_\varphi^n\\ &= \int_M (\dot\varphi - c(t) - \sqrt{-1}X'(\varphi))(\triangle\theta'_Y + < \overline\partial\theta'_Y, \overline\partial\theta_X >)\\ &\times e^{\theta_X + X(\varphi) + \ln\cos(X'(\varphi))}\omega_\varphi^n + O(\epsilon_0^2)\\ &= \int_M (\dot\varphi - c(t) - \sqrt{-1}X'(\varphi))(\triangle_{\omega_{KS}}\theta'_Y + < \overline\partial\theta'_Y, \overline\partial\theta_X >_{\omega_{KS}})\\ &\times e^{\theta_X + X(\varphi) + \ln\cos(X'(\varphi))}\omega_\varphi^n + O(\epsilon_0^2).\end{aligned} \tag{2.13}$$

Note that

$$< \overline\partial\theta'_Y, \overline\partial\theta_X >_{\omega_{KS}} = Y(\theta_X) = X(\theta'_Y) = < \overline\partial\theta_X, \overline\partial\theta'_Y >_{\omega_{KS}}$$

is a real-valued function ([TZ1]). Thus

$$\triangle_{\omega_{KS}}\theta'_Y + < \overline\partial\theta'_Y, \overline\partial\theta_X >_{\omega_{KS}} = \triangle_{\omega_{KS}}\theta'_Y + < \overline\partial\theta_X, \overline\partial\theta'_Y >_{\omega_{KS}} = -\theta'_Y.$$

Therefore, inserting (2.13) into (2.12), one will get (2.11). □

By using Lemma 2.2 and Lemma 2.3, we can complete the L^2-estimate of $\dot\varphi$.

Lemma 2.4. *Let $\epsilon_0 << 1$. Then*

$$\tilde H_0(t) \le \tilde H_0(0)e^{-\theta t} + \frac{B_0}{\theta}\epsilon_0^3, \ \forall\ t \in [0, T), \tag{2.14}$$

if φ_t lies in $\overline{\mathcal{K}}(\epsilon_0)$ and $\tilde{H}_0(t) \geq \frac{B_0}{\theta}\epsilon_0^3$ for any t in $[0,T)$, where the constant $B_0 = B_0(\|X'\|_{C^0})$ depends only on $\|X'\|_{C^0}$ and the constant $\theta > 0$ depends only on the gap of the first two eigenvalues of the operator P associated to the metric ω_{KS} in Lemma 2.2.

Proof. Let ψ^i be the first eigenvalue functions of the operator (P, ω_φ) with respect to the metric ω_φ with satisfying $\int_M |\psi^i|^2 e^{\tilde{h}}\omega_\varphi^n = 1$. Then by the continuity of eigenvalue functions and (2.11), one sees that there exists two constants C and A_0 such that

$$\begin{aligned} &|\int_M \psi^i(\dot{\varphi}_t - c(t))e^{\tilde{h}}\omega_\varphi^n \\ &\leq C\epsilon_0 \int_M |\dot{\varphi}_t - c(t)|e^{\tilde{h}}\omega_\varphi^n + A_0\epsilon_0^2. \end{aligned} \tag{2.15}$$

Now as same as in the proof of Lemma 1.3, we decompose $\dot{\varphi}_t - c(t)$ as $\psi + \psi^\perp$ with $\psi \in \Lambda_1(M, \omega_\varphi)$ and $\psi^\perp \in \Lambda_1^\perp(M, \omega_\varphi)$, where $\Lambda_1(M, \omega_\varphi)$ is a linear space spanned by a basis $\{\psi^i\}$ and $\Lambda_1^\perp(M, \omega_\varphi)$ be a subspace of L^2-weighted integral functions which are orthogonal to $\Lambda_1(M, \omega_\varphi) \cap \mathbb{R}$ in the sense of

$$\int_M \psi\psi' e^{\tilde{h}}\omega_\varphi^n = 0, \ \forall\ \psi \in \Lambda_1(M, \omega_\varphi), \psi' \in \Lambda_1^\perp(M, \omega_\varphi).$$

Then we get

$$\int_M |\psi|^2 e^{\tilde{h}}\omega_\varphi^n \leq C'\epsilon_0^2 \int_M (\dot{\varphi} - c(t))^2 e^{\tilde{h}}\omega_\varphi^n + nA_0^2\epsilon_0^4,$$

and so

$$\begin{aligned} &\int_M |\psi^\perp|^2 e^{\tilde{h}}\omega_\varphi^n \\ &\geq (1 - C'\epsilon_0^2)\int_M (\dot{\varphi} - c(t))^2 e^{\tilde{h}}\omega_\varphi^n - nA_0^2\epsilon_0^4. \end{aligned} \tag{2.16}$$

On the other hand, by using the continuity of eigenvalues and Lemma 2.2, there exists a number $\delta_0 > 0$ (compared to Lemma 1.2), which depends only on the gap of the first two eigenvalues of the operator (P, ω_{KS}) with respect to the metric ω_{KS} in Lemma 2.2, such that for any $\varphi \in \overline{\mathcal{K}}(\epsilon_0)$, we have

$$\int_M \|\nabla\psi^\perp\|^2 e^{\tilde{h}}\omega_\varphi^n \geq (1 + \delta_0)\int_M (\psi^\perp)^2 e^{\tilde{h}}\omega_\varphi^n.$$

Thus by (2.16), we get

$$\begin{aligned} &\int_M \|\nabla(\dot{\varphi}_t - c(t))\|^2 e^{\tilde{h}}\omega_\varphi^n \\ &\geq (1 + \delta_0)(1 - C'\epsilon_0^2)\int_M (\dot{\varphi}_t - c(t))^2 e^{\tilde{h}}\omega_\varphi^n - nA_0^2\epsilon_0^4. \end{aligned} \tag{2.17}$$

By inserting (2.17) into (2.7), we obtain

$$\begin{aligned}&\frac{d\tilde{H}_0(t)}{dt}\\&\leq -2[(1-2\epsilon_0)(1+\delta_0)(1-C'\epsilon_0^2)-(1+\epsilon_0)]\tilde{H}_0(t)+B_0\epsilon_0^3\\&\leq -\theta\tilde{H}_0(t)+B_0\epsilon_0^3, \qquad (2.18)\end{aligned}$$

where the constant $B_0 = B_0(\|X'\|_{C^0})$ depends only on $\|X'\|_{C^0}$ and $\theta = 2[(1-2\epsilon_0)(1+\delta_0)(1-C'\epsilon_0^2)-(1+\epsilon_0)] \geq \delta_0$ as ϵ_0 is small enough.

By (2.18), we have

$$\frac{d(\tilde{H}_0(t)-\frac{B_0\epsilon_0^3}{\theta})}{dt} \leq -\theta(\tilde{H}_0(t)-\frac{B_0\epsilon_0^3}{\theta}).$$

Since $\tilde{H}_0(t) \geq \frac{B_0\epsilon_0^3}{\theta}$, we get

$$\begin{aligned}\tilde{H}_0(t) \leq \ & e^{-\theta t}(\tilde{H}_0(0)-\tfrac{B_0\epsilon_0^3}{\theta})+\tfrac{B_0\epsilon_0^3}{\theta}\\ &\leq e^{-\theta t}\tilde{H}_0(0)+\tfrac{B_0\epsilon_0^3}{\theta}.\end{aligned}$$

□

Remark 2.5. *From (2.7) and (2.12), we see that if in addition that the initial Kähler potential ψ is K_{X_0}-invariant, then (2.11) can be improved as*

$$\tilde{H}_0(t) \leq \tilde{H}_0(0)e^{-\theta t}, \ \forall \ t \in [0,T)$$

whenever φ_t lies in $\overline{\mathcal{K}}(\epsilon_0)$.

To get a C^0-estimate and C^α-estimate for $\dot{\varphi}$, we use a method as in Appendix 1 to estimate $W^{k,2}$-estimates ($k \geq 1$) for $\dot{\varphi}$. Let

$$\tilde{H}_k(t) = \int_M \|\nabla^k\dot{\varphi}\|^2 e^{\tilde{h}}\omega_\varphi^n.$$

Then we have

Proposition 2.6. *Let $\epsilon_0 << 1$. Then under the same condition in Lemma 2.4, there exist two uniform constants $\theta', B > 0$ which depend only on the metric ω_{KS} and integer number k such that*

$$\tilde{H}_k(t) \leq e^{-\theta' t}(\tilde{H}_k(0)+B\tilde{H}_0(0))+\frac{B_0B}{\theta'}\epsilon_0^3, \ \forall \ t \in [0,T), \qquad (2.19)$$

if $\tilde{H}_k(t)+B\tilde{H}_0(t)) \geq \frac{B_0B}{\theta'}\epsilon_0^3$ for any $t \leq T$, where B_0 is the constant determined in Lemma 2.4.

Proof. First note that similarly to (4.1) in Appendix 1, we can obtain

$$\begin{aligned}&\frac{d\|\nabla^k\dot{\varphi}\|^2}{dt}\\&\leq -2\|\nabla^{k+1}\dot{\varphi}\|^2+C_1\|\nabla^k\dot{\varphi}\|^2+C_2\|\dot{\varphi}-c(t)\|^2.\end{aligned}$$

It follows

$$\begin{aligned}\frac{d\tilde{H}_k(t)}{dt} &= \int_M \frac{d\|\nabla^k\dot{\varphi}\|^2}{dt} e^{\tilde{h}}\omega_\phi^n + \int_M \|\nabla^k\dot{\varphi}\|^2(\triangle\dot{\varphi} + X(\dot{\varphi}) - \ddot{\varphi})e^{\tilde{h}}\omega_\varphi^n \\ &\leq -2\tilde{H}_{k+1}(t) + C_1'\tilde{H}_k(t) + C_2'\|\dot{\varphi} - c(t)\|^2 \\ &\leq -\theta'\tilde{H}_k(t) + C_3\tilde{H}_0(t).\end{aligned}$$

On the other hand, by (2.18), we have

$$\frac{d\tilde{H}_0(t)}{dt} \leq -\theta\tilde{H}_0(t) + B_0\epsilon_0^3, \ \forall\ t \in [0,T),$$

since we may also assume that $\tilde{H}_0(t) \geq \frac{B_0}{\theta}\epsilon_0^3$ for any t in $[0,T)$, Thus combining the above two inequalities, we get

$$\begin{aligned}\frac{d(\tilde{H}_k(t) + B\tilde{H}_0(t))}{dt} &\leq -\theta'[\tilde{H}_k(t) + \frac{(B\theta - C_3)}{\theta'}\tilde{H}_0(t)] + B_0B\epsilon_0^3 \\ &\leq -\theta'(\tilde{H}_k(t) + B\tilde{H}_0(t)) + B_0B\epsilon_0^3,\end{aligned} \tag{2.20}$$

where B is a sufficiently large number independent of ϵ_0. From (2.20), one can easily get

$$(\tilde{H}_k(t) + B\tilde{H}_0(t)) \leq e^{-\theta' t}(\tilde{H}_k(0) + B\tilde{H}_0(0)) + \frac{B_0B}{\theta'}\epsilon_0^3, \ \forall\ t \in [0,T),$$

and so (2.19) is true.

□

By the embedding theory of Sobolev spaces, we get

Corollary 2.7. *Let $\epsilon_0 << 1$. Then under the same condition in Lemma 2.4, there exist two uniform constants $\theta_0, C_0 > 0$ which depend only on the metric ω_{KS} such that*

$$\|\tilde{\dot{\varphi}}_t\|_{C^\alpha} \leq C_0[e^{-\theta_0 t}\|\psi - \underline{\psi}\|_{C^{2,\alpha}} + \epsilon_0^{\frac{3}{2}}], \ \forall\ t \in [0,T), \tag{2.21}$$

if $\|\tilde{\dot{\varphi}}_t\|_{C^\alpha} \geq C_0\epsilon_0^{\frac{3}{2}}$ for any $t \leq T$.

Remark 2.8. *By Remark 2.5, according to the proof of Proposition 2.6, we see that if in addition that the initial Kähler potential ψ is K_{X_0}-invariant, then (2.19) can be improved as*

$$\tilde{H}_k(t) \leq (\tilde{H}_0(0) + B\tilde{H}_k(0))e^{-\theta' t}, \ \forall\ t \in [0,T)$$

whenever φ_t lies in $\overline{\mathcal{K}}(\epsilon_0)$. Thus (2.21) can be improved as

$$\|\tilde{\dot{\varphi}}_t\|_{C^\alpha} \leq C_0 e^{-\theta_0 t}\|\psi - \underline{\psi}\|_{C^{2,\alpha}}.$$

The following lemma can be easily proved by using apriori estimates for solution $\varphi(t,\cdot)$ of (2.4) at finite time (cf [TZ3]).

Lemma 2.9. *Let $\psi \in \mathcal{K}(\frac{\epsilon_0}{N})$. Then there exists $T = T_N$ such that evolved Kähler potentials φ_t of (2.4) with ψ as an initial potential lies in $\mathcal{K}(\epsilon_0)$ for any $t < T$.*

We are now going to do a key estimate for the proof of Theorem 2.1.

Proposition 2.10. *There exist a small ϵ_0 and a large number N such that if the initial data $\psi \in \mathcal{M}(\omega_{KS})$ in (2.4) satisfies $\|\psi - \underline{\psi}\|_{C^{2,\alpha}} \le \frac{\epsilon_0}{N}$, then there there exist a family of $\sigma_t \in Aut_r(M)$ such that*

$$\|\varphi_{\sigma_t} - \underline{\varphi_{\sigma_t}}\|_{C^{2,\alpha}} \le \epsilon_0, \ \forall\ t > 0, \tag{2.22}$$

where $\varphi_{\sigma_t} = (\sigma_t)^*\varphi_t + \rho_t$ and ρ_t are Kähler potentials defined by (2.5) in Theorem 2.1.

Proof. The proof is a modification of one of Theorem 1.1. Let N_0 be a very big number and choose another big number N with $N_0 \le N \le \frac{1}{\epsilon_0^{1/4}}$ such that $C_0 e^{-\theta_0 T_N} \le \frac{1}{N_0}$, where C_0 and T_N are two uniform numbers determined in Corollary 2.7 and Lemma 2.9, respectively. Now we consider the solution $\varphi = \varphi_{T_N}$ of (2.4) at time T_N. By Lemma 5.1 in Appendix 2, we see that there exists $\sigma = \sigma_{T_N}$ such that for any $Y \in \eta_r(M)$ with $\int_M \|Y\|^2 \omega_{KS}^n = 1$, it holds

$$|\int_M \theta'_Y \varphi_\sigma e^{\theta_X} \omega_{KS}^n| \le O(\epsilon_0^2), \tag{2.23}$$

where $\varphi_\sigma = \sigma^*\varphi + \rho_\sigma$. By adding a constant to φ_σ so that $\widetilde{\varphi_\sigma} = \varphi_\sigma + const.$ satisfies $\int_M \widetilde{\varphi_\sigma} e^{\theta_X} \omega_{KS}^n = 0$, then we can decompose $\widetilde{\varphi_\sigma}$ into $\widetilde{\varphi_\sigma} = \phi + \phi^\perp$ with $\phi \in \Lambda_1(M, \omega_{KE})$ and $\phi^\perp \in \Lambda_1^\perp(M, \omega_{KE})$, where $\Lambda_1^\perp(M, \omega_{KS})$ is a subspace of weighted L^2-integral functions which are orthogonal to $\Lambda_1(M, \omega_{KS}) \cup \mathbb{R}$. Then $\phi = \sum_i a_i \theta_i$ for some constants a_i, where θ_i is a basis of the space $\Lambda_1(M, \omega_{KS})$. As a consequence, by (2.23), we see that $a_i = O(\epsilon_0^2)$ and so

$$\|\phi\|_{C^{2,\alpha}} \le O(\epsilon_0^2). \tag{2.24}$$

Since $\rho = \rho_\sigma$ satisfies equation,

$$\omega_\rho^n = \omega_{KS}^n e^{-\rho - X(\rho)},$$

by equation (2.4), we have

$$\omega_{\widetilde{\varphi_\sigma}}^n = \omega_{KS}^n e^{\sigma^*(\tilde{\varphi}) - \widetilde{\varphi_\sigma} - X(\widetilde{\varphi_\sigma}) + b}, \tag{2.25}$$

where b is a constant. Then $\phi^\perp$ is a solutions of equation

$$P[\log(\frac{[\omega_{\phi+\phi^\perp}]^n}{\omega_{KS}^n})] + \varphi^\perp + X(\phi^\perp) = P[\sigma^*(\tilde{\varphi}) + b - X(\phi)],$$

where P is a projection from Banach space $H^{2,\alpha}(M)$ to Banach space $H^{\alpha}(M)\cap\Lambda_1^{\perp}(M,\omega_{KS})$, and ϕ and $\sigma^*(\dot{\tilde{\varphi}})$ are regarded as two peturbation functions. Without of the generality, we may assume that

$$\|\sigma^*(\dot{\tilde{\varphi}})\|_{C^\alpha}\geq C_0\epsilon_0^{\frac{3}{2}},$$

where C_0 is the constant in Corollary 2.7. Then according to Corollary 2.6 and (2.24), we have

$$\|P[\sigma^*(\dot{\tilde{\varphi}})+b-X(\phi)]\|_{C^\alpha}=\|P[\sigma^*(\dot{\tilde{\varphi}})-X(\phi)]\|_{C^\alpha}\leq\frac{2\epsilon_0}{NN_0}.$$

Thus we can use the Implicity Functional Theorem to get

$$\|\phi^{\perp}\|_{C^{2,\alpha}}\leq 2(\|P[\sigma^*(\dot{\tilde{\varphi}})+b-X(\phi)]\|_{C^\alpha}+\|\phi\|_{C^{2,\alpha}})\leq\frac{8\epsilon_0}{NN_0}. \tag{2.26}$$

(2.24) and (2.26) implies

$$\|\widetilde{\varphi_\rho}\|_{C^{2,\alpha}}\leq\frac{16\epsilon_0}{NN_0}, \tag{2.27}$$

so $\widetilde{\varphi_\rho}\in\mathcal{K}(\frac{\epsilon_0}{N})$.

At the next step (Step 2) we consider equation (2.4) with $\widetilde{\varphi_\rho}$ as an initial potential to replace ψ. By Lemma 2.9, one sees that equation is solvable for any $t\in T_N$ with evolved Kähler potentials $\varphi_t^{(2)}\in\mathcal{K}(\epsilon_0)$ for any $t\leq T_N$. So by the argument at the last step (Step 1), we can also show that there exists $\sigma^{(2)}=\sigma_{T_N}^{(2)}\in\mathrm{Aut}_r(M)$ such that

$$\begin{aligned}\|\widetilde{\varphi_\rho^{(2)}}\|_{C^{2,\alpha}}&=\|(\sigma^{(2)})^*\widetilde{\varphi_{T_N}^{(2)}}+\rho_{\sigma^{(2)}}\|_{C^{2,\alpha}}\\&\leq(\frac{16}{N_0})^2\frac{\epsilon_0}{N}<\frac{\epsilon_0}{N}.\end{aligned} \tag{2.28}$$

Repeating to use the above step for finite times, we can obtain

$$\|\widetilde{\dot{\varphi}^{(k)}}\|_{C^\alpha}\leq C_0\epsilon_0^{\frac{3}{2}}$$

for some integer k. Then also by using the argument in Step 1, we can find $\sigma^{(k+1)}=\sigma_{T_N}^{(k+1)}\in\mathrm{Aut}_r(M)$ such that

$$\|\widetilde{\varphi_\rho^{(k)}}\|_{C^{2,\alpha}}=O(\epsilon_0^{\frac{3}{2}}).$$

Now (Step 3) we considering equation (2.4) with $\widetilde{\varphi_\rho^{(k)}}$ as an initial potential. Then we conclude that either evolved Kähler potentials $\varphi_t^{(k+1)}$ lies in $\mathcal{K}(\frac{\epsilon_0}{N})$ for any t or there exists some time T such that $\|\varphi_T^{(k+1)}\|_{C^{2,\alpha}}=\frac{\epsilon_0}{N}$ for an evolved Kähler potential $\varphi_T^{(k+1)}$ at time T. If the first case happens, then we will finish all steps. If the second case happens, then we can repeat the Step1-3 and we can finally prove that there exist a family of $\sigma_t\in\mathrm{Aut}_r(M)$

such that (2.22) satisfies for any evolved Kähler potential φ_t of (2.4) at t as long as the initial potential ψ lies in $\mathcal{K}(\frac{\epsilon_0}{N})$.

□

Proof of Theorem 2.1. We suffice to do higher-order estimates for the modified evolved Kähler potentials $((\sigma_t)^*\varphi_t + \rho_t)$ of equation (2.4) in Proposition 2.10. Here we use a trick in [CT2] to choose a modified family of holomorphic transformations $\overline{\sigma}_t \in \mathrm{Aut}_r(M)$ $(0 < t < \infty, \overline{\sigma}_0 = \mathrm{Id})$ to replace σ_t such that for any $t \in (0, \infty)$ (cf. [TZ3]),

$$\|\sigma_t^{-1}\overline{\sigma}_t - Id\| \le C,$$

and

$$\|(\overline{\sigma}_t^{-1})_* \frac{\partial \overline{\sigma}_t}{\partial t}\|_g \le C,$$

where $(\overline{\sigma}_t^{-1})_* \frac{\partial \overline{\sigma}_t}{\partial t} = \overline{X}_t \in \eta_r(M)$ is a family of holomorphic vector fields on M. Furthermore, for any $k \ge 0$, we may assume that there is a constant C_k such that

$$\|\frac{\partial^k \overline{X}_t}{\partial t^k}\|_g \le C_k.$$

Note that the choice of such $\overline{\sigma}_t$ just depends on the C^0-estimate of $\widetilde{\varphi_\sigma} = \widetilde{((\sigma_t)^*\varphi_t + \rho_t)}$. Thus by Proposition 2.10, we also have $(\overline{\varphi} - \frac{1}{V}\int_M \overline{\varphi}\omega_{KS}^n) \in \mathcal{K}(\epsilon_0)$. On the other hand, by equation (2.4), the new modified potential $\overline{\varphi} = \varphi_{\overline{\sigma}_t} = (\overline{\sigma}_t)^*(\varphi_t + \overline{\rho}_t)$ will satisfy equation,

$$\frac{\partial \overline{\varphi}}{\partial t} = \log \frac{\omega_{\overline{\varphi}}^n}{\omega_{KS}^n} + \overline{\varphi} + \overline{X}(\overline{\varphi}),$$

$$\overline{\varphi}(0) = \psi - \underline{\psi}. \tag{2.29}$$

Now for each t, we can consider solution φ' of equation (2.29) on the interval $[t-1, t+1]$ with $(\overline{\varphi}_{t-1} - \frac{1}{V}\int_M \overline{\varphi}_{t-1}\omega_{KS}^n)$ as an initial data. By the Maximal Principle, it is easy to see that both φ_s' and $\dot{\varphi}_s'$ are uniformly bounded in $[t-1, t+1]$. Since

$$\|\overline{\varphi}_s' - \frac{1}{V}\int_M \overline{\varphi}_s'\omega_{KS}^n\|_{C^{2,\alpha}} = \|\overline{\varphi}_s - \frac{1}{V}\int_M \overline{\varphi}_s\omega_{KS}^n\|_{C^{2,\alpha}},$$

by the regularity theory of parabolic equation, we get all bounded C^k-estimates for $\overline{\varphi}_t'$. This implies that all C^k-norms of $(\overline{\varphi}_t - \frac{1}{V}\int_M \overline{\varphi}_t\omega_{KS}^n)$ are uniformly bounded, and so are $\widetilde{\varphi_\sigma}$.

From the above estimates, we see that for any sequence of Kähler metrics $\omega_{\varphi_{\sigma_i}}$, there exists a limit Kähler metric ω_∞ of subsequence of $\omega_{\varphi_{\sigma_i}}$ in the sense of C^k-convergence. By applying Perelman's W-function in [Pe] to the normalized Ricci equation (0.1), one concuIdes that ω_∞ must be a Kähler-Ricci soliton (cf. [Se]). Since the Kähler-Ricci solition is unique, we see that there exists an element $\tau_\infty \in \mathrm{Aut}_0(M)$ such that $\omega_\infty = \tau_\infty^*\omega_{KS}$. By using the fact that the convergent sequence is arbitary, the above implies that there exists a family of $\tau = \tau_t \in \mathrm{Aut}_0(M)$ such that evolved Kähler metrics $\tau^* g$ converge to g_{KS} smoothly.

If in addition that the initial Kähler potential ψ is K_{X_0}-invariant, by Remark 2.8, we can follow the argument in the proof of Theorem 1.1 to apply the Implicity Functional Theorem to equation (2.25) in the proof of Proposition 2.10 to show that there exists a family of $\sigma = \sigma_t \in \mathrm{Aut}_r(M)$ such that the modified solution $\varphi_\sigma = ((\sigma_t)^*\varphi_t + \rho_t)$ of equation (2.25) satisfy

$$\|\widetilde{\varphi_\sigma}\|_{C^{2,\alpha}} = \|((\sigma_t)^*\varphi_t + \rho_t - \frac{1}{V}\int_M (\sigma_t)^*\varphi_t + \rho_t))\omega_{KS}^n)\|_{C^{2,\alpha}}$$
$$\leq 2\|P(\sigma^*(\tilde{\tilde{\varphi}})\|_{C^\alpha} \leq Ce^{-\theta' t}, \ \forall\ t > 0. \tag{2.30}$$

Similarly, we can also prove that for any k it holds

$$\|\widetilde{\varphi_\sigma}\|_{C^{k+2,\alpha}} \leq C_k e^{-\theta' t}, \ \forall\ t > 0,$$

since by Remark 2.8 and the embedding theory of Sobolev spaces we have

$$\|\tilde{\tilde{\varphi}}\|_{C^{k,\alpha}} \leq C_k' e^{-\theta' t}, \ \forall\ t > 0,$$

where C_k and C_k' are uniform constants which depends only on k, ϵ_0 and higher-order derivatives of the initial Kähler potential ψ. Therefore we prove that Kähler metrics $\sigma^*(\omega_\varphi)$ converge exponentially to the Kähler-Ricci soliton ω_{KS}. □

3. Uniqueness of the limit of Kähler Ricci flow

By Theorem 1.1 and Theorem 2.2 in Section 1 and 2, we complete the proof of Theorem 0.1. As an application of Theorem 0.1, we have the following uniqueness result about the limit of Kähler-Ricci flow.

Theorem 3.1. *Let g_t be the evolved Kähler metrics of Kähler-Ricci flow (0.1) on M. Suppose that there exists a sequence g_i of g_t and a sequence of holomorphic transformations $\sigma_i \in Aut(M)$ such that $\sigma_i^* g_i$ converge to a limit Kähler metric g_∞ in the sense of $C^{2,\alpha}$-norm of Kähler potentials. Then the Kähler-Ricci flow converges to g_∞ smoothly in the sense of Cheeger-Gromov.*

Proof. First we note that by applying Perelman's W-function in [Pe], the limit Kähler metric g_∞ must be a Kähler-Ricci soliton g_{KS} on M. On the other hand, by the convergence of g_i, one sees that for any $\epsilon << 1$ there exists a big index i such that the potential $\psi = \psi_i$ of g_i satisfies

$$\|\psi - \underline{\psi}\|_{C^{2,\alpha}} \leq \epsilon,$$

where $\omega_\psi = \omega_{KS} + \sqrt{-1}\partial\bar{\partial}\psi$. Now we consider the Kähler-Ricci flow (0.1) with $\omega_g = \omega_\psi$ as an initial Kähler metric. Then by Theorem 0.1, this flow converges to g_{KS} smoothly in the sense of Cheeger-Gromov, so the theorem is proved.

□

Remark 3.2. *In a subsequence paper, we will prove the uniqueness of the limit of Kähler-Ricci flow in more general. Namely, Theorem 3.1 is still true if we assume that there exists a sequence g_i of g_t of equation (0.1) which converge to a limit Riemanian metric g_∞ in $C^{2,\alpha}$-norm in the sense of Cheeger-Gromov.*

4. Appendix 1

In this appendix, we prove a lemma about $W^{k,2}$-estimates of $\dot{\varphi}$ for evolved Kähler metrics φ of flow (1.1) under the assumption $\varphi \in \mathcal{K}(\epsilon_0)$. Recall that a $k-norm$ $\|\nabla^k \dot{\varphi}\|^2$ is defined by

$$\|\nabla^k \dot{\varphi}\|^2 = \sum g^{i_1 j_1} ... g^{i_k j_k} \dot{\varphi}_{i_1 ... i_k} \dot{\varphi}_{j_1 ... j_k},$$

where $\dot{\varphi}_{i_1...i_k}$ are components of the k-covariant derivative of $\dot{\varphi}$ with respect to $g = \omega_\varphi$ as a Riemannian metric.

Since

$$\dot{\varphi}_{i_1...i_k} = \frac{\partial^k \dot{\varphi}}{\partial x^{i_1} ... \partial x^{i_k}} + \Phi_1(\dot{\varphi}, ..., \dot{\varphi}_{i_1 ... i_{k-1}}),$$

we have

$$\begin{aligned}\frac{d\dot{\varphi}_{i_1...i_k}}{dt} &= \frac{\partial^k \ddot{\varphi}}{\partial x^{i_1} ... \partial x^{i_k}} + \frac{d\Phi_1}{dt} \\ &= \ddot{\varphi}_{i_1...i_k} + \Phi_2(\dot{\varphi}_i, ..., \dot{\varphi}_{i_1...i_{k-1}}) + \frac{d\Phi_1}{dt},\end{aligned}$$

where Φ_1 and Φ_2 are two polynomials with variables $\dot{\varphi}_i, ..., \dot{\varphi}_{i_1...i_{k-1}}$ and coefficients $g_{ij}, \partial^l g_{ij}, l = 1, ..., k$. Note that $\frac{d\Phi_1}{dt}$ is uniformly bounded. Then by equations (0.1) and (1.1), one can estimate

$$\begin{aligned}&\frac{d\|\nabla^k \dot{\varphi}\|^2}{dt} \\ &= \sum_{i_1,...,i_k} \sum_{\alpha} (R_{i_\alpha i_\alpha} - g_{i_\alpha i_\alpha}) \dot{\varphi}_{i_1,...,i_\alpha,...,i_k} \dot{\varphi}_{i_1,...,i_\alpha,...,i_k} \\ &+ 2 \sum g^{i_1 j_1} ... g^{i_k j_k} \frac{d\dot{\varphi}_{i_1...i_k}}{dt} \dot{\varphi}_{j_1...j_k} \\ &\le C_1 \|\nabla^k \dot{\varphi}\|^2 + C_2 \|\nabla \dot{\varphi}\|^2 + 2(\ddot{\varphi})_{i_1...i_k} \dot{\varphi}_{j_1...j_k} \\ &\le -2\|\nabla^{k+1}\dot{\varphi}\|^2 + C_1' \|\nabla^k \dot{\varphi}\|^2 + C_2' \|\dot{\varphi} - c(t)\|^2. \qquad (4.1)\end{aligned}$$

Let

$$H_k(t) = \int_M \|\nabla^k \dot{\varphi}\|^2 \omega_\varphi^n.$$

Then by (4.1), we have

Lemma 4.1. *Let T be any positive number. Suppose that φ_t lies $\mathcal{K}(\epsilon_0)$ for any $t \in [0,T)$. Then*

$$H_k(t) \le Ce^{-\theta' t}, \ \forall \ t \in [0,T). \tag{4.2}$$

Proof. By (4.1), we have

$$\begin{aligned} \frac{dH_k(t)}{dt} &= \int_M \frac{d\|\nabla^k \dot{\varphi}\|^2}{dt}\omega_\varphi^n + \int_M \|\nabla^k \dot{\varphi}\|^2 \triangle \dot{\varphi}\omega_\varphi^n \\ &\le -2H_{k+1}(t) + C_3 H_k(t) + C_2' \|\dot{\varphi} - c(t)\|^2 \\ &\le -\theta' H_k(t) + C_4 H_0(t). \end{aligned} \tag{4.3}$$

On the other hand, from the proof of Lemma 1.3, we in fact prove that

$$\frac{dH_0(t)}{dt} \le -\theta H_0(t), \ \forall \ t \in [0,T),$$

if $\varphi \in \mathcal{K}(\epsilon_0), \ \forall \ t \in [0,T)$. Thus Combining the above inequality with (4.3), we get

$$\frac{d(H_k(t) + AH_0(t))}{dt} \le -\theta' [H_k(t) + \frac{(A\theta - C_4)}{\theta'} H_0(t)],$$

where A is a sufficiently large number. It follows

$$\frac{d\ln(H_k(t) + AH_0(t))}{dt} \le -\theta' \frac{H_k(t) + \frac{(A\theta - C_4)}{\theta'} H_0(t)}{H_k(t) + AH_0(t)} \le -\theta'.$$

Thus

$$H_k(t) + AH_0(t) \le (H_k(0) + AH_0)e^{-\theta' t}$$

and so (4.2) follows. □

5. Appendix 2

The following lemma is about the existence of almost orthonormality of a Kähler potential to the space of first eigenvalue-functions of operator (P, ω_{KS}) defined in Lemma 2.2 in Section 2. The lemma is crucial in the proof of Proposition 2.10.

Lemma 5.1. *Let M be a compact Kähler manifold M with $c_1(M) > 0$ which admits a Kähler-Ricci soliton (ω_{KS}, X_0). Then for any Kähler potential $\phi \in \mathcal{K}(\epsilon_0)$ there exists a $\sigma \in Aut_r(M)$ with bounded dist(σ, Id) such that for any $Y \in \eta_r(M)$ with $\int_M \|Y\|^2 \omega_{KS}^n = 1$, it holds*

$$| \int_M \theta_Y' (\sigma^* \phi + \rho_\sigma) e^{\theta_X} \omega_{KS}^n | \le C \|X'(\phi)\|_{C^0}^2 = O(\epsilon_0^2),$$

where $\theta_Y \in ker(P, \omega_{KS})$ and ρ_σ is a Kähler potential defined by (2.5) in Section 2.

Proof. This lemma was proved in [TZ1] if ϕ is K_0-invariant. The key point in the proof is to use a functional defined on a space of Kähler-Ricci solitons

$$\{\omega'_{KS} = \sigma^*(\omega_{KE}) = \omega_{KS} + \sqrt{-1}\partial\bar{\partial}\rho_\sigma |\ \sigma \in \mathrm{Aut}_r(M)\},$$

which was introduced in [Zh] by

$$\begin{aligned}&(I-J)(\omega_\phi, \omega'_{KS})\\ &= \int_0^1 dt \int_M \dot{\phi}_t e^{\theta_{X_0}(\phi_t)}\omega_{\phi_t}^n - \int_M (-\phi+\rho)e^{\theta_{X_0}+X(\rho)}(\omega'_{KS})^n,\end{aligned}$$

where ϕ_t is a K_{X_0}-invariant path in $\mathcal{M}(\omega_{KS})$ which connects 0 and $-\phi+\rho$, and $\theta_{X_0}(\phi_t)$ are potentials of X_0 associated to metric ω_{ϕ_t} defined by (2.1). It is proved in [Zh] that this well-defined for a K_0-invariant ϕ, i.e., the functional is independent of the choice of a K_0-invariant path. But for a general Kähler potential ϕ, one can also show that $(I-J)(\omega_\phi, \omega'_{KS})$ is not well-defined (to see (5.4) below), so we shall introduce another functional defined on whole space $\mathcal{M}(\omega_{KS})$ to replace it. In fact, we consider the following functional

$$\begin{aligned}\mathcal{F}(\omega_\phi, \omega'_{KS}) = \mathrm{Re}[&\int_0^1 dt \int_M (-\phi+\rho_\sigma)e^{\theta_{X_0}(t(-\phi+\rho_\sigma))}\omega_{t(-\phi+\rho_\sigma)}^n\\ &- \int_M (-\phi+\rho)e^{\theta'_{X_0}}(\omega'_{KS})^n].\end{aligned} \tag{5.1}$$

Clearly, the definition of $\mathcal{F}$ just uses a real part of $(I-J)(\omega_\phi, \omega'_{KS})$ while a Kähler potentials path is chosen by $\phi_t = t(-\phi+\rho_\sigma)$. We now consider a Kähler potentials path ρ_t induced by an one-parameter subgroup σ_t generated by the real part of $Y \in \eta_r(M)$, i.e. ρ_t are defined by $\omega_t = \sigma_t^*\omega'_{KS} = \omega'_{KS} + \sqrt{-1}\partial\bar{\partial}\rho_t$. Let

$$f_Y(t) = \mathrm{Re}[\int_0^{1+t} ds \int_M (\dot{\phi}_s)e^{\theta_{X_0}(\phi_s)}\omega_{\phi_s}^n - \int_M (-\phi+\rho_t)e^{\theta_{X_0}(\omega_t)}\omega_t^n], \tag{5.2}$$

where ϕ_s is a path in $\mathcal{M}(\omega_{KS})$ defined by $\phi_s = s(-\phi+\rho_\sigma),\ \forall\ 0 \le s \le 1$ and $\phi_s = -\phi+\rho_\sigma+\rho_t$, $1 \le s \le 1+t$. It is easy to see

$$\frac{d}{dt} f_Y(t)|_{t=0} = \int_M \theta'_Y(-\varphi+\rho_\sigma)e^{\theta'_X}(\omega'_{KS})^n.$$

This implies

$$\frac{d}{dt} f_Y(t)|_{t=0} = -\int_M \theta'_Y((\sigma^{-1})^*\varphi+\rho_{\sigma^{-1}})e^{\theta_X}\omega_{KS}^n \tag{5.3}$$

The gap between $f_Y(t)$ and $\mathcal{F}(\omega_\phi, \omega_t)$ can be computed as follows. Let $\Delta = \{(\tau, s)|\ 0 \le \tau \le 1,\ 0 \le s \le \tau + (1-\tau)(1+t)\}$ be a domain in $\mathbb{R}^2$. Let

$\Phi=\Phi(\tau,s;\cdot)$ be Kähler potentials with two parameters $(\tau,s)\in\Delta$ which satisfy:

$$\Phi=s(-\phi+\rho_\sigma+\rho_t),\ 0\le s\le 1,\ \text{as } \tau=1;$$
$$\Phi=\phi_s,\ 0\le s\le 1+t,\ \text{as } \tau=0;$$
$$\Phi=0,\ \text{as } s=0; \Phi=-\phi+\rho_\sigma+\rho_t,\ s=\tau+(1-\tau)(1+t).$$

Then by using the Stoke's formula, we have

$$\begin{aligned}
&|f_Y(t)-\mathcal{F}(\omega_\phi,\omega_t)|\\
&=|\mathrm{Re}\{\int_{\partial\Delta}\int_M d_{\tau,s}\Phi(\tau,s;\cdot)e^{\theta_{X_0}(\phi_s)}\omega_{\phi_s}^n\}|\\
&=|\mathrm{Re}\{\int_\Delta d\tau ds\int_M \dot{\Phi}_\tau(<\overline{\partial}\dot{\Phi}_s,\overline{\partial}\theta_{X_0}(\Phi)>-\\
&<\overline{\partial}\theta_{X_0}(\Phi),\overline{\partial}\dot{\Phi}_s>)e^{\theta_{X_0}(\Phi)}\omega_\Phi^n\}|\\
&=2|\mathrm{Re}\{\int_\Delta d\tau ds\int_M \dot{\Phi}_\tau\mathrm{Im}(X_0(\Phi_s))e^{\theta_{X_0}(\Phi)}\omega_\Phi^n\}|\\
&\le C\|X'(\phi)\|_{C^0}^2. \qquad (5.4)
\end{aligned}$$

At the last inequality, we used a fact that $X_0(\rho_\sigma)$ and $X_0(\rho_t)$ are both real-valued. Similarly, we can get

$$|\frac{d}{dt}(f_Y(t)-\mathcal{F}(\omega_\phi,\omega_t))|_{t=0}\le C\|X'(\phi)\|_{C^0}^2. \qquad (5.5)$$

Next we claim

$$\mathcal{F}(\sigma)=\mathcal{F}(\omega_\phi,\omega'_{KS})\ge 0. \qquad (5.6)$$

To prove the claim, we let

$$\begin{aligned}
g(t)=\mathrm{Re}[&\int_0^t ds\int_M(-\phi+\rho_\sigma)e^{\theta_{X_0}(s(-\phi+\rho_\sigma))}\omega_{s(-\phi+\rho_\sigma)}^n\\
&-\int_M(-\phi+\rho)e^{\theta_{X_0}(t(-\phi+\rho_\sigma))}\omega_{t(-\phi+\rho_\sigma)}^n].
\end{aligned}$$

Then

$$\mathcal{F}(\sigma)=g(1)=\int_0^1 g(t)'dt.$$

On the other hand, we have

$$\begin{aligned}
g(t)'&=\mathrm{Re}[n\sqrt{-1}\int_M\partial(-\phi+\rho_\sigma)\wedge\overline{\partial}(-\phi+\rho_\sigma)e^{\theta_{X_0}(t(-\phi+\rho_\sigma))}\omega_{t(-\phi+\rho_\sigma)}^n]\\
&\ge 0.
\end{aligned}$$

Thus we get $g(1)\ge 0$ and prove the claim.

By the above claim, we can take a minimizing sequence of $\mathcal{F}(\sigma)$ in $\mathrm{Aut}_r(M)$ and we see that for any small $\epsilon\le\epsilon_0$, there exists a $\sigma\in\mathrm{Aut}_r(M)$ with bounded $\mathrm{dist}(\sigma, Id)$ such that for any $Y \in \eta_r(M)$ with $\int_M \|Y\|^2\omega_{KS}^n = 1$, we have

$$|D\mathcal{F}(\sigma)(Y)| \leq \epsilon. \tag{5.7}$$

Therefore combining (5.3), (5.5) and (5.7), we prove the lemma while σ is replaced by σ^{-1}.

□

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