Lecture notes on the Ein-Popa extension result

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Abstract

These are lecture notes on a recent remarkable preprint of Ein-Popa, which simplifies the algebraic proof of the finite generation of the canonical ring given by the team BCHM. The Ein-Popa extension result has been translated in the analytic language by Berndson-Paun and Paun. In these notes we follow the analytic language used in Berndson-Paun and Paun. The author of this manuscript does not claim any originality of the main ideas and arguments which are due to Ein-Popa, based in their turn in the ideas of Hacon-McKernan, Takayama and Siu.

1 The Ein-Popa extension result

Let X be a complex manifold. The multiplier ideal sheaf $\mathcal{I}(\theta) \subset \mathcal{O}_X$ associated to a closed positive $(1,1)$-current θ is the sheaf of germs of holomorphic functions $f \in \mathcal{O}_x$ such that

$$\int_{U_x} |f|^2 e^{-\varphi} < +\infty\,,$$

where φ is a local potential of the current $\theta = \frac{i}{2\pi}\partial\bar{\partial}\varphi$ over some neighborhood U_x of the point x. Let $Z \subset X$ be a smooth hypersurface. If the restriction of the local potentials of θ to Z is not identically $-\infty$ on any local connected component of Z we can also define the multiplier ideal sheaf $\mathcal{I}(\theta_{|Z}) \subset \mathcal{O}_Z$ in a similar way. In this setting we introduce the adjoint ideal sheaf $\mathcal{I}_Z(\theta) \subset \mathcal{I}(\theta)$ of germs of holomorphic functions $f \in \mathcal{I}(\theta)_x$ such that

$$\int_{Z \cap U_x} |f|^2 e^{-\varphi} < +\infty\,.$$

We will use also the analogue notations $\mathcal{I}(\psi_{|Z}) \subset \mathcal{O}_Z\,, \mathcal{I}_Z(\psi) \subset \mathcal{I}(\psi)$ with respect to a global quasi-plurisubharmonic function ψ which is not identically $-\infty$ on any connected component of Z. We observe now the following claim.

Claim 1 *Let $Z \subset X$ be a smooth and irreducible hypersurface inside a complex projective manifold X and let L be a line bundles over X such that the class $c_1(L)$ admits a Kähler current θ with well defined restriction $\theta_{|Z}$. Then the restriction map*

$$H^0\left(X\,, \mathcal{O}_X(K_X + Z + L) \otimes \mathcal{I}_Z(\theta)\right) \longrightarrow H^0\left(Z\,, \mathcal{O}_Z(K_Z + L_{|Z}) \otimes \mathcal{I}(\theta_{|Z})\right)\,,$$

is surjective.

This claim is a direct consequence of the adjunction formula and the singular version of the Ohsawa-Takegoshi-Manivel extension theorem [Pal], [Mc-Va], [Man], [Dem2],[Oh-Ta], [Oh].

We will note by $\lambda(\theta, A) := \inf_{x\in A} \lambda(\theta)_x$ the generic Lelong number of θ along an irreducible complex analytic set $A \subset X$. Similar notations $\lambda(\psi)_x$ and $\lambda(\psi, A)$ will be employed also for global quasi-plurisubharmonic functions. We remind that a quasi-plurisubharmonic functions ψ is called with analytic singularities if it can locally be expressed as

$$\psi = c \log \sum_j |h_j|^2 + \rho \,,$$

with $c \in \mathbb{R}_{>0}$, h_j holomorphic functions and ρ a bounded function. A closed positive $(1,1)$-current is called with analytic singularities is its local potentials has this property. We remind the following well known (from algebraic geometers) claim (see also [Be-Pa], [Pau2] for a similar statement).

Claim 2 *Let $B \subset \mathbb{C}^n$ be an open ball, let $V \subset B$ be a hyperplane, let $v = 0$ be its equation and let ψ be a quasi-plurisubharmonic function with analytic singularities over B which is not identically $-\infty$ over V and let*

$$\Lambda_\Omega := \sup_{x\in\Omega} \lambda(\psi)_x < +\infty \,,$$

for any relatively compact open set $\Omega \subset\subset B$. Then

$$I_{\varepsilon,\delta} := \int_\Omega |v|^{-2(1-\varepsilon)} e^{-\delta\psi} < +\infty \,,$$

for all $\varepsilon \in (0,1)$ and $\delta \in (0, 2/\Lambda_\Omega)$.

Proof. The assumptions on the restriction to V of the function

$$\psi = c \log \sum_j |h_j|^2 + \rho \,,$$

implies the existence of a blow-up map $\mu : (v,\zeta) \mapsto (v,z)$ such that

$$\psi \circ \mu = c \log |\zeta^\alpha|^2 + R \,,$$

with $\alpha \in \mathbb{Z}^{n-1}_{\geq 0}$ and R a bounded function. This last equality follows from the fact that we can construct the blow-up map μ in a way that the sheaf $\mu^* \sum_j \mathcal{O} \cdot h_j$ is invertible. Moreover we can also assume that the Jacobian $J(\mu)$ of μ equal to a monomial ζ^β, $\beta \in \mathbb{Z}^{n-1}_{\geq 0}$ up to an invertible factor. On the other hand Skoda's lemma implies

$$+\infty \;>\; \int_\Omega e^{-\delta\psi} \;=\; \int_{\mu^{-1}(\Omega)} |\zeta^\alpha|^{-2c\delta} \, |J(\mu)|^2 \, e^{-\delta R} \,,$$

for all $\delta \in (0, 2/\Lambda_\Omega)$. Thus Fubini's formula implies that the integrability of the function $f_\delta := |\zeta^\alpha|^{-2c\delta} |\zeta^\beta|^2$ is a sufficient condition for the convergence of the integrals $I_{\varepsilon,\delta}$. □

We prove now the following analytic version of the Ein-Popa [Ei-Po] extension result (see [Be-Pa] and [Pau2] for similar statements). The idea of the proof is due to Ein-Popa [Ei-Po], based in their turn in the ideas of Hacon-McKernan [Ha-Mc1], [Ha-Mc2], Takayama [Tak] and Siu [Siu1], [Siu2]. We will follow closely the translation in the analytic language by Berndson-Paun [Be-Pa] and Paun [Pau1], [Pau2].

Lemma 1 *Let X be a complex projective manifold, let $Z \subset X$ be a smooth irreducible hypersurface and let L be a holomorphic $\mathbb{Q}$-line bundle over X such that;*

I) *there exists a closed positive $(1,1)$-current $\Theta \in c_1(K_X + Z + L)$ with well defined restriction $\Theta_{|Z}$,*

II) *there exists a decomposition of $\mathbb{Q}$-line bundles $L = \mathcal{O}_X(\Delta) + \mathcal{O}_X(D) + R$, where;*

• *$\Delta = \sum_{j=1}^{N} \lambda_j Z_j$ is a divisor over X with $\lambda_j \in \mathbb{Q} \cap [0,1)$ and $Z_j \subset X$ distinct irreducible smooth hypersurfaces with normal crossing intersection with Z such that $Z \cap Z_j \cap Z_l = \emptyset$ for all $j \neq l$.*

• *D is an effective $\mathbb{Q}$-divisor over X such that Z is not one of its components and R is a holomorphic $\mathbb{Q}$-line bundle over X which admits a Kähler current $\rho \in c_1(R)$ with bounded local potentials over Z.*

Let also $(V_t)_{t=1}^{N'}$ be the irreducible components of the family $(Z_j \cap Z)_{j=1}^{N}$ and let $m \in \mathbb{N}_{>1}$ such that $m\lambda_j \in \mathbb{N}$ for all j, mD is integral and mR is a holomorphic line bundle. Then for any section

$$u \in H^0\Big(Z\,,\, m\left(K_Z + L_{|Z}\right)\Big)\,,$$

with the vanishing property

$$\operatorname{div} u \;-\; m\left(\sum_{t=1}^{N'} \lambda(\Theta_{|Z}\,, V_t)\, V_t \;+\; D_{|Z}\right) \;\geq\; 0\,, \tag{1.1}$$

there exists a section

$$U \in H^0\Big(X\,,\, m(K_X + Z + L)\Big)\,, \qquad U_{|Z} = u \otimes (d\zeta)^m\,,$$

with $\zeta \in H^0(X, \mathcal{O}(Z))$ such that $\operatorname{div}\zeta = Z$.

Proof.
Notations. For all $\nu \in \mathbb{N}$ we define the integers

$$k_\nu := \max\{k \in \mathbb{N} \,:\, km \leq \nu\}\,,$$

and $q_\nu := \nu - k_\nu m = 0, ..., m-1$. Let $\omega > 0$ be a Kähler form over X and set

$$\Omega_X := \frac{\omega^n}{n!}\,, \qquad \Omega_Z := \frac{\omega^{n-1}_{|Z}}{(n-1)!}\,,$$

We consider now the crucial Ein-Popa decomposition [Ei-Po]

$$\mathcal{O}_X(m\Delta) = L_1 + \cdots + L_{m-1}\,,$$

with $L_k := \mathcal{O}_X(\Delta_k)$ and with

$$\Delta_k := \sum_{m\lambda_j = k} Z_j\,,$$

for all $k = 1, ..., m-1$. Let h_{Z_j} be smooth hermitian metrics over $\mathcal{O}_X(Z_j)$, let h_{L_k} be the induced smooth hermitian metric over L_k and let denote by $h_{L_k} e^{-\varphi_k}$ the canonical singular hermitian metric associated to the divisor Δ_k. We equip the line bundle

$$L_m := \mathcal{O}_X(mD) + mR\,,$$

with the singular hermitian metric $h_{L_m} e^{-\varphi_m}$ such that

$$2\pi\, m\, (\,[D] + \rho) = i\,\mathcal{C}_{h_{L_m}}(L_m) + i\partial\bar{\partial}\varphi_m\,.$$

(As before h_{L_m} is smooth.) Let also h_Z be an arbitrary smooth hermitian metric on $\mathcal{O}(Z)$. We equip the line bundle

$$F_m \;:=\; m(K_X + Z) + \sum_{j=1}^{m} L_j \;=\; m(K_X + Z + L)\,,$$

with the smooth hermitian metric

$$h_m := \Omega_X^{-m} \otimes h_Z^m \otimes h_{L_1} \otimes \cdots \otimes h_{L_m}\,.$$

Let (A, h_A) be an ample line bundle over X with $0 < \omega_A := i\,\mathcal{C}_{h_A}(A)$ and let define for all $\nu \in \mathbb{N}$ the Siu-Demailly [Dem3], [Siu1], [Siu2] type line bundle

$$\mathcal{L}_\nu := k_\nu F_m + q_\nu (K_X + Z) + \sum_{j=0}^{q_\nu} L_j\,,$$

with $L_0 := A$. We equip the line bundle $\mathcal{L}_\nu$ with the smooth hermitian metric

$$H_\nu := h_m^{k_\nu} \otimes \Omega_X^{-q_\nu} \otimes h_Z^{q_\nu} \otimes h_{L_0} \otimes h_{L_1} \otimes \cdots \otimes h_{L_{q_\nu}}\,.$$

We choose the line bundle (A, h_A) sufficiently ample such that;

(A1) for all $q = 0, ..., m-1$ the line bundle $\mathcal{L}_{q|Z} \equiv qK_Z + (L_0 + \cdots + L_q)_{|Z}$ is base point free, globally generated by some family $(s_{q,j})_{j=1}^{N_q} \subset H^0(Z, \mathcal{L}_{q|Z})$,

(A2) the restriction map $H^0(X, F_m + A) \longrightarrow H^0(Z, F_m + A)$ is surjective,

(A3) for all $q = 0, ..., m-1$ hold the inequality $i\,\mathcal{C}_{H_q}(\mathcal{L}_q) \geq 2\pi m\omega$.

We note by $|\cdot|_\nu$ the norm of the smooth hermitian metric

$$\Omega_Z^\nu \otimes h_{L_1}^{k_\nu} \otimes \cdots \otimes h_{L_m}^{k_\nu} \otimes h_{L_0} \otimes h_{L_1} \otimes \cdots \otimes h_{L_{q_\nu}}\,,$$

over the line bundle

$$\mathcal{L}_{\nu|Z} \;\equiv\; \nu K_Z + k_\nu \sum_{j=1}^{m} L_{j|Z} + \sum_{j=0}^{q_\nu} L_{j|Z} \;=\; \nu K_Z + k_\nu m L_{|Z} + \sum_{j=0}^{q_\nu} L_{j|Z}\,.$$

The assumption (A1) implies

$$\max_{0 \leq p,q \leq m-1} \max_{Z} \frac{\sum_{j=1}^{N_q} |s_{q,j}|_q^2}{\sum_{t=1}^{N_p} |s_{p,t}|_p^2} \;=\; C \;<\; +\infty\,.$$

We prove now the following claim (see also [Ei-Po], [Tak], [Be-Pa], [Pau1] and [Pau2]).

Claim 3 *Let u and ζ as in the statement of the lemma 1 and let*

$$\sigma_{\nu,j} := u^{k_\nu} \otimes s_{q_\nu,j} \in H^0(Z, \mathcal{L}_{\nu|Z}), \qquad j = 1, ..., M_\nu := N_{q_\nu}.$$

Then for all $\nu \in \mathbb{N}_{\geq m}$ there exists a family of sections $(S_{\nu,j})_{j=1}^{M_\nu} \subset H^0(X, \mathcal{L}_\nu)$ such that $S_{\nu,j|Z} = \sigma_{\nu,j} \otimes (d\zeta)^\nu$.

Proof. The proof of this claim goes by induction. The statement is obvious for $\nu = m$ by the assumption (A2). So we assume it true for ν and we prove it for $\nu + 1$. We have

$$\mathcal{L}_{\nu+1} = k_\nu F_m + (q_\nu + 1)(K_X + Z) + \sum_{j=0}^{q_\nu+1} L_j = K_X + Z + \mathcal{L}_\nu + L_{q_\nu+1},$$

if $q_\nu \leq m - 2$ and

$$\mathcal{L}_{\nu+1} \;=\; (k_\nu + 1)F_m + L_0 \;=\; K_X + Z + \mathcal{L}_\nu + L_m,$$

if $q_\nu = m - 1$. So in all cases hold the induction formula

$$\mathcal{L}_{\nu+1} = K_X + Z + \mathcal{L}_\nu + L_{q_\nu+1}.$$

We will equip the line bundle $\mathcal{L}_\nu + L_{q_\nu+1}$ with an adequate singular hermitian metric with strictly positive curvature. For this purpose let $(\varepsilon_\nu)_\nu, (\delta_\nu)_\nu \subset (0, 1)$ and consider the Hacon-McKernan decomposition [Ha-Mc1], [Ha-Mc2], [Be-Pa], [Pau2]

$$\begin{aligned} \mathcal{L}_\nu + L_{q_\nu+1} \;&=\; (1 - \varepsilon_\nu)L_{q_\nu+1} + \varepsilon_\nu L_{q_\nu+1} \\ &+\; (1 - \delta_\nu)\mathcal{L}_\nu + \delta_\nu(k_\nu F_m + \mathcal{L}_{q_\nu}), \end{aligned} \tag{1.2}$$

in the case $q_\nu \leq m - 2$. The case $q_\nu = m - 1$ will not present any difficulty. Let $\tau_\nu \in (0, k_\nu^{-1})$. According to Demailly's regularising process [Dem1], we can replace the current Θ with a family of closed and real $(1, 1)$-currents with analytic singularities $\Theta_\nu \in \{\Theta\}$, $\Theta_\nu \geq -\tau_\nu\, \omega$, such that the restrictions $\Theta_{\nu|Z}$ are also well defined and

$$\lambda(\Theta_{\nu|Z})_z \;\leq\; \lambda(\Theta_{|Z})_z,$$

for all $z \in Z$ and ν. This combined with the condition (1.1) implies

$$\operatorname{div} u - m \sum_{t=1}^{N'} \lambda(\Theta_{\nu|Z}, V_t)\, V_t \geq 0. \tag{1.3}$$

Let now ψ_ν be a quasi-plurisubharmonic function with analytic singularities such that

$$2\pi\, m\, \Theta_\nu = i\, \mathcal{C}_{h_m}(F_m) + i\partial\bar{\partial}\psi_\nu,$$

let $B_\nu := \sum_{j=1}^{M_\nu} |S_{\nu,j}|^2_{H_\nu}$, let $\Phi_\nu := \log B_\nu$ and set

$$\Psi_\nu \;:=\; \begin{cases} (1 - \varepsilon_\nu)\varphi_{q_\nu+1} + (1 - \delta_\nu)\Phi_\nu + \delta_\nu\, k_\nu\, \psi_\nu, & \text{if} \quad q_\nu \leq m - 2, \\ \varphi_m + \Phi_\nu, & \text{if} \quad q_\nu = m - 1. \end{cases}$$

We show now that for adequate choices of the parameters $\varepsilon_\nu, \delta_\nu$ the singular hermitian line bundle

$$(\mathcal{L}_\nu + L_{q_\nu+1}\,,\, H_\nu \otimes h_{L_{q_\nu+1}} e^{-\Psi_\nu})\,, \tag{1.4}$$

is big and

$$\sigma_{\nu+1,j} \in H^0\left(Z\,,\, \mathcal{O}_Z(\mathcal{L}_{\nu+1|Z}) \otimes \mathcal{I}(\Psi_{\nu|Z})\right)\,. \tag{1.5}$$

Then the conclusion of the claim 3 will follow by applying the claim 1 to the section $\sigma_{\nu+1,j} \otimes (d\zeta)^\nu$ in order to obtain the required extensions $S_{\nu+1,j}$. We distinguish again two cases.

Case $q_\nu \leq m-2$. By (1.2) we infer the decomposition of the $(1,1)$-current

$$\begin{aligned}
& i\,\mathcal{C}_{H_\nu}(\mathcal{L}_\nu) + i\,\mathcal{C}_{h_{L_{q_\nu+1}}}(L_{q_\nu+1}) + i\partial\bar\partial\Psi_\nu \\
= \;& (1-\varepsilon_\nu)\left[i\,\mathcal{C}_{h_{L_{q_\nu+1}}}(L_{q_\nu+1}) + i\partial\bar\partial\varphi_{q_\nu+1}\right] + \varepsilon_\nu\, i\,\mathcal{C}_{h_{L_{q_\nu+1}}}(L_{q_\nu+1}) \\
+ \;& (1-\delta_\nu)\left[i\,\mathcal{C}_{H_\nu}(\mathcal{L}_\nu) + i\partial\bar\partial\Phi_\nu\right] + \delta_\nu\left[i\,\mathcal{C}_{H_\nu}(\mathcal{L}_\nu) + k_\nu i\partial\bar\partial\psi_\nu\right] \\
\geq \;& \varepsilon_\nu\, i\,\mathcal{C}_{h_{L_{q_\nu+1}}}(L_{q_\nu+1}) + \delta_\nu\left[2\pi\, k_\nu m\,\Theta_\nu + i\,\mathcal{C}_{H_{q_\nu}}(\mathcal{L}_{q_\nu})\right] \\
\geq \;& \varepsilon_\nu\, i\,\mathcal{C}_{h_{L_{q_\nu+1}}}(L_{q_\nu+1}) + 2\pi\, m\,\delta_\nu(1-k_\nu\tau_\nu)\omega\,,
\end{aligned}$$

by the assumption (A3). We infer that if $C_\omega > 0$ is a constant such that $i\,\mathcal{C}_{h_j}(L_j) \geq -2\pi C_\omega\,\omega$ for all $j = 1, ..., m-1$ then the bundle (1.4) is big as soon as

$$\varepsilon_\nu < m(1-k_\nu\tau_\nu)\delta_\nu / C_\omega\,. \tag{1.6}$$

On the other hand the relation $\sigma_{\nu+1,j} = u^{k_\nu} \otimes s_{q_\nu+1,j}$, $j = 1, ..., M_{\nu+1}$ combined with the fact that

$$B_{\nu\,|Y} = |d\zeta|^{2\nu}_{\omega,h_Z} \sum_{t=1}^{M_\nu} |u^{k_\nu} \otimes s_{q_\nu,t}|^2_\nu\,,$$

and with the definition of the constant C implies

$$I_{\nu+1} := \int_Z |\sigma_{\nu+1,j}|^2_{\nu+1}\, e^{-\Psi_\nu} \leq C' \int_Z |u|^{2k_\nu\delta_\nu}_{h'_m}\, e^{-(1-\varepsilon_\nu)\varphi_{q_\nu+1} - \delta_\nu k_\nu \psi_\nu}\,, \tag{1.7}$$

with $h'_m := \Omega_Z^{-m} \otimes h_{L_1} \otimes \cdots \otimes h_{L_m}$. We consider now the decomposition

$$2\pi\, m\Theta_{\nu|Z} = 2\pi[W_\nu] + \alpha_\nu + i\partial\bar\partial g_\nu\,, \tag{1.8}$$

with

$$W_\nu := m\sum_{t=1}^{N'} \lambda(\Theta_{\nu|Z}, V_t)\, V_t\,,$$

with α_ν a smooth closed and real $(1,1)$-form and with g_ν a quasi-plurisub harmonic function with analytic singularities such that $\lambda(g_\nu, V_t) = 0$ for all

$t = 1, ..., N'$.
In particular g_ν is not identically $-\infty$ over the sets V_t.
Then the the decomposition (1.8) combined with the Lelong-Poincaré formula implies

$$2\pi([\operatorname{div} u] - [W_\nu]) = 2\pi([\operatorname{div} u] - m\Theta_{\nu\,|Z}) + \alpha_\nu + i\partial\bar{\partial} g_\nu$$

$$= \beta_\nu + i\partial\bar{\partial} f_\nu\,,$$

with β_ν a smooth closed and real $(1,1)$-form and with

$$f_\nu := \log |u|^2_{h'_m} - \psi_\nu + g_\nu\,.$$

The condition (1.3) rewrites as $0 \le \operatorname{div} u - W_\nu$. We infer that f_ν is a quasi-plurisubharmonic function, thus bounded from above. We infer by (1.7) the inequality

$$I_{\nu+1} \le C' \int_Z e^{-(1-\varepsilon_\nu)\varphi_{q_\nu+1} + \delta_\nu k_\nu (f_\nu - g_\nu)} \le C'' \int_Z e^{-(1-\varepsilon_\nu)\varphi_{q_\nu+1} - \delta_\nu k_\nu g_\nu}\,. \qquad (1.9)$$

On the other hand

$$\Lambda_\nu := \sup_{z\in Z} \lambda(g_\nu)_z < +\infty\,,$$

since Z is compact. Thus the last integral in (1.9) is convergent for all values $\varepsilon_\nu \in (0,1)$ and

$$0 < \delta_\nu < 2(k_\nu \Lambda_\nu)^{-1}\,, \qquad (1.10)$$

by the claim 2 and so the condition (1.5) is satisfied in the case $q_\nu \le m-2$.

Case $q_\nu = m-1$. In this case the condition (1.4) is obviously satisfied. On the other hand the relation $\sigma_{\nu+1,j} = u^{k_\nu+1} \otimes s_{0,j}$ combined with the fact that

$$B_{\nu\,|Y} = |d\zeta|^{2\nu}_{\omega,h_Z} \sum_{t=1}^{N_{m-1}} |u^{k_\nu} \otimes s_{m-1,t}|^2_\nu\,,$$

and the definition of the constant C implies

$$I_{\nu+1} \le C' \int_Z |u|^2_{h'_m} e^{-\varphi_m} < +\infty\,, \qquad (1.11)$$

The convergence follows from the condition (1.1) and the fact that ρ has bounded local potentials along Z. This concludes the proof of the claim 3. □

End of the proof. The claim 3 implies that the singular hermitian line bundle

$$\left(\mathcal{L}_{km}\,,\, H_{km} B^{-1}_{km}\right) \equiv \left(kF_m + A\,,\, h^k_m \otimes h_A\, B^{-1}_{km}\right)\,,$$

is pseudoeffective. So we have obtain the following;

$$i\,\mathcal{C}_{h_m}(F_m) + \frac{1}{k}\, i\,\partial\bar{\partial}\Phi_{km} \ge -\frac{1}{k}\,\omega_A\,, \qquad (1.12)$$

$$\frac{1}{k}\,\Phi_{km\,|Z} = \log |u|^2_{h'_m} + \frac{1}{k}\log\left(|d\zeta|^{2km}_{\omega,h_Z} \sum_{j=0}^{N_0} |s_{0,j}|^2_{h_A}\right)\,. \qquad (1.13)$$

Let $h_{mF} := h_{L_1} \otimes \cdots \otimes h_{L_m}$, let $\varphi_\Delta := \frac{1}{m}\sum_{j=1}^{m-1} \varphi_j$ and set

$$\Xi_k := \frac{m-1}{mk}\,\Phi_{km} + \varphi_\Delta + \frac{1}{m}\varphi_m\,.$$

Then the $\mathbb{Q}$-decomposition

$$(m-1)(K_X+Z)+mL = \frac{m-1}{m}\,F_m + \Delta + \frac{1}{m}\,L_m\,,$$

combined with the inequality (1.12) shows that the singular hermitian line bundle

$$\Big((m-1)(K_X+Z)+mL\,,\,\Omega_X^{-(m-1)}\otimes h_Z^{m-1}\otimes h_{mL}\,e^{-\Xi_k}\Big)\,,$$

is big as soon as

$$k > (m-1)C_A/\varepsilon\,, \tag{1.14}$$

with $\varepsilon\,, C_A \in \mathbb{R}_{>0}$ such that $\rho \geq \varepsilon\,\omega$ and $\omega_A \leq 2\pi\,mC_A\,\omega$. On the other hand the expression (1.13) and the condition (1.1) imply

$$\int_Z |u|^2_{h'_m}\,e^{-\Xi_k} \;\leq\; C_k \int_Z |u|^{2/m}_{h'_m} e^{-\varphi_\Delta-\varphi_m/m} \;\leq\; C'_k \int_Z e^{-\varphi_\Delta} \;<\; +\infty\,,$$

since $h_{L_1}\otimes\cdots\otimes h_{L_{m-1}} e^{-m\varphi_\Delta}$ is the canonical metric associated to the integral divisor $m\Delta$ and $\lambda_j < 1$. In conclusion we can apply the claim 1 to the section

$$u \otimes (d\zeta)^{m-1} \in H^0\Big(Z\,,\,K_Z + (m-1)(K_X+Z)+mL\Big)\,,$$

in order to obtain the required lifting U of the section u. $\square$

1.1 A perturbed extension statement

The Ein-Popa extension result [Ei-Po] previously explained modifies quite directly in a perturbed extension statement due to Paun [Pau2]. We explain now this statement. For any $\mathbb{Q}$-line bundle/divisor E we fix a smooth form $\theta_E \in c_1(E)$. We observe the following quite elementary fact.

Claim 4 *Let A_0 be an ample line bundle over a complex projective variety X of complex dimension n, let $\omega \in c_1(A_0)$, $\omega > 0$, let Z, $Z_j \subset X$, $j = 1, ..., N$ be irreducible divisors and let D be a $\mathbb{Q}$-divisor over X. Let also $C_0 \in \mathbb{N}_{>0}$ such that*

$$\theta_Z\,, \theta_{Z_j}\,, \theta_D\,, \theta_{K_X}\,, \frac{n-1}{\pi}\, i\partial\bar{\partial} \log \operatorname{dist}_\omega(x, \cdot) \;\geq\; -C_0\,\omega\,,$$

for all $j = 1, ..., N$ and $x \in X$. Then for any holomorphic $\mathbb{Q}$-line bundle R as in the statement of the lemma 1, any $m \in \mathbb{N}_{>1}$ such that mD, mR are integral and any subset

$$\mathcal{S} \subset \{Z_j \,:\, j = 1, ..., N\} \times \{1, ..., m-1\}\,,$$

the family of holomorphic line bundles $(L_k)_{k=1}^m$ defined by

$$L_k := \mathcal{O}_X(\Delta_k)\,, \qquad \Delta_k := \sum_{Z \in \mathcal{S}_k} Z\,, \qquad \mathcal{S}_k := \{Z_j \,:\, j = 1, ..., N\} \times \{k\}\,,$$

for all $k = 1, ..., m-1$ and $L_m := \mathcal{O}_X(mD) + m\,R$, satisfies the properties (AI), $I = 1, 2, 3$ in the proof of the lemma 1 with respect to

$$A := m[2 + (N+3)C_0]A_0\,.$$

Proof. The inequality $\theta_{L_k} \geq -NC_0\,\omega$ for all $k = 1, ..., m-1$ implies

$$\theta_{\mathcal{L}_q} \;\geq\; \theta_A - (m-1)(N+2)C_0\,\omega\,, \quad \forall q = 0, ..., m-1\,.$$

For $q = m$ hold the inequality

$$\Theta_{\mathcal{L}_m} \;\geq\; \theta_A - (m-1)(N+2)C_0\,\omega - mC_0\omega\,,$$

where $\Theta_{\mathcal{L}_m} \in c_1(\mathcal{L}_m)$ is a current with bounded potentials along Z. On the other hand the Kawamata-Viehweg-Nadel vanishing theorem and the claim 1 imply that the properties $(A1)$ and $(A2)$ in the proof of the lemma 1 are satisfied with respect to A in the statement of the claim 4. This choice of A satisfies also the property $(A3)$. □

Corollary 1 *Let X be a complex projective manifold, let $Z \subset X$ be a smooth irreducible hypersurface, let A_0 be an ample line bundle over X, let $\omega \in c_1(A_0)$, $\omega > 0$ and let L be a holomorphic $\mathbb{Q}$-line bundle over X which admits a decomposition as*

$$L = \mathcal{O}_X(\Delta) + \mathcal{O}_X(D) + R\,,$$

where;
▶ $\Delta = \sum_{j=1}^{N} \lambda_j Z_j$ *is a divisor over X with $\lambda_j \in \mathbb{Q} \cap [0,1)$ and $Z_j \subset X$ distinct irreducible smooth hypersurfaces with normal crossing intersection with Z such that $Z \cap Z_j \cap Z_l = \emptyset$ for all $j \neq l$,*

▶ *D is an effective $\mathbb{Q}$-divisor over X such that Z is not one of its components and the components $(\Gamma_p)_{p=1}^{Q}$ of the restricted divisor $D_{|Z}$ does not intersect the irreducible components $(V_t)_{t=1}^{N'}$ of the family $(Z_j \cap Z)_{j=1}^{N}$,*

▶ *R is a holomorphic $\mathbb{Q}$-line bundle over X such that there exists a Kähler current $\rho \in c_1(R)$ with $\rho \geq \varepsilon\omega$, $\varepsilon \in \mathbb{R}_{>0}$ and with bounded local potentials along Z.*

• *Let $C_0 \in \mathbb{N}_{>0}$ as in the statement of the claim 4, let*

$$C_1 := 2 + (N+3)C_0\,, \qquad C_2 := NC_0C_1\,, \qquad \lambda \;:=\; \max_{1\leq j\leq N} \lambda_j\,.$$

• *Let $m \in \mathbb{N}_{>1}$, such that $m\Delta$, mD are integral, mR is a holomorphic line bundle and*

$$m \;\geq\; \frac{1}{2C_2(1-\lambda)\lceil 1/\varepsilon\rceil}\,.$$

• *Let $V := \sum_{t=1}^{N'} V_t$, let $\Gamma := \sum_{p=1}^{Q} \Gamma_p$ and let $\eta \in \mathbb{R}_{>0}$ such that $\eta < 1/\operatorname{mult}(\Gamma)$.*

Assume the existence of a closed $(1,1)$-current $\Theta \in c_1(K_X+Z+L)$ with analytic singularities and with well defined restriction $\Theta_{|Z}$ such that

$$\Theta \;\geq\; -\,\frac{1}{2C_1\lceil 1/\varepsilon\rceil}\,\frac{1}{m}\,\omega\,. \tag{1.15}$$

Then for any $u \in H^0\Big(Z\,,\, m\left(K_Z + L_{|Z}\right)\Big)$ with the vanishing property

$$\operatorname{div} u \;-\; m\left(\sum_{t=1}^{N'} \lambda(\Theta_{|Z}\,, V_t)\, V_t \;+\; D_{|Z}\right) \;\geq\; -\,\frac{1}{3C_2\lceil 1/\varepsilon\rceil}\, V \;-\; \eta\Gamma\,, \tag{1.16}$$

there exists a section

$$U \in H^0\Big(X\,,\, m(K_X + Z + L)\Big)\,, \qquad U_{|Z} = u \otimes (d\zeta)^m\,,$$

with $\zeta \in H^0(X, \mathcal{O}(Z))$ such that $\operatorname{div}\zeta = Z$.

Proof. We repeat the proof of the lemma 1 with some very little modifications. The data $(\lambda_j)_j$ determines a set $\mathcal{S}$ as in the statement of the claim 4. Thus the conditions (AI), $I = 1, 2, 3$ in the proof of the lemma 1 are satisfied with respect to A in statement of the claim 4. We perform the induction of the claim 3 for the steps $\nu = m, ..., \bar{k}m$, with $\bar{k} := m\, C_1 \lceil 1/\varepsilon \rceil$.

The case $q_\nu \leq m-2$**.** We replace the currents Θ_ν in the proof of the lemma 1 with the current $\Theta \geq -\tau\, \omega$,

$$\tau := \frac{1}{2\bar{k}}\,,$$

and we reconsider the conditions needed for the parameters $\varepsilon_\nu \equiv \bar{\varepsilon} > 0$, $\delta_\nu \equiv \bar{\delta} > 0$. With the notations of the claim 4 hold the inequality $\theta_{L_k} \geq -NC_0\, \omega$. We infer that in our setting the condition (1.6) on the bigness of the singular hermitian line bundle (1.4) becomes

$$0 \,<\, \bar{\varepsilon} \,<\, \frac{m(1-k_\nu \tau)\bar{\delta}}{NC_0}\,.$$

We observe that the inequality $1 - k_\nu \tau > 0$ is satisfied for all $\nu = m, ..., \bar{k}m$ by our definition of $\tau > 0$. So a first condition on $\bar{\varepsilon}$ is

$$\bar{\varepsilon} \,<\, \frac{m(1-\bar{k}\tau)\bar{\delta}}{NC_0}\,.$$

Let now ψ, W, α and g correspond respectively to ψ_ν, W_ν, α_ν and g_ν in the proof of the claim 3 and let φ_V, φ_Γ such that

$$2\pi[V] = \theta_V + i\partial\bar{\partial}\varphi_V\,, \qquad 2\pi[\,\Gamma\,] = \theta_\Gamma + i\partial\bar{\partial}\varphi_\Gamma\,,$$

for some smooth $(1,1)$-forms θ_V and θ_Γ. Let

$$\mu := \frac{1}{3C_2\lceil 1/\varepsilon \rceil}\,.$$

This definition implies the inequality

$$\mu < m \min\left\{\frac{1/\bar{k} - \tau}{NC_0}\,,\, 1-\lambda\right\}\,, \tag{1.17}$$

by our choice of m. By the vanishing condition (1.16) and the Lelong-Poincaré formula we infer

$$\begin{aligned} 0 \;&\leq\; 2\pi\left([\operatorname{div} u - W + \mu V + \eta\Gamma]\right) \\ &=\; 2\pi\left([\operatorname{div} u + \mu V + \eta\Gamma] - m\Theta_{|Z}\right) + \alpha + i\partial\bar{\partial} g \\ &=\; \beta + i\partial\bar{\partial} f\,, \end{aligned}$$

with β a smooth $(1,1)$-form and with

$$f \;:=\; \log|u|^2_{h'_m} \;-\; \psi \;+\; g \;+\; \mu\varphi_V \;+\; \eta\varphi_\Gamma\,,$$

quasi-plurisubharmonic, thus bounded from above. We infer

$$\begin{aligned} I_{\nu+1} \;&\leq\; C' \int_Z e^{-(1-\bar{\varepsilon})\varphi_{q_\nu+1} \,+\, \bar{\delta}k_\nu(f - g - \mu\varphi_V - \eta\varphi_\Gamma)} \\ &\leq\; C'' \int_Z e^{-(1-\bar{\varepsilon})\varphi_{q_\nu+1} \,-\, \bar{\delta}k_\nu\mu\varphi_V \,-\, \bar{\delta}k_\nu(g+\eta\varphi_\Gamma)}\,. \end{aligned} \tag{1.18}$$

Let

$$\Lambda_\eta := \sup_{z\in Z} \lambda(g+\eta\varphi_\Gamma)_z \, < \, +\infty \, .$$

By the claim 2 the integral (1.18) is finite if $\bar{\delta} k_\nu \mu < \bar{\varepsilon}$ and $\bar{\delta} \, < \, 2(k_\nu \Lambda_\eta)^{-1}$ for all $\nu = m, ..., \bar{k}m$. So we take $\bar{\varepsilon}$ and $\bar{\delta}$ such that

$$\bar{\delta}\bar{k}\mu \, < \, \bar{\varepsilon} \, < \, \frac{m(1-\bar{k}\tau)\bar{\delta}}{NC_0} \, , \qquad 0 \, < \, \bar{\delta} \, < \, 2(\bar{k}\Lambda_\eta)^{-1} \, .$$

The existence of $\bar{\varepsilon}$ follows from the inequality (1.17).

The case $q_\nu = m-1$. We consider the decomposition $\varphi_m = \varphi_{mD} + \varphi_{m\rho}$, where φ_{mD} and $\varphi_{m\rho}$ are potentials corresponding respectively to the closed positive currents $2\pi[mD]$ and $2\pi m\rho$. The vanishing condition (1.16) and the Lelong-Poincaré formula imply

$$0 \, \leq \, 2\pi \left([\operatorname{div} u - mD_{|Z} + \mu V + \eta \Gamma] \right) \, = \, \tilde{\beta} + i\partial\bar{\partial}\tilde{f} \, ,$$

with $\tilde{\beta}$ a smooth $(1,1)$-form and with

$$\tilde{f} := \log |u|^2_{h'_m} - \varphi_{mD} + \mu\, \varphi_V + \eta\, \varphi_\Gamma \, ,$$

quasi-plurisubharmonic, thus bounded from above. We infer

$$I_{\nu+1} \leq \tilde{C}' \int_Z |u|^2_{h'_m} e^{-\varphi_m} = C' \int_Z e^{\tilde{f} - \varphi_{m\rho} - \mu\,\varphi_V - \eta\,\varphi_\Gamma} \leq \tilde{C}'' \int_Z e^{-\mu\,\varphi_V - \eta\,\varphi_\Gamma} < +\infty \, ,$$

since $\mu < 1$, since the singular part of φ_V does not intersect with the singular part of φ_Γ and since $\varphi_{m\rho}$ is bounded along Z by assumption.

End of the proof. The constant $C_A > 0$ in the proof of the lemma 1 corresponds to C_1. We infer that the condition (1.14) becomes $k > (m-1)C_1/\varepsilon$, which is satisfied by our choice of the integer $\bar{k}$. On the other hand

$$\begin{aligned} \int_Z |u|^2_{h'_m} e^{-\Xi_{\bar{k}}} \quad &\leq \quad C_k \int_Z |u|^{2/m}_{h'_m} e^{-\varphi_\Delta \, - \, \varphi_m/m} \\ &= \quad C_k \int_Z e^{-\varphi_\Delta \, + \, (\tilde{f} \, - \, \varphi_{m\rho} \, - \, \mu\varphi_V \, - \, \eta\,\varphi_\Gamma)/m} \\ &\leq \quad C'_k \int_Z e^{-\varphi_\Delta \, - \, \mu\varphi_V/m \, - \, \eta\,\varphi_\Gamma/m} \, < \, +\infty \, , \end{aligned}$$

since $\lambda_j + \mu/m < 1$ for all $j = 1, ..., N$ by the inequality (1.17) and since the singular part of $\varphi_\Delta + \mu\varphi_V/m$ does not intersect with the singular part of φ_Γ by our assumption on the components of the divisor $D_{|Z}$. □

2 Applications to the Non-Vanishing

In this section we will discuss the issues related with the application of the perturbed extension statement of Ein-Popa type to the following fundamental Non-Vanishing result due to the team BCHM [BCHM]. Let X be a complex projective manifold. We will consider the Neron-Severi lattice and the associated real Neron-Severi space

$$\begin{aligned} \mathrm{NS}(X) &:= H^{1,1}(X,\mathbb{R}) \cap (H^2(X,\mathbb{Z})/\{\text{torsion}\}), \\ \mathrm{NS}_{\mathbb{R}}(X) &:= \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}. \end{aligned}$$

Theorem 1 (Non-Vanishing result).
Let X be a complex projective manifold and let $\alpha \in \mathrm{NS}_{\mathbb{R}}(X)$ admitting a Kähler current $\theta \in \alpha$ with $\mathcal{I}(\theta) = \mathcal{O}_X$ and such that the $(1,1)$-cohomology class $c_1(K_X) + \alpha$ is pseudoeffective. Then there exist a non zero effective $\mathbb{R}$-divisor D such that

$$[D] \in c_1(K_X) + \alpha\,.$$

2.1 Shokurov's modification of the manifold

The following construction is essentially due to Shokurov. The slight modification explained here is similar to the one explained in [Pau2].

Lemma 2 *Under the assumptions of the Non-Vanishing theorem 1 and under the assumption that the Non-Vanishing theorem 1 hold true over any complex projective manifold of dimension* $\dim_{\mathbb{C}} X - 1$ *there exist;*

- *a birational morphism $\mu : \widehat{X} \to X$ of complex projective manifolds and a Kähler form $\widehat{\omega}$ over $\widehat{X}$,*
- *finitely many smooth and irreducible hypersurfaces $Z\,, Z_j \subset \widehat{X}$, $j \in I$ as in the in the statement of the lemma 1 such that the sets $V_j := Z_j \cap Z$ are irreducible for all $j \in I$.*
- *a $(1,1)$-cohomology class $\widehat{\alpha} := \{[\Delta]\} + \kappa$, with*

$$\Delta = \sum_{j\in I} a_j\, Z_j\,, \qquad (a_j)_{j\in I} \subset (0,1)$$

and with $\kappa \in \mathrm{NS}_{\mathbb{R}}(\widehat{X})$ a Kähler class, such that the following hold:

► *there exist a sequence $(\varepsilon_l)_l \subset (0,1)$, $\varepsilon_l \downarrow 0$ as $l \to +\infty$ and closed $(1,1)$-currents*

$$\widehat{\Theta}_l \in c_1(K_{\widehat{X}} + Z) + \widehat{\alpha}\,, \qquad \widehat{\Theta}_l \;\geq\; -\varepsilon_l\, \widehat{\omega}\,,$$

with well defined restriction $\widehat{\Theta}_{l|Z}$,

► *there exist an effective $\mathbb{R}$-divisor G over Z such that $[G] \in c_1(K_Z) + \widehat{\alpha}_{|Z}$ and such that for all $l \in \mathbb{N}$,*

$$G \;-\; \sum_{j\in I_-} \lambda(\widehat{\Theta}_{l|Z}, V_j)\, V_j \;-\; \sum_{j\in I_+} a_j\, V_j \;\geq\; 0\,, \tag{2.1}$$

with respect to a decomposition $I = I_+ \amalg I_-$ independent of l,

▶ *there exist an effective $\mathbb{R}$-divisor F over $\widehat{X}$ and $r \in \mathbb{R}_{>1}$ such that*

$$c_1(K_{\widehat{X}} + Z) + \widehat{\alpha} + \{[F]\} \;=\; r\,\mu^*\Big(c_1(K_X) + \alpha\Big) + \{[E]\}\,, \tag{2.2}$$

with $\mathcal{O}_{\widehat{X}}(E) := K_{\widehat{X}} - \mu^* K_X$.

Proof.

(A) Set up and notations.

Let $\omega > 0$ be a Kähler form inside a rational class and let $\varepsilon_0 \in \mathbb{R}_{>0}$ such that $\theta \geq \varepsilon_0\,\omega$. By means of Demailly's regularisation process we can assume without lost of generality that the local potentials of the current θ are with regular analytic singularities and its Lelong numbers are rational. In fact Demailly's regularisation process preserves the conditions $\theta \geq \varepsilon'\omega$, for some $\varepsilon' \in (0, \varepsilon_0)$ and $\mathcal{I}(\theta) = \mathcal{O}_X$. Let $\Theta \in c_1(K_X) + \alpha$ be a closed positive $(1,1)$-current and fix now a point $x_0 \in X$ such that the local potentials of θ and Θ are bounded at the point x_0. By standard facts one infers the existence of $p \in \mathbb{N}_{>0}$ and of a closed positive current

$$H \in \beta := p\Big(c_1(K_X) + \alpha\Big) + \alpha\,,$$

with regular analytic singularities and rational Lelong numbers such that

$$\lambda(H, x_0) \geq n + 1\,,$$

$n = \dim_{\mathbb{C}} X$. By means of the Hironaka desingularisation result we can find a birational morphism $\mu : \widehat{X} \to X$ of projective manifolds which factors through the blow up map of x_0 and such that;

$$K_{\widehat{X}} = \mu^* K_X + E\,, \qquad E := \sum_{j \in \bar{J}} e_j Z_j\,, \qquad e_j \in \mathbb{Z}_{\geq 0}\,, \qquad Z_{j_0} := \mu^{-1}(x_0)\,,$$

with $(Z_j)_{j \in \bar{J}}$ a finite family of distinct and smooth irreducible hypersurfaces with simple normal crossings such that $Z_j \cap Z_k \cap Z_l = \emptyset$ for all $j \neq k \neq l \neq j$, such that $Z_j \cap Z_k$ are irreducible for all j, k and such that,

$$\begin{aligned}
\mu^*\theta \;&=\; \sum_{j \in \bar{J}} \theta_j [Z_j] + R_\theta\,, \quad \theta_j \in \mathbb{Q}_{\geq 0}\,, \quad \theta_{j_0} = 0\,, \quad R_\theta \geq \varepsilon' \mu^*\omega \quad \text{smooth,}\\
\mu^* H \;&=\; \sum_{j \in \bar{J}} h_j [Z_j] + R_H\,, \quad h_j \in \mathbb{Q}_{\geq 0}\,, \quad h_{j_0} \geq n + 1\,, \quad R_H \geq 0 \quad \text{smooth.}
\end{aligned}$$

Notice also that $e_{j_0} = n - 1$ since μ factors through the blow up map of x_0. Consider now the decomposition

$$\mu^*\Theta = \sum_{j \in \bar{J}} \lambda_j [Z_j] + R_\Theta\,, \quad \lambda_j := \lambda(\mu^*\Theta\,, Z_j) \in \mathbb{R}_{\geq 0}$$

whith R_Θ a closed positive $(1,1)$-current such that $\lambda(R_\Theta\,, Z_j) = 0$ by construction. Notice that $\lambda_{j_0} = 0$ by our choice of the point x_0. Let $\widehat{\omega} > 0$ be a Kähler form over $\widehat{X}$. Let now $\tau \in \mathbb{Q} \cap (0,1)$ and apply the μ^*-functor to the decomposition

$$c_1(K_X) + \tau\beta + (1 - \tau)\alpha \;=\; (\tau p + 1)\Big(c_1(K_X) + \alpha\Big)\,,$$

(also considered in Paun [Pau2]) we infer the cohomology identity

$$c_1(K_{\widehat{X}}) + \mu^*\Big(\tau\beta + (1-\tau)\alpha\Big) - \{[E]\} = (\tau p + 1)\mu^*\Big(c_1(K_X) + \alpha\Big)\,. \tag{2.3}$$

Let $\zeta_j \in \{Z_j\}$ be a smooth representative, let $\delta_j \in \mathbb{Q}_{>0}$ and set

$$\begin{aligned}
\widehat{\omega}_\delta &:= \tau R_H + (1-\tau)R_\theta - \sum_{j\in\bar{J}} \delta_j\,\zeta_j \;\geq\; \varepsilon'\mu^*\omega - \sum_{j\in\bar{J}} \delta_j\,\zeta_j\,,\\
a_j &:= \tau(h_j - \theta_j - p\lambda_j) - \lambda_j + \theta_j - e_j + \delta_j\,,\\
b_j &:= \tau(h_j - \theta_j) + \theta_j - e_j + \delta_j\,,
\end{aligned}$$

We remind that there exist $\delta_{\omega,\zeta} \in \mathbb{R}_{>0}$ such that

$$\varepsilon'\mu^*\omega - \sum_{j\in\bar{J}} \delta_j\,\zeta_j > 0\,,$$

for all $\delta_j \in (0, \delta_{\omega,\zeta})$. We infer by (2.3) the identities

$$c_1(K_{\widehat{X}}) + \sum_{j\in\bar{J}} a_j\,\{[Z_j]\} + \{\widehat{\omega}_\delta\} \;=\; (\tau p + 1)\{R_\Theta\}\,, \tag{2.4}$$

$$c_1(K_{\widehat{X}}) + \sum_{j\in\bar{J}} b_j\,\{[Z_j]\} + \{\widehat{\omega}_\delta\} \;=\; (\tau p + 1)\mu^*\Big(c_1(K_X) + \alpha\Big)\,. \tag{2.5}$$

(B) Tie breaking.
The assumption $\mathcal{I}(\theta) = \mathcal{O}_X$ is equivalent to the inequality $\theta_j - e_j < 1$ for all $j \in \bar{J}$, which implies

$$-\lambda_j + \theta_j - e_j < 1\,, \qquad \forall j \in \bar{J}\,. \tag{2.6}$$

On the other hand the inequality $h_{j_0} - \theta_{j_0} - p\lambda_{j_0} = h_{j_0} \geq n+1 > 0$ implies

$$j_0 \in I := \{j \in \bar{J} \,:\, h_j - \theta_j - p\lambda_j > 0\} \neq \emptyset\,.$$

Let define now

$$c_j^\delta := \frac{1 + \lambda_j - \theta_j + e_j - \delta_j}{h_j - \theta_j - p\lambda_j} > 0\,,$$

for all $j \in I$. We choose $\delta_j \in \mathbb{Q}_{>0}$, $j \in \bar{J}$ such that

$$-\lambda_j + \theta_j - e_j + \delta_j < 1\,, \qquad \forall j \in \bar{J}\,, \tag{2.7}$$

there exist a unique $j_1 \in I$ such that

$$0 \;<\; \tau := \min_{j\in I} c_j^\delta \;<\; c_j^\delta\,, \qquad \forall j \in I \smallsetminus \{j_1\} \tag{2.8}$$

and such that

$$\frac{\varepsilon'}{n+1}\,\mu^*\omega - \sum_{j\in J} \delta_j\,\zeta_j > 0\,. \tag{2.9}$$

We notice that

$$0 \;<\; \tau \;\leq\; c_{j_0}^{\delta} \;=\; \frac{n-\delta_{j_0}}{h_{j_0}} \;<\; \frac{n}{n+1} \;<\; 1\,,$$

and that $\widehat{\omega}_\delta > 0$ by the condition (2.9). The main up-shot of this choice of δ (usually called tie break) is that the condition (2.8) implies $a_{j_1}^{\delta} = 1$ and $a_j^{\delta} < 1$ for all $j \in J := \bar{J} \smallsetminus \{j_1\}$. This will allow us to restrict certain currents in an adequate way over the hypersurface $Z := Z_{j_1}$.

(C) Restriction and weak limit extraction.
Let $\kappa := \{\widehat{\omega}_\delta\}$ be the required Kähler class, let $I := \{j \in J \,:\, a_j > 0\}$ and let define the class $\alpha = \{[\Delta]\} + \kappa$ with respect to this data as in the statement of the lemma 2. We consider now a family of closed and real $(1,1)$-currents with analytic singularities

$$(R_\Theta^\varepsilon)_{\varepsilon\in(0,1)} \in \{R_\Theta\}\,, \qquad R_\Theta^\varepsilon \;\geq\; -\varepsilon\,\widehat{\omega}\,, \qquad \lambda(R_\Theta^\varepsilon\,, Z_j) = 0\,, \quad \forall j \in \bar{J}\,.$$

obtained by regularising the current R_Θ. We infer in particular the existence of the restriction

$$R^\varepsilon_{\Theta\,|Z} \geq -\varepsilon\,\widehat{\omega}_{|Z}\,. \tag{2.10}$$

The fact that

$$(R^\varepsilon_{\Theta\,|Z})_{\varepsilon\in(0,1)} \in \{R_\Theta\}_{|Z}\,,$$

combined with the weak compactness of the mass imply the existence of a sequence $(\varepsilon_l)_l \subset (0,1)$, $\varepsilon_l \downarrow 0$ as $l \to +\infty$ and a closed positive current

$$\Xi \in \{R_\Theta\}_{|Z}\,,$$

such that $R^{\varepsilon_l}_{\Theta\,|Z} + \varepsilon_l\,\widehat{\omega}_{|Z} \longrightarrow \Xi$ weakly as $l \to +\infty$. Then the semi-continuity of the Lelong numbers implies

$$\lambda(\Xi, V_j) \;\geq\; \lambda(R^{\varepsilon_l}_{\Theta\,|Z}, V_j)\,, \tag{2.11}$$

for all $j \in J$ and all l. On the other hand the identity (2.4) implies that the current

$$\widehat{\Theta}_l \;:=\; \sum_{j\in J\smallsetminus I} -a_j[Z_j] \;+\; (\tau p + 1)\,R_\Theta^{\varepsilon_l}\,,$$

satisfies all the requirements in the statement of the lemma 2. Moreover by restricting the identity (2.4) to Z we infer

$$\widehat{\Theta}_Z \;:=\; \sum_{j\in J\smallsetminus I} -a_j[V_j] \;+\; (\tau p + 1)\,\Xi \;\in\; c_1(K_Z) \;+\; \alpha_{|Z}\,.$$

We decompose the current Ξ as

$$\Xi \;=\; \sum_{j\in I} \lambda(\Xi, V_j)\,[V_j] \;+\; R_\Xi\,,$$

and we rewrite the closed positive current $\widehat{\Theta}_Z$ as

$$\widehat{\Theta}_Z \;=\; \sum_{j\in I} \xi_j\,[V_j] \;+\; \widehat{R}_\Xi \;\in\; c_1(K_Z) \;+\; \alpha_{|Z}\,, \tag{2.12}$$

with $\xi_j := (\tau p+1)\lambda(\Xi, V_j)$ and with

$$\widehat{R}_{\Xi} := \sum_{j\in J\smallsetminus I} -a_j[V_j] + (\tau p+1)\, R_{\Xi}\,.$$

(D) Application of the non-vanishing in dimension one less.
Let $\mu_j := \xi_j - a_j$ for all $j\in I$. The identity (2.12) implies

$$\sum_{j\in I}\mu_j\,[V_j] + \widehat{R}_{\Xi} \in c_1(K_Z) + \kappa_{|Z}\,. \tag{2.13}$$

Let $I_+ := \{j\in I \,:\, \mu_j>0\}$, let $I_- := I\smallsetminus I_+$ and let define the $(1,1)$-cohomology class

$$\alpha_Z := \sum_{j\in I_-} -\mu_j\,\{[V_j]\} + \kappa_{|Z}\,.$$

We observe that $0\le -\mu_j\le a_j<1$ for all $j\in I_-$ and that the $(1,1)$-cohomology class $c_1(K_Z)+\alpha_Z$ is pseudoeffective since

$$0 \le \sum_{j\in I_+}\mu_j\,[V_j] + \widehat{R}_{\Xi} \in c_1(K_Z) + \alpha_Z\,,$$

by the identity (2.13). Thus the non-vanishing assumption in dimension on less implies the existence of an effective $\mathbb{R}$-divisor Γ over Z such that $[\Gamma]\in c_1(K_Z)+\alpha_Z$. On the other hand the decomposition

$$\widehat{\alpha}_{|Z} = \alpha_Z + \sum_{j\in I_-}\xi_j\,\{[V_j]\} + \sum_{j\in I_+} a_j\,\{[V_j]\}\,,$$

implies that the effective $\mathbb{R}$-divisor

$$G := \Gamma + \sum_{j\in I_-}\xi_j\,V_j + \sum_{j\in I_+} a_j\,V_j\,,$$

satisfies $[G]\in c_1(K_Z)+\widehat{\alpha}_{|Z}$. It also satisfies the vanishing condition (2.1) since

$$\xi_j \ge (\tau p+1)\lambda(R^{\varepsilon_l}_{\Theta\,|Z}, V_j) = \lambda(\widehat{\Theta}_{l|Z}, V_j)\,,$$

for all $j\in I_-$ and l, by the inequality 2.11.

(E) End of the proof of the Shokurov's lemma.
The identity (2.5) decomposes as (2.2) with respect to $r:=\tau p+1$ and

$$F := \sum_{j\in I\cup\{j_1\}}(r\lambda_j+e_j)\,Z_j + \sum_{j\in J\smallsetminus I}[\tau h_j+(1-\tau)\theta_j+\delta_j]\,Z_j\,.$$

□

2.2 Diophantine approximation

The following lemma follows from quite elementary facts from linear algebra and diophantine approximation. The elementary details are left to the reader.

Lemma 3 *Let*

• $\Delta = \sum_{j\in I} a_j Z_j$ *be an effective* $\mathbb{Q}$*-divisor over a complex projective manifold* $\widehat{X}$ *such that* $\lfloor \Delta \rfloor = 0$*,*

• $Z \subset \widehat{X}$ *be a smooth and irreducible hypersurface which is not a component of the divisor* Δ *and* $V_j := Z \cap Z_j$ *are distinct and irreducible for all* $j \in I$*,*

• $\widehat{\alpha} := \{[\Delta]\} + \kappa$ *be a* $(1,1)$*-cohomology class, with* $\kappa \in \mathrm{NS}_{\mathbb{R}}(\widehat{X})$ *a Kähler class such that there exist an effective* $\mathbb{R}$*-divisor* G *over* Z *with the properties*

$$[G] \in c_1(K_Z) + \widehat{\alpha}_{|Z}\,,$$

and

$$G \;-\; \sum_{t\in I_-} \xi_t\, V_t \;-\; \sum_{t\in I_+} a_t\, V_t \;\geq\; 0\,, \qquad \xi_t \in \mathbb{R}_{\geq 0}\,, \qquad I = I_+ \amalg I_-\,.$$

For any norm $\|\cdot\|$ *on the finite dimensional vector space* $\mathrm{NS}_{\mathbb{R}}(\widehat{X})$ *there exist a constant* $C > 0$*, a sequence* $(m_p)_{p\in\mathbb{N}_{>0}} \subset \mathbb{N}_{>0}$ *and finite families* $(A^{j,p})_{j=1}^{N_p}$ *of ample* $\mathbb{Q}$*-line bundles over* $\widehat{X}$ *such that for all* $p \in \mathbb{N}_{>0}$*,*

▶ $m_p\Delta$ *is integral,* $m_p A^{j,p}$ *is a line bundle and*

$$\left\| c_1(A^{j,p}) - \kappa \right\| \;\leq\; \frac{C}{p\, m_p}\,,$$

for all $j = 1, ..., N_p$*.*

▶ *there exist* $r_{j,p} \in \mathbb{R}_{>0}$ *such that*

$$\kappa \;=\; \sum_{j=1}^{N_p} r_{j,p}\, c_1(A^{j,p})\,, \qquad \text{and} \qquad \sum_{j=1}^{N_p} r_{j,p} = 1\,,$$

▶ *if we set* $\mathcal{L}^{j,p} := \mathcal{O}_{\widehat{X}}(\Delta) + A^{j,p}$ *for all* $j = 1, ..., N_p$*, there exist an effective* $\mathbb{R}$*-divisor*

$$[G^{j,p}] \in c_1(K_Z + \mathcal{L}^{j,p})\,,$$

with the vanishing property

$$G^{j,p} \;-\; \sum_{t\in I_-} \xi_t\, V_t \;-\; \sum_{t\in I_+} a_t\, V_t \;\geq\; -\,\frac{C}{m_p p} \sum_{t\in I_-} V_t\,.$$

2.3 Shokurov's construction of sections

We remind now a well known fact due to Shokurov (see for example [Pau2]).

Claim 5 *Let* L *be a holomorphic* $\mathbb{Q}$*-line bundle over a polarised connected complex projective manifold* (Z, ω) *which admits a closed positive* $(1,1)$*-current* $\theta \in c_1(L)$ *such that* $\theta \geq \varepsilon\omega$ *for some* $\varepsilon \in \mathbb{R}_{>0}$ *and such that* $\mathcal{I}(\theta) = \mathcal{O}_Z$*. If there exists an effective* $\mathbb{Q}$*-divisor* G *over* Z *such that* $[G] \in c_1(K_Z + L)$ *then*

$$h^0(Z, m(K_Z + L)) > 0\,,$$

for all $m \in \mathbb{N}_{>0}$ *such that* mL *is a holomorphic line bundle and* mG *is integral. Moreover if there exists an effective and simple normal crossing* $\mathbb{R}$*-divisor* V *over* Z *such that* $G - V \geq 0$*, then there exists a non zero section*

$$u \in H^0(Z, m(K_Z + L))\,, \qquad \operatorname{div} u \;-\; \lfloor (m-1)V \rfloor \;\geq\; 0\,. \tag{2.14}$$

Proof. There exist a flat hermitian line bundle F and a non zero section

$$\sigma \in H^0(Z, m(K_Z + L) + F)\,,$$

such that $mG = \operatorname{div}\sigma$. Set

$$\mathcal{L} := (m-1)(K_Z + L) + L\,,$$

and observe the obvious identity $m(K_Z + L) = K_Z + \mathcal{L}$. We define the current

$$\theta_G := (m-1)[G] + \theta \in c_1(\mathcal{L}) = c_1(\mathcal{L} + F)\,, \quad \theta_G \geq \varepsilon\omega\,,$$

and we observe that $\sigma \in H^0(Z\,,\mathcal{S}_F)$, with

$$\mathcal{S}_F \;:=\; \mathcal{S} \otimes_{\mathcal{O}_Z} \mathcal{O}_Z(F)\,, \qquad \mathcal{S} \;:=\; \mathcal{O}_Z(K_Z + \mathcal{L}) \otimes_{\mathcal{O}_Z} \mathcal{I}(\theta_G)\,.$$

In fact let h be a smooth hermitian metric over

$$m(K_Z + L) + F \;=\; K_Z + \mathcal{L} + F\,,$$

and let $\gamma_h \in c_1(K_Z+\mathcal{L})$ be its normalised curvature form. Let $\alpha \in c_1(L)$ smooth and let write $\theta = \alpha + \frac{i}{2\pi}\partial\bar{\partial}\varphi_\theta$. The Lelong-Poincaré formula implies

$$\theta_G \;=\; \frac{m-1}{m}\,\gamma_h + \alpha + \frac{i}{2\pi}\,\partial\bar{\partial}\varphi_G\,, \qquad \varphi_G := \frac{m-1}{m}\log|\sigma|^2_h + \varphi_\theta\,.$$

Then

$$\int_Z |\sigma|^2_h\, e^{-\varphi_G} = \int_Z |\sigma|^{2/m}_h\, e^{-\varphi_\theta} \leq C\int_Z e^{-\varphi_\theta} < +\infty\,,$$

implies $\sigma \in H^0(Z\,,\mathcal{S}_F)$. By applying the Kawamata-Viehweg-Nadel vanishing theorem to the line bundles $\mathcal{L}$ and $\mathcal{L} + F$ we infer

$$h^q(Z\,,\mathcal{S}) \;=\; h^q(Z\,,\mathcal{S}_F) \;=\; 0\,, \qquad \forall q > 0\,.$$

Thus

$$h^0(Z\,,\mathcal{S}) = \chi(Z\,,\mathcal{S}) = \chi(Z\,,\mathcal{S}_F) = h^0(Z\,,\mathcal{S}_F) > 0\,,$$

since F is topologically trivial. Moreover the inclusion

$$\mathcal{I}(\theta_G) \subset \mathcal{I}((m-1)[V]) = \mathcal{O}_Z(-\lfloor (m-1)V \rfloor)\,,$$

implies the existence of the required section u. □

Conclusion. It seem clear at this point that the magnitude of the vanishing error of type (2.14) produced by a combination of diophantine approximation with Shokurov's construction of sections is much bigger than the magnitude of the vanishing error allowed by the extension condition (1.16) of corollary 1. To be more precise we want to apply claim 5 in the setting of lemma 2 after application of the diophantine approximation of lemma 3. Therefore we want to apply the claim 5 with $L := \mathcal{L}^{j,p}$ and with

$$V \;=\; \sum_{t\in I}\beta_t\, V_t \;:=\; \sum_{t\in I_-}\left(\xi_t \;+\; \frac{C}{m_p p}\right)V_t \;+\; \sum_{t\in I_+} a_t\, V_t\,.$$

The trivial identity

$$\beta_t \ - \ \frac{1}{m_p} \lfloor (m_p - 1)\beta_t \rfloor \ = \ \frac{1}{m_p}\beta_t \ + \ \frac{1}{m_p}\Big[(m_p - 1)\beta_t \ - \ \lfloor (m_p - 1)\beta_t \rfloor \Big] \, ,$$

shows that the error

$$\frac{1}{m_p}\beta_t \ \leq \ \beta_t \ - \ \frac{1}{m_p} \lfloor (m_p - 1)\beta_t \rfloor \, ,$$

does not allow to apply the condition (1.16) in corollary 1 to the section u produced by claim 5. We observe furthermore that if one let $m \to +\infty$ the condition (1.15) in corollary 1 force a priori a blow-up of the generic Lelong numbers $\xi_t = \lambda(\Theta_{|Z}, V_t)$ and therefore a blow-up of β_t, $t \in I_-$. We observe finally that the situation $I_+ = \emptyset$ my occur in lemma 2.

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