Stability of the parametric fundamental equation of information for nonpositive parameters

Eszter Gselmann and Gyula Maksa

Abstract. In this note we prove that the parametric fundamental equation of information is stable in the sense of Hyers and Ulam provided that the parameter is nonpositive. We also prove, as a corollary, that the system of equations that defines the recursive and semi-symmetric information measures depending on a nonpositive parameter is stable in a certain sense.

Mathematics Subject Classification (2000). 39B82, 39B72.

Keywords. Stability, fundamental equation of information, entropy of degree α.

1. Introduction

The basic problem in the stability theory of functional equations is whether an approximate solution of a functional equation or a system of functional equations can be approximated by a solution of the equation or the system of equations in question.

In this paper we prove that the parametric fundamental equation of information

$$f(x) + (1-x)^{\alpha} f\left(\frac{y}{1-x}\right) = f(y) + (1-y)^{\alpha} f\left(\frac{x}{1-y}\right) \tag{1.1}$$

is stable in the sense of Hyers and Ulam (see the expository papers Forti [5], Ger [6], Moszner [10]), provided that α is nonpositive. Equation (1.1) arises in a natural way in characterizing information measures based on the properties of α–recursivity and semi–symmetry (see Aczél–Daróczy [3]). In the investigations (1.1) is supposed to hold on

$$D = \left\{(x, y) \in \mathbb{R}^2 | x, y \in [0, 1[, x + y \leq 1\right\}$$

This research has been supported by the Hungarian Scientific Research Fund (OTKA) Grants NK 68040 and K 62316.

with $f:[0,1]\to\mathbb{R}$ or only on the interior of D,

$$D^{\circ}=\left\{(x,y)\in\mathbb{R}^2|x,y,x+y\in]0,1[\right\},$$

with $f:]0,1[\to\mathbb{R}$ (see Aczél–Daróczy [3], Aczél [1], [2] and their references). In [9] we proved that (1.1) is stable on D, moreover it is superstable (see Forti [5]) if $\alpha>0$ and $\alpha\neq 1$. The question of the stability of (1.1) in the exceptional case $\alpha=1$ on D° was raised by Székelyhidi [11]. The method we used in [9] can not be applied neither in this case nor in the case $\alpha\leq 0$ neither on D nor on D°. Finally, we should remark that the ideas we use in this paper to prove the stability of (1.1) on D° as well as on D do not work if $\alpha>0$.

2. The main result

First we prove the following.

Theorem 2.1. *Let $\alpha,\varepsilon\in\mathbb{R}$ be fixed, $\alpha\leq 0$ and $\varepsilon\geq 0$. Suppose that the function $f:]0,1[\to\mathbb{R}$ satisfies the inequality*

$$\left|f(x)+(1-x)^{\alpha}f\left(\frac{y}{1-x}\right)-f(y)-(1-y)^{\alpha}f\left(\frac{x}{1-y}\right)\right|\leq\varepsilon \tag{2.1}$$

for all $(x,y)\in D^{\circ}$. Then, in case $\alpha<0$ there exist $a,b\in\mathbb{R}$ such that

$$|f(x)-[ax^{\alpha}+b(1-x)^{\alpha}-b]|\leq 15\varepsilon,\quad (x\in]0,1[) \tag{2.2}$$

furthermore, in case $\alpha=0$, there exists a logarithmic function $l:]0,1[\to\mathbb{R}$ and $c\in\mathbb{R}$ such that

$$|f(x)-[l(1-x)+c]|\leq 63\varepsilon.\quad (x\in]0,1[) \tag{2.3}$$

Proof. Define the function F on $]0,+\infty[^2$ by

$$F(u,v)=(u+v)^{\alpha}f\left(\frac{v}{u+v}\right). \tag{2.4}$$

Then

$$F(tu,tv)=t^{\alpha}F(u,v)\quad (t,u,v\in]0,+\infty[) \tag{2.5}$$

and

$$f(x)=F(1-x,x),\quad (x\in]0,1[) \tag{2.6}$$

furthermore, with the substitutions

$$x=\frac{w}{u+v+w},\quad y=\frac{v}{u+v+w}\quad (u,v,w\in]0,+\infty[)$$

inequality (2.1) implies that

$$\begin{aligned}\Big|f\left(\tfrac{w}{u+v+w}\right)&+\tfrac{(u+v)^{\alpha}}{(u+v+w)^{\alpha}}f\left(\tfrac{v}{u+v}\right)\\ -f\left(\tfrac{v}{u+v+w}\right)&-\tfrac{(u+w)^{\alpha}}{(u+v+w)^{\alpha}}f\left(\tfrac{w}{u+w}\right)\Big|\leq\varepsilon\end{aligned}$$

whence, by (2.4)

$$|F(u+v,w)+F(u,v)-F(u+w,v)-F(u,w)|\leq\varepsilon(u+v+w)^{\alpha} \tag{2.7}$$

follows for all $u, v, w \in]0, +\infty[$.

In the next step we define the functions g and G on $]0, +\infty[$ and on $]0, +\infty[^2$, respectively by

$$g(u) = F(u, 1) - F(1, u) \tag{2.8}$$

and

$$G(u, v) = F(u, v) + g(v). \tag{2.9}$$

We will show that

$$|G(u, v) - G(v, u)| \leq 3\varepsilon(u + v + 1)^\alpha. \quad (u, v \in]0, +\infty[) \tag{2.10}$$

Indeed, with the substitution $w = 1$, inequality (2.7) implies that

$$|F(u + v, 1) + F(u, v) - F(u + 1, v) - F(u, 1)| \leq \varepsilon(u + v + 1)^\alpha. \tag{2.11}$$

Interchanging u and v, it follows from (2.11) that

$$|-F(u + v, 1) - F(v, u) + F(v + 1, u) - F(v, 1)| \leq \varepsilon(u+v+1)^\alpha. \quad (u, v \in]0, +\infty[)$$

This inequality, together with (2.11) and the triangle inequality imply that

$$|F(u, v) - F(v, u) - F(u + 1, v) - F(u, 1) + F(v + 1, u) + F(v, 1)| \leq 2\varepsilon(u+v+1)^\alpha \tag{2.12}$$

holds for all $u, v \in]0, +\infty[$. On the other hand, with $u = 1$, we get from (2.7) that

$$|F(1 + v, w) + F(1, v) - F(1 + w, v) - F(1, w)| \leq \varepsilon(1 + v + w)^\alpha.$$

Replacing here v by u and w by v, respectively, we have that

$$|F(u + 1, v) + F(1, u) - F(v + 1, u) - F(1, v)| \leq \varepsilon(u + v + 1)^\alpha. \quad (u, v \in]0, +\infty[)$$

Again, the triangle inequality and the definitions (2.8) and (2.9), (2.12) imply (2.10).

In what follows we will investigate the function g. At this point of the proof we have to distinguish two cases.

In case $\alpha < 0$ we will determine the function g by proving that

$$g(u) = c(u^\alpha - 1) \quad (u \in]0, +\infty[) \tag{2.13}$$

with some $c \in \mathbb{R}$. Indeed, (2.10) implies that

$$|G(tu, tv) - G(tv, tu)| \leq 3\varepsilon(tu + tv + 1)^\alpha, \quad (t, u, v \in]0, +\infty[)$$

therefore by (2.9) and (2.5)

$$|t^\alpha F(u, v) + g(tv) - t^\alpha F(v, u) - g(tu)| \leq 3\varepsilon(tu + tv + 1)^\alpha \quad (t, u, v \in]0, +\infty[)$$

or

$$\left|F(u, v) - F(v, u) - t^{-\alpha} (g(tu) - g(tv))\right| \leq 3\varepsilon(u + v + t^{-1})^\alpha \quad (t, u, v \in]0, +\infty[)$$

whence

$$\lim_{t\to 0} t^{-\alpha} (g(tu) - g(tv)) = F(u, v) - F(v, u) \quad (t, u, v \in]0, +\infty[)$$

follows. Particularly, with $v = 1$, by (2.8), we have that

$$g(u) = \lim_{t\to 0} t^{-\alpha} (g(tu) - g(t)) . \quad (u \in]0, +\infty[) \tag{2.14}$$

Let now $u, v \in]0, +\infty[$. Then, by (2.14), we obtain that

$$\begin{aligned} g(uv) &= \lim_{t\to 0} t^{-\alpha}\left[g(tuv) - g(t)\right] \\ &= \lim_{t\to 0}\left[(tv)^{-\alpha}\left(g((tv)u) - g(tv)\right)v^{\alpha} + t^{-\alpha}(g(tv) - g(t))\right] \\ &= g(u)v^{\alpha} + g(v). \end{aligned}$$

Therefore, $g(u)v^{\alpha} + g(v) = g(v)u^{\alpha} + g(u)$, that is,

$$g(u)\left(v^{\alpha} - 1\right) = g(v)\left(u^{\alpha} - 1\right) \quad (u, v \in]0, +\infty[)$$

which implies (2.13) with $c = g(2)\left(2^{\alpha} - 1\right)^{-1}$.

Thus, by (2.6), (2.13), (2.9) and (2.10), we have that

$$\begin{aligned} &|f(x) - c(1-x)^{\alpha} - (f(1-x) - cx^{\alpha})| \\ &= |F(1-x, x) + cx^{\alpha} - (F(x, 1-x) + c(1-x)^{\alpha})| \\ &= |G(1-x, x) - G(x, 1-x)| \leq 3 \cdot 2^{\alpha}\varepsilon \end{aligned} \tag{2.15}$$

holds for all $x \in]0, 1[$.

In the next step we define the functions f_0 and F_0 on $]0, 1[$ and on $]0, 1[^2$ by

$$f_0(x) = f(x) - c\left[(1-x)^{\alpha} - 1\right] \tag{2.16}$$

and

$$F_0(p, q) = f_0(p) + p^{\alpha} f_0(q) - f_0(pq) - (1-pq)^{\alpha} f_0\left(\frac{1-p}{1-pq}\right), \tag{2.17}$$

respectively. Then (2.1) and (2.15) imply that

$$\left| f_0(x) + (1-x)^{\alpha} f_0\left(\frac{y}{1-x}\right) - f_0(y) - (1-y)^{\alpha} f_0\left(\frac{x}{1-y}\right)\right| \leq \varepsilon \tag{2.18}$$

for all $(x, y) \in D^{\circ}$ and

$$|f_0(x) - f_0(1-x)| \leq 3 \cdot 2^{\alpha}\varepsilon. \quad (x \in]0, 1[) \tag{2.19}$$

Furthermore, with the substitutions $x = 1-p$, $y = pq$ $(p, q \in]0, 1[)$, (2.18) implies that

$$\left| f_0(1-p) + p^{\alpha} f_0(q) - f_0(pq) - (1-pq)^{\alpha} f_0\left(\frac{1-p}{1-pq}\right)\right| \leq \varepsilon \tag{2.20}$$

holds for all $p, q \in]0, 1[$. Therefore, due to (2.19) and the triangle inequality, (2.18) implies that

$$|F_0(p, q)| \leq (1 + 3 \cdot 2^{\alpha})\,\varepsilon. \quad (p, q \in]0, 1[) \tag{2.21}$$

It can easily be checked that

$$\begin{aligned} &f_0(p)\left[q^{\alpha} + (1-q)^{\alpha} - 1\right] - f_0(q)\left[p^{\alpha} + (1-p)^{\alpha} - 1\right] \\ &= F_0(q, p) - F_0(p, q) + (1-pq)^{\alpha}\left[F_0\left(\tfrac{1-q}{1-pq}, p\right) + f_0\left(1 - \tfrac{1-p}{1-pq}\right) - f_0\left(\tfrac{1-p}{1-pq}\right)\right] \end{aligned}$$

holds for all $p, q \in]0, 1[$. Thus, by (2.21) and (2.19) we get that

$$\begin{aligned} &|f_0(p)\left[q^{\alpha} + (1-q)^{\alpha} - 1\right] - f_0(q)\left[p^{\alpha} + (1-p)^{\alpha} - 1\right]| \\ &\leq 2(1 + 3 \cdot 2^{\alpha})\varepsilon + (1-pq)^{\alpha}\left[(1 + 3 \cdot 2^{\alpha})\varepsilon + 3 \cdot 2^{\alpha}\varepsilon\right], \end{aligned}$$

that is,

$$\begin{aligned}&\left|f_0(p) - \tfrac{f_0(q)}{q^\alpha+(1-q)^\alpha-1}\left[p^\alpha + (1-p)^\alpha - 1\right]\right| \\ &\leq \tfrac{2(1+3\cdot 2^\alpha)+(1-pq)^\alpha(1+6\cdot 2^\alpha)}{q^\alpha+(1-q)^\alpha-1}\varepsilon.\end{aligned} \qquad (p, q \in]0, 1[)$$

Taking into consideration (2.16), with $q = \frac{1}{2}$ with the definitions $a = f_0\left(\frac{1}{2}\right)\left(2^{1-\alpha} - 1\right)^{-1}$, $b = a + c$, this inequality implies that

$$|f(x) - [ax^\alpha + b(1-x)^\alpha - b]| \leq \frac{8 + 6 \cdot 2^\alpha + 2^{-\alpha}}{2^{1-\alpha} - 1}\varepsilon. \quad (x \in]0, 1[)$$

Since

$$\sup_{\alpha<0} \frac{8 + 6 \cdot 2^\alpha + 2^{-\alpha}}{2^{1-\alpha} - 1} = 15$$

we get (2.2).

In case $\alpha = 0$ we will show that there exists a logarithmic function $l :]0, +\infty[\to \mathbb{R}$ such that

$$|g(u) - l(u)| \leq 6\varepsilon$$

for all $u \in]0, +\infty[$. Indeed, (2.10) yields in this case that

$$|G(u, v) - G(v, u)| \leq 3\varepsilon. \quad (u, v \in]0, +\infty[)$$

Due to (2.5) and (2.9) we obtain that

$$\begin{aligned} G(tu, tv) &= F(tu, tv) + g(tv) \\ &= F(u, v) + g(tv) \\ &= G(u, v) - g(v) + g(tv) \end{aligned}$$

that is,

$$G(tu, tv) - G(u, v) = g(tv) - g(v), \quad (t, u, v \in]0, +\infty[)$$

therefore

$$\begin{aligned}&|g(tv) - g(v) + g(u) - g(tu)| \\ &= |G(tu, tv) - G(u, v) - G(tv, tu) + G(v, u)| \\ &\leq |G(tu, tv) - G(tv, tu)| + |G(v, u) - G(u, v)| \leq 6\varepsilon\end{aligned} \tag{2.22}$$

for all $t, u, v \in]0, +\infty[$. Now (2.22) with the substitution $u = 1$ implies that

$$|g(tv) - g(v) - g(t)| \leq 6\varepsilon$$

holds for all $t, v \in]0, +\infty[$, since obviously $g(1) = 0$. This means that the function g is approximately logarithmic on $]0, +\infty[$. Thus (see e.g. Forti [5]) there exists a logarithmic function $l :]0, +\infty[\to \mathbb{R}$ such that

$$|g(u) - l(u)| \leq 6\varepsilon$$

holds for all $u \in]0, +\infty[$.

Furthermore,

$$\begin{aligned}
&|f(x)-l(1-x)-(f(1-x)-l(x))| \\
&=|F(1-x,x)-l(1-x)-F(x,1-x)+l(x)| \\
&=|F(1-x,x)+g(x)-g(x)-l(1-x) \\
&-F(x,1-x)+g(1-x)-g(1-x)+l(x)\ | \\
&\leq |F(1-x,x)+g(x)-(F(x,1-x)+g(1-x))| \\
&+|g(1-x)-l(1-x)|+|l(x)-g(x)| \\
&=|G(1-x,x)-G(x,1-x)| \\
&+|g(1-x)-l(1-x)|+|l(x)-g(x)| \\
&\leq 3\varepsilon+6\varepsilon+6\varepsilon=15\varepsilon.
\end{aligned} \tag{2.23}$$

As in the first part of the proof define the functions f_0 and F_0 on $]0,1[$ and on $]0,1[^2$, respectively, by

$$f_0(x)=f(x)-l(1-x)$$

and

$$F_0(p,q)=f_0(p)+f_0(q)-f_0(pq)-f_0\left(\frac{1-p}{1-pq}\right).$$

Due to (2.23)

$$|f_0(x)-f_0(1-x)|\leq 15\varepsilon \tag{2.24}$$

holds for all $x\in]0,1[$. Furthermore, inequality (2.1) implies, with the substitutions $x=1-p$, $y=pq$ $(p,q\in]0,1[)$, that

$$\left|f_0(1-p)+f_0(q)-f_0(pq)-f_0\left(\frac{1-p}{1-pq}\right)\right|\leq\varepsilon \tag{2.25}$$

holds for all $p,q\in]0,1[$. Inequalities (2.24) and (2.25) and the triangle inequality imply that

$$|F_0(p,q)|\leq 16\varepsilon \tag{2.26}$$

for all $p,q\in]0,1[$. An easy calculation shows that

$$f_0(p)-f_0(q)=F_0(q,p)-F_0(p,q)+F_0\left(\frac{1-p}{1-pq},p\right)+f_0\left(1-\frac{1-p}{1-pq}\right)-f_0\left(\frac{1-p}{1-pq}\right)$$

therefore,

$$\begin{aligned}
&|f_0(p)-f_0(q)| \\
&\leq |F_0(q,p)|+|F_0(p,q)|+\left|F_0\left(\tfrac{1-p}{1-pq},p\right)\right|+\left|f_0\left(1-\tfrac{1-p}{1-pq}\right)-f_0\left(\tfrac{1-p}{1-pq}\right)\right| \\
&\leq 3\cdot 16\varepsilon+15\varepsilon=63\varepsilon
\end{aligned} \tag{2.27}$$

holds for all $p,q\in]0,1[$. With the substitution $q=\frac{1}{2}$ (2.27) implies that

$$\left|f_0(p)-f_0\left(\frac{1}{2}\right)\right|\leq 63\varepsilon.$$

Using the definition of the function f_0, we obtain that

$$|f(x)-l(1-x)-c|\leq 63\varepsilon$$

holds for all $x\in]0,1[$, where $c=f_0\left(\frac{1}{2}\right)$. Hence inequality (2.3) holds, indeed. □

Remark 2.2. Applying Theorem 2.1 in the case $\varepsilon = 0$ we get the general solution of (1.1) on D° (see also Maksa [8]).

3. Two corollaries of the main result

The first corollary says that equation (1.1) is stable on D, as well.

Theorem 3.1. *Let $\alpha, \varepsilon \in \mathbb{R}$ be fixed, $\alpha \leq 0$, $\varepsilon \geq 0$. Suppose that the function $f : [0,1] \to \mathbb{R}$ satisfies inequality (2.1) for all $(x,y) \in D$. Then, in case $\alpha < 0$ there exist $a, b \in \mathbb{R}$ such that the function h_1 defined on $[0,1]$ by*

$$h_1(x) = \begin{cases} 0, & \text{if } \ x = 0 \\ ax^\alpha + b(1-x)^\alpha - b, & \text{if } \ x \in \,]0,1[\\ a - b, & \text{if } \ x = 1 \end{cases}$$

is a solution of (1.1) on D and

$$|f(x) - h_1(x)| \leq 15\varepsilon, \quad (x \in [0,1]) \tag{3.1}$$

furthermore, in case $\alpha = 0$, there exist $a, b, c \in \mathbb{R}$ such that the function h_2 defined on $[0,1]$ by

$$h_2(x) = \begin{cases} a, & \text{if } \ x = 0 \\ c, & \text{if } \ x \in \,]0,1[\\ b, & \text{if } \ x = 1 \end{cases}$$

is a solution of (1.1) on D and

$$|f(x) - h_2(x)| \leq 63\varepsilon. \tag{3.2}$$

Proof. An easy calculation shows that the functions h_1 and h_2 are the solutions of (1.1) on D in case $\alpha < 0$ and in case $\alpha = 0$, respectively. Firstly, we investigate the case $\alpha < 0$. Theorem 2.1. implies that (3.1) holds for all $x \in]0,1[$. Therefore, it is enough to prove that (3.1) holds for $x = 0$ and for $x = 1$. It follows from (2.1) with $x = 0$, that $((1-x)^\alpha - 1)\,|f(0)| \leq \varepsilon$ if $x \in]0,1[$. Since $\alpha < 0$, $f(0) = 0$ follows, that is, (3.1) is valid for $x = 0$. Let now $x \in]0,1[$ and $y = 1 - x$ in (2.1). Then

$$|f(1-x) - f(x) - f(1)\,((1-x)^\alpha - x^\alpha)| \leq \varepsilon. \tag{3.3}$$

Apply (2.2) to $1 - x$ instead of x. Hence we get that

$$|-f(1-x) + a(1-x)^\alpha + bx^\alpha - b| \leq 15\varepsilon \tag{3.4}$$

Adding the inequalities (3.3), (2.2) and (3.4) up and using the triangle inequality to obtain that

$$|a - b - f(x)| \cdot |(1-x)^\alpha - x^\alpha| \leq 31\varepsilon. \quad (x \in]0,1[)$$

Since $\alpha < 0$ we get that $f(1) = a - b$ and so (3.1) holds also for $x = 1$.

Now, we fall to deal with the case $\alpha = 0$. Let $x \in]0,1[$ and $y = 1-x$ in (2.1), then we obtain that

$$|f(x) - f(1-x)| \leq \varepsilon, \quad (x \in]0,1[) \tag{3.5}$$

furthermore, let us observe that (2.1) does not pose any restriction on the value of $f(0)$ as well as $f(1)$. Thus $f(0)=a$, $f(1)=b$, where $a,b\in\mathbb{R}$ are certain constants. Therefore (3.2) holds if $x=0$ or $x=1$.

In case $x\in]0,1[$, due to Theorem 2.1. there exist a logarithmic function $l:]0,1[\to\mathbb{R}$ and $c\in\mathbb{R}$ such that

$$|f(x)-l(1-x)-c|\leq 63\varepsilon \tag{3.6}$$

holds for all $x\in]0,1[$. Hence it is enough to prove that the function l is identically zero on $]0,1[$. Indeed, due to (2.3), (3.5) and (3.6)

$$\begin{aligned}
&|l(1-x)-l(x)|\\
&=|l(1-x)-f(1-x)+f(1-x)+c\\
&-l(x)+f(x)-f(x)-c\,|\\
&\leq|l(1-x)+c-f(x)|+|f(1-x)-l(x)-c|\\
&+|f(x)-f(1-x)|\\
&\leq 127\varepsilon
\end{aligned} \tag{3.7}$$

holds for all $x\in]0,1[$, using that the function l is logarithmic, the last inequality can be written as

$$\left|l\left(\frac{1-x}{x}\right)\right|\leq 127\varepsilon. \quad (x\in]0,1[)$$

By substitution $x=\frac{1}{1+p}$ $(p\in]0,+\infty[)$ into the last inequality, we obtain that

$$|l(p)|\leq 127\varepsilon$$

holds for all $p\in]0,+\infty[$, where we used the fact that every logarithmic function on $]0,1[$ is uniquely extendable to a logarithmic function on $]0,+\infty[$.

Thus l is bounded on $]0,+\infty[$. However, the only bounded, logarithmic function on $]0,+\infty[$ is the identically zero function. Therefore,

$$|f(x)-c|\leq 63\varepsilon$$

holds for all $x\in]0,1[$, i.e., (3.2) is proved. □

The second corollary concerns the stability of a system of equations.

Theorem 3.2. *Let $n\geq 2$ be a fixed positive integer,*

$$\Gamma_n^\circ=\left\{(p_1,\ldots,p_n)|p_i>0, i=1,\ldots,n,\sum_{i=1}^n p_i=1\right\}$$

and (I_n) be the sequence of functions $I_n:\Gamma_n^\circ\to\mathbb{R}$ and suppose that there exist a sequence (ε_n) of nonnegative real numbers and a real number $\alpha<0$ such that

$$\left|I_n(p_1,\ldots,p_n)-I_{n-1}(p_1+p_2,p_3,\ldots,p_n)-(p_1+p_2)^\alpha I_2\left(\frac{p_1}{p_1+p_2},\frac{p_2}{p_1+p_2}\right)\right|\leq\varepsilon_{n-1} \tag{3.8}$$

for all $n\geq 3$ and $(p_1,\ldots,p_n)\in\Gamma_n^\circ$, and

$$|I_3(p_1,p_2,p_3)-I_3(p_1,p_3,p_2)|\leq\varepsilon_1 \tag{3.9}$$

holds on Γ_n°*. Then, in case* $\alpha<0$ *there exist* $c,d\in\mathbb{R}$ *such that*

$$\begin{aligned}&\left|I_n(p_1,\ldots,p_n)-(cH_n^\alpha(p_1,\ldots,p_n)+d(p_1^\alpha-1))\right|\\&\leq\sum_{k=2}^{n-1}\varepsilon_k+15(2\varepsilon_2+\varepsilon_1)\left(1+\sum_{k=2}^{n-1}\left(\sum_{i=1}^k p_i^\alpha\right)\right)\end{aligned}\tag{3.10}$$

for all $n\geq 2$ *and* $(p_1,\ldots,p_n)\in\Gamma_n^\circ$*. Furthermore, in case* $\alpha=0$ *there exists a logarithmic function* $l:]0,1[\to\mathbb{R}$ *and* $c\in\mathbb{R}$ *such that*

$$\begin{aligned}&\left|I_n(p_1,\ldots,p_n)-\left(cH_n^0(p_1,\ldots,p_n)+l(p_1)\right)\right|\\&\sum_{k=2}^{n-1}\varepsilon_k+63(n-1)(2\varepsilon_2+\varepsilon_1)\end{aligned}\tag{3.11}$$

for all $n\geq 2$ *and* $(p_1,\ldots,p_n)\in\Gamma_n^\circ$*, where the convention* $\sum_{k=2}^1\varepsilon_k=\sum_{k=2}^1\left(\sum_{i=1}^k p_i^\alpha\right)=0$ *is adapted and*

$$H_n^\alpha(p_1,\ldots,p_n)=\left(2^{1-\alpha}-1\right)^{-1}\left(\sum_{i=1}^n p_i^\alpha-1\right).\quad((p_1,\ldots,p_n)\in\Gamma_n^\circ)\tag{3.12}$$

Proof. As in Maksa [9], it can be proved that, due to (3.8) and (3.9), for the function f defined on $]0,1[$ by $f(x)=I_2(1-x,x)$ we get that

$$\left|f(x)+(1-x)^\alpha f\left(\frac{y}{1-x}\right)-f(y)-(1-y)^\alpha f\left(\frac{x}{1-y}\right)\right|\leq 2\varepsilon_2+\varepsilon_1$$

for all $(x,y)\in D^\circ$, i.e., (2.1) holds with $\varepsilon=2\varepsilon_2+\varepsilon_1$. Therefore, applying Theorem 2.1. we obtain (2.2) and (2.3), respectively, with some $a,b,c\in\mathbb{R}$ and a logarithmic function $l:]0,1[\to\mathbb{R}$ and $\varepsilon=2\varepsilon_2+\varepsilon_1$, i.e.,

$$\left|I_2(1-x,x)-(ax^\alpha+b(1-x)^\alpha-b)\right|\leq 15(2\varepsilon_2+\varepsilon_1),\quad(x\in]0,1[)$$

in case $\alpha<0$, and

$$\left|I_2(1-x,x)-(l(1-x)+c)\right|\leq 63(2\varepsilon_2+\varepsilon_1)\quad(x\in]0,1[)$$

in case $\alpha=0$.

Therefore (3.10) holds with $c=(2^{1-\alpha}-1)a$, $d=b-a$ in case $\alpha<0$, and (3.11) holds in case $\alpha=0$, respectively, for $n=2$.

We continue the proof by induction on n. Suppose that (3.10) and (3.11) holds, resp., and for the sake of brevity, introduce the notation

$$J_n(p_1,\ldots,p_n)=\begin{cases}cH_n^\alpha(p_1,\ldots,p_n), & \text{if}\quad\alpha<0\\ cH_n^0(p_1,\ldots,p_n)+l(p_1), & \text{if}\quad\alpha=0\end{cases}$$

for all $n\geq 2$, $(p_1,\ldots,p_n)\in\Gamma_n^\circ$. It can easily be seen that (3.10) and (3.11) hold on Γ_n° for J_n instead of I_n $(n\geq 3)$ with $\varepsilon_n=0$ $(n\geq 2)$. Thus, for all

$(p_1, \dots, p_{n+1}) \in \Gamma^{\circ}_{n+1}$, we get that

$$
\begin{aligned}
&I_{n+1}(p_1, \dots, p_{n+1}) - J_{n+1}(p_1, \dots, p_{n+1}) \\
&= I_{n+1}(p_1, \dots, p_{n+1}) - J_n(p_1 + p_2, p_3, \dots, p_{n+1}) - (p_1 + p_2)^{\alpha} J_2 \left(\tfrac{p_1}{p_1+p_2}, \tfrac{p_2}{p_1+p_2}\right) \\
&= I_{n+1}(p_1, \dots, p_{n+1}) - I_n(p_1 + p_2, p_3, \dots, p_{n+1}) - (p_1 + p_2)^{\alpha} I_2 \left(\tfrac{p_1}{p_1+p_2}, \tfrac{p_2}{p_1+p_2}\right) \\
&+I_n(p_1 + p_2, p_3, \dots, p_{n+1}) - J_n(p_1 + p_2, p_3, \dots, p_{n+1}) \\
&+(p_1 + p_2)^{\alpha} \left(I_2 \left(\tfrac{p_1}{p_1+p_2}\right) - J_2 \left(\tfrac{p_1}{p_1+p_2}, \tfrac{p_2}{p_1+p_2}\right)\right).
\end{aligned}
$$

Therefore, if $\alpha < 0$, (3.8) (with $n + 1$ instead of n), (3.10) with $n = 2$ and the induction hypothesis (applying to $(p_1 + p_2, \dots, p_{n+1})$ instead of $(p_1, \dots, p_n)$) imply that

$$
\begin{aligned}
&|I_{n+1}(p_1, \dots, p_{n+1}) - J_{n+1}(p_1, \dots, p_{n+1})| \\
&\leq \varepsilon_n + \textstyle\sum_{k=2}^{n-1} \varepsilon_k + 15(2\varepsilon_2 + \varepsilon_1) \left(1 + \sum_{k=2}^{n-1} \left(\sum_{i=1}^{k+1} p_i\right)^{\alpha}\right) \\
&+15(2\varepsilon_2 + \varepsilon_1)(p_1 + p_2)^{\alpha} \\
&= \textstyle\sum_{k=2}^{n} \varepsilon_k + 15(2\varepsilon_2 + \varepsilon_1) \left(1 + \sum_{k=2}^{n} \left(\sum_{i=1}^{k} p_i\right)^{\alpha}\right),
\end{aligned}
$$

that is (3.10) holds for $n + 1$ instead of n.

Finally, if $\alpha = 0$, (3.9) (with $n + 1$ instead of n), (3.11) with $n = 2$ and the induction hypothesis (applying to $(p_1 + p_2, \dots, p_{n+1})$ instead of $(p_1, \dots, p_n)$) imply that

$$
\begin{aligned}
&|I_{n+1}(p_1, \dots, p_{n+1}) - J_{n+1}(p_1, \dots, p_{n+1})| \\
&\leq \varepsilon_n + \textstyle\sum_{k=2}^{n-1} \varepsilon_k + 63(n - 1)(2\varepsilon_2 + \varepsilon_1) + 63(2\varepsilon_2 + \varepsilon_1) \\
&= \textstyle\sum_{k=2}^{n} \varepsilon_k + 63n(2\varepsilon_2 + \varepsilon_1),
\end{aligned}
$$

this yields that (3.11) holds for $n + 1$ instead of n. □

Remark 3.3. Applying Theorem 3.2 with the choice $\varepsilon_n = 0$ for all $n \in \mathbb{N}$ we get the α–recursive, 3–semisymmetric information measures.

Remark 3.4. The sequence H_n^{α} defined in (3.12) is the entropy of degree α (see Aczél–Daróczy [3], Havrda–Charvát [7], Daróczy [4], Tsallis [12]) for $\alpha \leq 0$. Since the sequence J_n satisfies (3.8) and (3.9) with $\varepsilon_n = 0$ $(n \geq 1)$, our theorem says that (J_n) is stable in a certain sense.

References

[1] J. Aczél, *Notes on generalized information functions*, Aequationes Math., **22** (1981), 91–107.

[2] J. Aczél, *Characterizing information measures: approaching the end of an era*, Lectures Notes in Computer Science, **286**, Uncertainty in Knowledge–Based Systems, Springer–Verlag, (1986), 359–384.

[3] J. Aczél, Z. Daróczy, *On measures of information and their characterization*, Academic Press, New York – San Francisco – London, 1975.

[4] Z. Daróczy, *Generalized information functions*, Information and Control **16** (1970), 36–51.

[5] Z. L. Forti, *Hyers–Ulam stability of functional equations in several variables*, Aequationes Math. **50** (1995), no. 1–2, 143–190.

[6] R. Ger, *A survey of recent results on stability of functional equations, Proceeding of the* 4^{th} *International Conference on Functional Equations and Inequalities*, Pedagogical University in Cracow (1994), 5–36.

[7] J. Havrda and F. Charvát, *Quantification Method of Classification Processes. Concept of Structural* α*-Entropy*, Kybernetika **3** (1967), 30–35.

[8] Gy. Maksa, *Solution on the open triangle of the generalized fundamental equation of information with four unknown functions*, Utilitas Math. **21** (1982), 267–282.

[9] Gy. Maksa, *The stability of the entropy of degree alpha*, J. Math. Anal. Appl. **346** (2008), 17–21.

[10] Z. Moszner, *Sur les définitions différentes de la stabilité des équations fonctionnelles*, Aequationes Math. **68** (2004), no. 3, 260–274.

[11] L. Székelyhidi, *38. Problem*, Aequationes Math. **41** (1991), 302.

[12] C. Tsallis, *Possible Generalization of Boltzmann-Gibbs Statistics*, Journal of Statistical Physics **52(1-2)** (1988), 479–487.

Eszter Gselmann
Institute of Mathematics
University of Debrecen
P. O. Box: 12.
Debrecen
Hungary
H–4010
e-mail: `gselmann@math.klte.hu`

Gyula Maksa
Institute of Mathematics
University of Debrecen
P. O. Box: 12.
Debrecen
Hungary
H–4010
e-mail: `maksa@math.klte.hu`