Γ-STABILITY AND VORTEX MOTION IN TYPE II SUPERCONDUCTORS

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Abstract. We consider a time-dependent Ginzburg-Landau equation for superconductors with a strictly complex relaxation parameter, and derive motion laws for the vortices in the case of a finite number of vortices in a bounded magnetic field. The motion laws correspond to the flux-flow Hall effect. As our main tool, we develop a quantitative Γ-stability result relating the Ginzburg-Landau energy to the renormalized energy.

1. Introduction

1.1. **Physical background.** The evolution of a superconducting material is usually described by means of the time-dependent Ginzburg-Landau equation first derived by Schmid [26] and by Gorkov-Eliashberg [8]. After suitably choosing units, this equation can be written as

$$\gamma \left(\frac{\partial}{\partial t} + i\Phi \right) u = (\nabla - iA)^2 u + \frac{1}{\varepsilon^2} u(1 - |u|^2), \tag{1.1}$$

where $u : \Omega \to \mathbb{C}$ is a complex order parameter, $A : \Omega \to \mathbb{R}^2$ the vector potential of the magnetic field and Φ the electric field potential for a superconducting sample $\Omega \subset \mathbb{R}^2$. The parameter ε is a material constant, and we will assume it to be small, which corresponds to *type II* superconductors.

In the works of Schmid and Gorkov-Eliashberg, the relaxation constant γ in (1.1) is a positive real number. However, that theory is insufficient to describe the so-called flux-flow Hall effect, and so Dorsey [6] and Kopnin et al. [14] suggested to replace γ by a *complex* relaxation rate $\tilde{\gamma} = \alpha + i\beta$. Following their lead, we examine the evolution law for the order parameter

$$(\alpha + i\beta) \left(\frac{\partial}{\partial t} + i\Phi \right) u = (\nabla - iA)^2 u + \frac{1}{\varepsilon^2} u(1 - |u|^2). \tag{1.2}$$

The natural evolution law for the electric field $E = -(\partial_t A + \nabla\Phi)$ and the magnetic field $h = \operatorname{curl} A$ is Ampère's law,

$$-\nabla^\perp h = j_A + j_{(n)}, \tag{1.3}$$

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where $j_A = \Im(\bar{u}(\nabla - iA)u)$ is the supercurrent and $j_{(n)}$ the *normal current* induced by the electric field:

$$j_{(n)} = \sigma E, \tag{1.4}$$

where σ is the conductivity of the normal state. For simplicity, we follow Kopnin et al. [14] and assume here that the normal-state Hall effect is negligible (else it would be necessary to replace σ by the nondiagonal normal state conductivity tensor).

For this system of equations, Dorsey [6] and Kopnin et al. [14] used formal asymptotics to calculate the flux-flow Hall conductivity by analyzing the vortex motion. A related matched asymptotic expansion was done by E [7], who derived the vortex motion law.

As our superconducting sample is finite, we need to specify boundary conditions. For a simply connected superconductor with cylindrical symmetry in a perpendicular external field h_{ex}, the usual boundary conditions are

$$\nu \cdot (\nabla - iA)u = 0 \tag{1.5}$$

$$h = h_{ex}, \tag{1.6}$$

where ν is the exterior normal to the boundary $\partial\Omega$.

We note that the equations enjoy a gauge invariance under the transformation $u \mapsto ue^{i\chi}$, $A \mapsto A + \nabla\chi$, $\Phi \mapsto \Phi - \partial_t\chi$. The quantities E, h, and j_A are invariant under this operation.

1.2. **Mathematical structure and related results.** The right-hand side of (1.2) is the negative L^2 gradient with respect to u of the energy functional

$$G_\varepsilon(u, A; h_{ex}) = \frac{1}{2}\int_\Omega |(\nabla - iA)u|^2 + |\operatorname{curl} A - h_{ex}|^2 + \frac{1}{2\varepsilon^2}(1 - |u|^2)^2. \tag{1.7}$$

We can therefore interpret (1.2) as the gauged gradient flow

$$(\alpha + i\beta)\left(\frac{\partial}{\partial t} + i\Phi\right) u = -\nabla_u^{L^2} G_\varepsilon(u, A; h_{ex}). \tag{1.8}$$

The evolution equation for A, (1.3), has a similar structure:

$$\sigma(\partial_t A + \nabla\Phi) = (iu, \nabla - iAu) + \nabla^\perp \operatorname{curl} A = -\nabla_A^{L^2} G_\varepsilon(u, A; h_{ex}), \tag{1.9}$$

where the right-hand side is the negative L^2 gradient of the energy with respect to variations in A.

We note that (1.8) is a linear combination of the purely dissipative gradient flow,

$$\alpha\left(\frac{\partial}{\partial t} + i\Phi\right) u = -\nabla_u^{L^2} G_\varepsilon(u, A; h_{ex}), \tag{1.10}$$

and the purely conservative gauged Schrödinger equation,

$$i\beta\left(\frac{\partial}{\partial t} + i\Phi\right) u = -\nabla_u^{L^2} G_\varepsilon(u, A; h_{ex}). \tag{1.11}$$

Both evolution laws for the order parameter (1.10) and (1.11) have been considered in the literature, typically combined with the field evolution equation (1.9).

Vortex motion laws for (1.10) on bounded domains were rigorously derived by Spirn [27] for the case of a bounded external field $h_{ex} = O(1)$ and by Sandier-Serfaty [23] for fields of order $h_{ex} = O(|\log\varepsilon|)$. The method of [27] relies on PDE estimates for the evolution of the energy density, while [23] relies more heavily on the gradient flow structure of the equation. Rigorous results for vortex motion in all of $\mathbb{R}^2$ with no applied field were derived by Gustafson-Sigal [9]. Their results are valid for finite ε as long as the vortex separation is sufficiently large, and yield an H^1 control on the solutions globally in time, both for the Gorkov-Eliashberg equations and for the wave equation analog, the Maxwell-Higgs equations. This approach is based on the study of the spectrum of the linearized operators around the fundamental vortex solutions, see [10].

The pure magnetic Schrödinger case (1.11) was studied by Spirn [28] using the evolution law for the Jacobian.

For a combined evolution equation such as (1.8), there is a little less structure that can be used to derive motion laws. The related equation without gauge field,

$$(\alpha + i\beta)\partial_t u = \Delta u + \frac{1}{\varepsilon^2}u(1-|u|^2), \tag{1.12}$$

was studied by Kurzke-Melcher-Moser-Spirn [15] on a bounded domain and by Miot [22] on the whole space $\mathbb{R}^2$. A related model for motion of magnetic vortices under the Landau-Lifshitz-Gilbert flow was studied in [16] under somewhat more restrictive assumptions. In all of these works, the vortex motion law is again a hybrid of a conservative motion and the gradient flow of the renormalized energy. The motion law is derived using the conservation laws for energy density and Jacobian. These conservation laws can be combined with a suitable choice of test functions to achieve a certain cancellation property.

1.3. **Statement of the results.** To state our results, we first introduce some notation and make our language more precise.

As a general assumption, Ω denotes a bounded and simply connected domain with C^1 boundary. For $a = (a_1, \dots, a_n) \in \Omega^n$, we define

$$\rho_a = \frac{1}{4}\min\left\{\min_{i\neq j}|a_i - a_j|, \min_i \operatorname{dist}(a_i, \partial\Omega)\right\}. \tag{1.13}$$

We set $\Omega^{n*} = \{a \in \Omega^n : \rho_a > 0\}$. For $s < \rho_a$, we define the corresponding domains with holes punched out,

$$\Omega_s(a) = \Omega \setminus \bigcup_{i=1}^n \overline{B_s(a_i)}. \tag{1.14}$$

We use the following weak norm on the dual of $W_0^{1,p}(\Omega)$:

$$\|\mu\|_{\dot{W}^{-1,q}} = \sup\left\{\mu(\phi) : \phi \in W_0^{1,p}(\Omega), \|\nabla\phi\|_{L^p} \leq 1\right\}, \quad \frac{1}{p} + \frac{1}{q} = 1. \tag{1.15}$$

We will work with functions (u, A, Φ) or sequences of such functions, $(u_\varepsilon, A_\varepsilon, \Phi_\varepsilon) : \Omega \times (0,T) \to \mathbb{C} \times \mathbb{R}^2 \times \mathbb{R}$ for some $T \in (0, \infty]$. Here ε will be used to denote any fixed sequence $\varepsilon_k \to 0$, and we will take subsequences of this sequence and still denote these as $\varepsilon \to 0$. We will call u the *order parameter*, A the *magnetic field potential* and Φ the *electric field potential*. We define the *covariant derivatives*

$$\nabla_A u = \nabla u - iAu \tag{1.16}$$

$$\partial_\Phi u = \partial_t u + i\Phi u. \tag{1.17}$$

The *electric field* E is given by

$$E = -(\partial_t A + \nabla\Phi) \tag{1.18}$$

and the *magnetic field* h by

$$h = \operatorname{curl} A \tag{1.19}$$

The *energy density* g_ε is given by

$$g_\varepsilon = g_\varepsilon(u, A; h_{ex}) = \frac{1}{2}|\nabla_A u|^2 + \frac{1}{2}|h - h_{ex}|^2 + \frac{1}{2\varepsilon^2}(1 - |u|^2)^2. \tag{1.20}$$

With $(\cdot,\cdot)$ denoting the real-valued scalar product on $\mathbb{C} \cong \mathbb{R}^2$ given by $(u, v) = \Re(\bar{u}v)$, we define the *supercurrent* $j_A(u)$, taking the scalar product in every component, as

$$j_A(u) = (iu, \nabla_A u) \tag{1.21}$$

and the *gauged Jacobian* $J_A(u)$ as

$$J_A(u) = \frac{1}{2}\operatorname{curl} j_A(u) + \frac{1}{2}\operatorname{curl} A. \tag{1.22}$$

The total Ginzburg-Landau energy G_ε is the integrated energy density,

$$G_\varepsilon(u, A; h_{ex}) = \int_\Omega g_\varepsilon(u, A; h_{ex}). \tag{1.23}$$

We give a quick overview of the definition of the so-called renormalized energy. More details about this are given in Section 5. For $a \in \Omega^{n*}$ and $d \in \mathbb{Z}^n$, the *canonical harmonic map* $u_\star = u_\star(a, d)$ is defined as an $\mathbb{S}^1$-valued map that solves the equations

$$\operatorname{div} j(u_*) = 0 \tag{1.24}$$

$$\operatorname{curl} j(u_*) = 2\pi \sum_{i=1}^n d_i \delta_{a_i} \tag{1.25}$$

with Neumann boundary conditions $\nu \cdot j(u_*) = 0$. This defines the canonical harmonic map up to a constant. Similarly, we define the limiting magnetic field $h_\star$ as the solution of the limiting London equation,

$$-\Delta h_\star + h_\star = 2\pi \sum d_i \delta_{a_i} \tag{1.26}$$

with Dirichlet boundary conditions $h_\star = h_{ex}$ on $\partial\Omega$. Letting $\Xi_\star$ denote the solution of $-\Delta \Xi_\star = h_\star$ with boundary conditions $\Xi_\star = 0$ on $\partial\Omega$, we define the limiting magnetic field potential $A_\star$ via $A_\star = \operatorname{curl} \Xi_\star$.

The *renormalized energy* of the configuration (a, d) is then given by (1.27)

$$W_\Omega(a, d; h_{ex}) = \lim_{\rho \to 0} \left(\int_{\Omega_\rho(a)} \frac{1}{2} |\nabla_{A_\star} u_*|^2 - \pi \sum_{i=1}^n d_i^2 \log \frac{1}{\rho} \right) + \frac{1}{2} \int_\Omega |h_\star - h_{ex}|^2.$$

Given $(u_\varepsilon, A_\varepsilon)$ and (a, d), we define the *excess energy* D_ε by (1.28)

$$D_\varepsilon = D_\varepsilon(u_\varepsilon, A_\varepsilon, a, d; h_{ex}) := G_\varepsilon(u_\varepsilon, A_\varepsilon; h_{ex}) - n \left(\log \frac{1}{\varepsilon} + \gamma_0 \right) - W_\Omega(a, d; h_{ex}),$$

where γ_0 is a constant defined in (5.9). The renormalized energy and the full Ginzburg-Landau energy are connected via Theorem 2.2, which essentially states that $\liminf_{\varepsilon \to 0} D_\varepsilon(u_\varepsilon, A_\varepsilon, a, d; h_{ex}) \geq 0$ if $J_{A_\varepsilon}(u_\varepsilon) \to \pi \sum d_i \delta_{a_i}$. If also $\limsup_{\varepsilon \to 0} D_\varepsilon(u_\varepsilon, A_\varepsilon, a, d; h_{ex}) = 0$, the sequence $(u_\varepsilon, A_\varepsilon)$ is said to be *well-prepared for the initial data* (a, d).

We now have the notation in place to state our main dynamic result:

Theorem 1.1. *Let* $(u_\varepsilon, A_\varepsilon, \Phi_\varepsilon) : \Omega \times [0, \infty) \to \mathbb{C} \times \mathbb{R}^2 \times \mathbb{R}$ *be a sequence of solutions of the system of equations*

$$(\alpha_\varepsilon + i\beta_\varepsilon) \partial_\Phi u = (\nabla_{A_\varepsilon})^2 u_\varepsilon + \frac{1}{\varepsilon^2} u_\varepsilon (1 - |u_\varepsilon|^2) \tag{1.29}$$

$$\sigma_\varepsilon E_\varepsilon = -\nabla^\perp h_\varepsilon - j_{A_\varepsilon}(u_\varepsilon) \tag{1.30}$$

with the boundary conditions

$$\nu \cdot \nabla_{A_\varepsilon} u_\varepsilon = 0 \quad \text{on } \partial\Omega \tag{1.31}$$

$$h_\varepsilon = h_{ex} \quad \text{on } \partial\Omega \tag{1.32}$$

and initial conditions $(u_\varepsilon^0, A_\varepsilon^0)$ *such that*

$$\left\| J_{A_\varepsilon^0}(u_\varepsilon^0) - \pi \sum_{j=1}^n d_j \delta_{a_j^0} \right\|_{\dot{W}^{-1,p}} \longrightarrow 0 \tag{1.33}$$

as $\varepsilon \to 0$ *for some* $p > 1$, *where* $a^0 = (a_j^0) \in \Omega^{n*}$ *and* $d = (d_j) \in \{\pm 1\}^n$. *Assume furthermore the well-preparedness condition,*

$$\lim_{\varepsilon \to 0} D_\varepsilon(u_\varepsilon^0, A_\varepsilon^0, a^0, d; h_{ex}) = 0. \tag{1.34}$$

Let the time scale factors α_ε, β_ε and the normal-state conductivity σ_ε satisfy

$$\alpha_\varepsilon = \frac{\alpha_0}{|\log \varepsilon|}, \quad \beta_\varepsilon = \beta_0, \quad \sigma_\varepsilon = \frac{\sigma_0}{|\log \varepsilon|} \tag{1.35}$$

for some $\alpha_0, \beta_0, \sigma_0 > 0$.

Define $a(t)$ as the solution of the system of ordinary differential equations

$$\alpha_0 \dot{a}_j + d_j \beta_0 \dot{a}_j^\perp = -\frac{1}{\pi} \nabla_{a_j} W_\Omega(a(t), d; h_{ex}) \tag{1.36}$$

with initial conditions $a_j(0) = a_j^0$.

Then for all $t < T_M$, where T_M is the maximal time of existence for the system (1.36), *the following holds:*

(1) *The rescaled energy density $\frac{1}{|\log \varepsilon|} g_\varepsilon(u_\varepsilon(t), A_\varepsilon(t); h_{ex})$ converges in the weak topology of measures to $\pi \sum \delta_{a_j(t)}$.*
(2) *The gauged Jacobian $J_{A_\varepsilon}(u_\varepsilon(t))$ converges in $\dot{W}^{-1,p}$ to $\pi \sum d_j \delta_{a_j(t)}$.*
(3) *The well-preparedness propagates, i.e.*

$$\limsup_{\varepsilon \to 0} D_\varepsilon(u_\varepsilon(t), A_\varepsilon(t), a(t), d; h_{ex}) = 0.$$

Remark 1.2. The existence of initial data satisfying the assumptions of Theorem 1.1 follows from Lemma 5.8 with e.g. $r = \varepsilon^\gamma$, $0 < \gamma < 1$. Note that the $\dot{W}^{-1,p}$ convergence follows from the combination of $\dot{W}^{-1,1}$ convergence and $\dot{W}^{-1,p}$ compactness.

Remark 1.3. The global existence of solutions of (1.29)–(1.30) in the Coulomb gauge can be established by a modification of the argument in Tang-Wang [29] for the corresponding problem with $\beta_\varepsilon = 0$ (note that their proof does not use the maximum principle). First one can generate the existence of a weak solution $(u, A) \in L^\infty(0, T; \mathcal{L}^2_{div}) \cap L^2(0, T; \mathcal{H}^1_{div})$ for all $T > 0$ by a Galerkin procedure and the *a priori* energy estimate, where $\mathcal{L}^2_{div} = L^2(\Omega) \otimes L^2_{div}(\Omega)$ and $\mathcal{H}^1_{div} = H^1(\Omega) \otimes H^1_{div}(\Omega)$. Here $L^2_{div}(\Omega)$ is the closure of $\mathcal{V} = \{B \in C^\infty(\Omega)$ such that $B \cdot n|_{\partial\Omega} = 0, \operatorname{div} B = 0\}$ in L^2 and $H^1_{div}(\Omega)$ is the closure of $\mathcal{V}$ in $H^1(\Omega)$. Higher regularity can then be established by smoother initial data and standard methods.

Remark 1.4. The system of ODEs (1.36) can be rewritten as

$$\dot{a}_j = -\frac{1}{\alpha_0^2 + \beta_0^2} \begin{pmatrix} \alpha_0 & d_j \beta_0 \\ -d_j \beta_0 & \alpha_0 \end{pmatrix} \nabla_{a_j} W_\Omega(a(t), d; h_{ex}). \tag{1.37}$$

Remark 1.5. In [6] and [14], the vortex motion is assumed to be driven by an applied transport current. In a purely dissipative regime, vortices move perpendicular to the transport current, leading to an induced average electric field that is parallel to the transport current.

The complex character of the motion law means that the vortices additionally have a component of motion that is parallel to the current (as in an ideal fluid). This leads to a flux-flow Hall effect. In the research presented here, we do not include an applied current, but instead examine a purely energy-driven system. We, however, expect that our results carry over to such a situation and yield a Hall angle depending only on the ratio of β_0 and α_0.

In a recent preprint of Tice [30] the motion of a vortex under the influence of both an applied magnetic field and an applied boundary current gives rise to a vortex motion law with components that are a composition of a gradient and a curl of limiting potential functions.

Remark 1.6. Essentially, the smallness of the energy excess means that $(u_\varepsilon, A_\varepsilon)$ stay close to the subset of configurations that are almost optimal given their vortex positions. The evolution can thus be well described by the point vortex system (1.36) plus an adiabatic process by which $(u_\varepsilon, A_\varepsilon)$ follows this point vortex motion.

Our main tool for the proof of Theorem 1.1 is the following result that gives us a way of comparing an energy excess and the distance from the set of almost optimal configurations:

Theorem 1.7. *If* $\|J(u)-\sum_{j=1}^n \pi d_j \delta_{\xi_j}\|_{\dot W^{-1,1}(\Omega)} \leq s_\varepsilon$ *and* $\|A-A_\star\|_{L^p(\Omega)} \leq t_\varepsilon$ *for some* $p>2$ *then*

$$\int_{\Omega_r(\xi)} e_\varepsilon(|u|) + \frac{1}{4}\left|\frac{j_A(u)}{|u|} - j_{A_\star}(u_\star)\right|^2 + \frac{1}{2}\int_\Omega |\mathrm{curl}\, A - \mathrm{curl}\, A_\star|^2 \leq D_\varepsilon(u, A, \xi, d; h_{ex}) + \mathbf{error}$$

where **error** $\ll 1$ *is explicit in terms of* $\varepsilon, \xi, d, r, s_\varepsilon, t_\varepsilon, \rho_\xi, n, p, h_{ex}$*, see Theorem 4.1 for the precise statement.*

This theorem is true for any pair $(u, A) \in H^1 \otimes H^1$ under the Coulomb gauge (not just for solutions), and it is more powerful than what we use here. It is possible to use Theorem 1.7 to develop quantitative (finite ε) bounds on the dynamical law, whereas we only use qualitative (limit as $\varepsilon \to 0$) results in Theorem 1.1. It should have other applications, as well, in static and dynamical settings. One can interpret Theorem 1.7 as obtaining L^2 convergence of $\nabla u - \nabla u_\star$ and $h - h_{ex}$ to zero merely from $W^{-1,p}$ convergence and control of the excess energy. For similar quantitative bounds in the non-gauged case, we mention Jerrard-Spirn [12, 13] and Kurzke-Spirn [17]. Finally, we note that there seem to be parallels to estimates found in [9] which measure the H^1 distance of solutions of (1.10) to a manifold of "multi-vortex configurations"; however, the assumptions are different than those required in Theorem 1.7.

The rest of this article is devoted to the proof of Theorems 1.1 and 1.7.

2. Γ-convergence and compactness results

2.1. **Static results.**

We recall the following compactness result:

Proposition 2.1. *Let* $(u_\varepsilon, A_\varepsilon)$ *be a sequence such that* $G_\varepsilon(u_\varepsilon, A_\varepsilon) \leq M|\log \varepsilon|$ *where* $h_{ex} = O(1)$*. Then* $(J_{A_\varepsilon}(u_\varepsilon))$ *is weakly compact in the dual space*

$(C_0^{0,\gamma}(\Omega))^*$. For a subsequence, $J_{A_\varepsilon}(u_\varepsilon) \rightharpoonup \overline{J} = \pi \sum d_i \delta_{a_i}$, $d_i \in \mathbb{Z}$, $a_i \in \Omega$, and the energy satisfies

$$\liminf_{\varepsilon\to 0} \frac{1}{|\log\varepsilon|} G_\varepsilon(u_\varepsilon, A_\varepsilon) \geq |J|(\Omega) \tag{2.1}$$

Note that the compact embedding $(C_0^{0,\gamma})^* \subset \dot{W}^{-1,p}$ implies that the compactness also holds in negative Sobolev spaces.

Proof. Several proofs of this exist in the literature; the most direct is probably the one in [25]. □

Proposition 2.1 gives a comparatively crude lower bound for the energy that does not depend on the vortex positions. These become visible in the next order of the energy through the so-called *renormalized energy.* We obtain the following limiting lower bound for the energy as a corollary of our more precise analysis in Section 4.

Theorem 2.2 (Second-order Γ-convergence)**.** *Let $(u_\varepsilon, A_\varepsilon)$ be a sequence satisfying for some $K > 0$ the energy bound*

$$G_\varepsilon(u_\varepsilon, A_\varepsilon) \leq n\pi|\log\varepsilon| + K$$

and suppose that the vortices are localized as

$$\|J(u_\varepsilon) - \pi \sum_{j=1}^n d_j \delta_{a_j}\|_{\dot{W}^{-1,1}} \to 0 \tag{2.2}$$

where the limit satisfies

$$a = (a_1, \dots, a_n) \in \Omega^{n*}, d_j \in \{\pm 1\}.$$

Then the limit in (2.2) is also true in $\dot{W}^{-1,p}$ for any $1 < p < \infty$ and

$$G_\varepsilon(u_\varepsilon, A_\varepsilon; h_{ex}) \geq n(\gamma_0 + \pi|\log\varepsilon|) + W_\Omega(a, d; h_{ex}) - o_\varepsilon(1) \tag{2.3}$$

as $\varepsilon \to 0$.

In addition, for every (a, d) such that $a \in \Omega^{n*}$, $d \in \{\pm 1\}^n$ there exist $(u_\varepsilon, A_\varepsilon)$ such that (2.3) holds with equality.

Proof. The $\dot{W}^{-1,p}$ compactness statement of Proposition 2.1 allows us to replace $\dot{W}^{-1,1}$ limits by $\dot{W}^{-1,p}$ limits. We now replace A_ε by $\tilde{A}_\varepsilon$ that minimizes $G_\varepsilon(u_\varepsilon, \cdot; h_{ex})$ It follows that $\tilde{A}_\varepsilon$ satisfies the Euler-Lagrange equation

$$j_{\tilde{A}_\varepsilon}(u_\varepsilon) + \operatorname{curl}\operatorname{curl}\tilde{A}_\varepsilon = 0 \tag{2.4}$$

with $\operatorname{curl}\tilde{A}_\varepsilon = h_{ex}$ on $\partial\Omega$. so setting $\tilde{h}_\varepsilon = \operatorname{curl} A_\varepsilon$, we obtain for the gauged Jacobian

$$2J_{\tilde{A}_\varepsilon}(u_\varepsilon) = -\Delta\tilde{h}_\varepsilon + \tilde{h}_\varepsilon \tag{2.5}$$

Using Lemma 4.5, we see that by choosing Coulomb gauge, we have that $J_{\tilde{A}_\varepsilon}(u_\varepsilon)$ converge to the same limit as $J(u_\varepsilon)$. It follows via elliptic estimates that A_ε satisfy (at least for a subsequence) (4.2) for some $t_\varepsilon \to 0$. Hence we can apply Theorem 4.1. As the left-hand side of (4.5) is nonnegative and the error on the right-hand side is $o_\varepsilon(1)$, the estimate (2.3) follows.

The construction can be done using Lemma 5.8 with $\varepsilon \ll r \ll 1$, e.g. $r = \sqrt{\varepsilon}$. □

2.2. **Vortex paths.**

Proposition 2.3. *Under the assumptions of Theorem 1.1, there is a time* $T_1 > 0$ *such that*

$$\int_0^{T_1} \int_\Omega \alpha_0 |\partial_\Phi u|^2 + \sigma_0 |E|^2 \le |\log \varepsilon| \tag{2.6}$$

Furthermore, there exists $\xi = (\xi_1, \dots, \xi_n) \in H^1(0, T_1; \mathbb{R}^{2n})$ *with*

$$\left\| J_{A_\varepsilon}(u_\varepsilon(t)) - \sum_{j=1}^n d_j \delta_{\xi_j(t)} \right\|_{\dot{W}^{-1,1}} \longrightarrow 0 \tag{2.7}$$

such that for all $T \le T_1$

$$\liminf_{\varepsilon \to 0} \alpha_\varepsilon \int_0^T \int_\Omega |\partial_\Phi u|^2 \ge \pi \alpha_0 \int \sum_{j=1}^n |\dot{\xi}_j|^2. \tag{2.8}$$

Proof. We follow the proof of Lemma 3.4 in [23]. From (A.24),

$$G_\varepsilon(u_\varepsilon(0), A_\varepsilon(0)) - G_\varepsilon(u_\varepsilon(t), A_\varepsilon(t)) = \frac{\alpha_0}{|\log \varepsilon|} \int_0^t \int_\Omega |\partial_\Phi u|^2 + \frac{\sigma_0}{|\log \varepsilon|} \int_0^t \int_\Omega |E|^2.$$

We rescale time via $\bar{u}_\varepsilon(|\log \varepsilon| t, x) = u_\varepsilon(t, x)$ and $\bar{A}_\varepsilon(|\log \varepsilon| t, x) = A_\varepsilon(t, x)$ and set $\bar{\Phi}_\varepsilon(|\log \varepsilon| t, x) = |\log \varepsilon| \Phi_\varepsilon(t, x)$. In this new variables, $(\bar{u}_\varepsilon, \bar{A}_\varepsilon, \bar{\Phi}_\varepsilon)$ and $\bar{E}_\varepsilon = -\partial_t \bar{A}_\varepsilon - \nabla \bar{\Phi}_\varepsilon$ satisfy, dropping the index ε, the rescaled equation

$$(\alpha_0 + i) \partial_{\bar{\Phi}} \bar{u} = (\nabla_{\bar{A}})^2 \bar{u} + \frac{1}{\varepsilon^2} \bar{u}(1 - |\bar{u}|^2) \tag{2.9}$$

$$\sigma_0 \bar{E} = -\nabla^\perp \operatorname{curl} \bar{A} - (i\bar{u}, \nabla_{\bar{A}} \bar{u}). \tag{2.10}$$

We now assume by contradiction that there exists $s_\varepsilon \ll |\log \varepsilon|$ such that

$$\frac{1}{|\log \varepsilon|} \int_0^{s_\varepsilon} \int_\Omega \alpha_0 |\partial_\Phi u|^2 + \sigma_0 |E|^2 = 1 \tag{2.11}$$

Testing (2.9) with $\partial_{\bar{\Phi}} u$ then leads, using the rescaling, to

$$1 = \int_0^{s_\varepsilon} \int_\Omega \alpha_0 |\partial_{\bar{\Phi}} \bar{u}|^2 + \sigma_0 |\bar{E}|^2 = G_\varepsilon(\bar{u}(0), \bar{A}(0)) - G_\varepsilon(\bar{u}(s_\varepsilon), \bar{A}(s_\varepsilon)) \tag{2.12}$$

Rescaling again to $(\tilde{u}, \tilde{A}, \tilde{\Phi})$ with $\tilde{u}(t, x) = \bar{u}(s_\varepsilon t, x)$, $\tilde{A}(t, x) = \bar{A}(s_\varepsilon t, x)$ and $\tilde{\Phi}(t, x) = s_\varepsilon^{-1} \bar{\Phi}(s_\varepsilon t, x)$. It follows that

$$\int_0^1 \int_\Omega \alpha_0 |\partial_{\tilde{\Phi}} \tilde{u}|^2 + \sigma_0 |\tilde{E}|^2 = s_\varepsilon \ll |\log \varepsilon|. \tag{2.13}$$

Now we can follow Proposition 4.3 of [23] (with $N_\varepsilon = 1$) and obtain that the vorticities $J_{\tilde{A}}(\tilde{u})$ converge to a measure $\mu(t)$ with $\partial_t \mu = 0$ in $[0, 1]$; in other words, the vortices do not move. This shows that $J_{\bar{A}}(\bar{u})(s_\varepsilon)$ converges to the same limit as $J_A(u)(0)$; from the Γ-convergence statement (2.3), we thus obtain using initial well-preparedness (1.34) that $G_\varepsilon(\bar{u}(s_\varepsilon), \bar{A}(s_\varepsilon)) \geq G_\varepsilon(u(0), A(0)) - o_\varepsilon(1)$, a contradiction to (2.12). Hence (2.6) is shown.

Proposition 4.3 of [23] now implies that $J_{A_\varepsilon}(u_\varepsilon)$ converges to a measure $\mu(t)$ for every t, and $t \mapsto \mu(t)(\zeta)$ is H^1 for every smooth ζ. By Proposition 2.1, $\mu(t) = \sum d_j(t)\delta_{\xi_j(t)}$. The continuity implies that for sufficiently short time, d_j are constant and the ξ_j stay away from each other, so can be chosen as H^1 functions, which implies (2.7). The bound (2.8) follows from (4.14) of [23] using the same reasoning as in Corollary 7 of [24]. □

Proposition 2.4. *Under the assumptions of Theorem 2.2, the energy concentrates at the same points as the Jacobian. In particular, if* $J_{A_\varepsilon}(u_\varepsilon) \to \pi \sum d_j \delta_{a_j}$ *in* $\dot{W}^{-1,1}$ *and* A_ε *satisfies the Coulomb gauge condition, then*

$$\frac{1}{|\log \varepsilon|} g_\varepsilon(u_\varepsilon, A_\varepsilon) \to \pi \sum \delta_{a_j} \tag{2.14}$$

in the weak topology of measures.

Proof. The proof of this can be found in [28] and follows from $D_\varepsilon \leq C$ and (4.16). Similar results for the ungauged problem can be found in [5, 20]. □

2.3. **The magnetic field.** From (A.25), the magnetic field h satisfies the parabolic equation

$$\sigma_\varepsilon \partial_t h_\varepsilon - \Delta h_\varepsilon + h_\varepsilon = 2J_{A_\varepsilon}(u_\varepsilon) \tag{2.15}$$

with the boundary condition $h_\varepsilon = h_{ex}$ on $\partial\Omega$ and the initial condition $h_\varepsilon(0) = \operatorname{curl} A_\varepsilon(0)$.

By the results of the previous subsection, $J_{A_\varepsilon}(u_\varepsilon)$ is bounded in $H^1(\dot{W}^{-1,p})$ for $p < 2$ and converges to $J_\star = \pi \sum d_j \delta_{\xi_j(t)}$.

Then it follows from standard L^p parabolic theory [21] that

$$\sup_t \|h_\varepsilon - h_\star\|_{W^{1,p}(\Omega)} \to 0 \tag{2.16}$$

as $\varepsilon \to 0$, where

$$-\Delta h_\star + h_\star = 2\pi \sum d_j \delta_{\xi_j(t)} \tag{2.17}$$

with the boundary condition $h_\star = h_{ex}$ on $\partial\Omega$.

In particular, if we let A_ε, $A_\star$ denote the Coulomb gauge vector potentials for h_ε and $h_\star$,

$$\sup_t \|A - A_\star\|_{W^{2,p}} \to 0 \tag{2.18}$$

and by Sobolev embedding, it follows that for every time, $A \to A_\star$ in all $W^{1,q}$, in particular the condition (4.2) can be fulfilled with $t_\varepsilon \to 0$ for any $p \in (1, \infty)$.

3. The Gronwall loop

This section is the heart of the proof of Theorem 1.1, and based on the analogous results for the gauge-free case in [15].

We always write $a(t) = (a_1(t), \ldots, a_n(t))$ for the solution of the system of ODEs

$$\alpha_0 \dot{a}_i + \beta_0 d_i \dot{a}_i^\perp = -\frac{1}{\pi} \nabla_{a_i} W(a, d) \tag{3.1}$$

with the initial condition

$$a_i(0) = a_i^0. \tag{3.2}$$

Here $W(a, d) = W_\Omega(a, d; h_{ex})$ denotes the renormalized energy defined of (1.27) and (5.6).

We let $\xi(t) = (\xi_1, \ldots, \xi_n(t))$ be the vortex positions found in Proposition 2.3, and abbreviate $D_\varepsilon(t) = D_\varepsilon(u_\varepsilon(t), A_\varepsilon(t), a(t), d; h_{ex})$.

For the difference between the ODE position and the true vortex position, we write $\eta(t) = \xi(t) - a(t) \in \mathbb{R}^{2n}$.

Considering the growth of $\eta(t)$ and $D_\varepsilon(t)$, we have the following asymptotic results:

Proposition 3.1. *For $0 \le t_1 \le t_2 \le T_1$, where T_1 is the time found in Proposition 2.3, the energy excess satisfies*

$$D_\varepsilon(t_2) + |\eta(t_2)| \le D_\varepsilon(t_1) + |\eta(t_1)| + o_\varepsilon(1) + C \int_{t_1}^{t_2} \sum_{j=1}^n |\mathscr{R}_j|, \tag{3.3}$$

where C only depends on $\min_{t \in [t_1, t_2]} \rho_{a(t)}$

$$\mathscr{R}_j = \alpha_0 \dot{\xi}_j + \beta_0 d_j \dot{\xi}_j^\perp + \frac{1}{\pi} \nabla_{a_j} W(a(t), d)). \tag{3.4}$$

Proof. We differentiate the energy $W(a(t), d)$. By (3.1),

$$\frac{d}{dt} W(a(t), d) = \dot{a} \cdot \nabla_a W(a(t), d) = -\alpha_0 \pi \sum_{j=1^n} |\dot{a}_j|^2. \tag{3.5}$$

We calculate the difference in energy losses for G_ε and W_ε using (A.24) and obtain by means of (3.5)

$$\begin{aligned} D_\varepsilon(t_2) &= G_\varepsilon(u_\varepsilon(t_2), A_\varepsilon(t_2)) - W_\varepsilon(a(t_2), d) \\ &= G_\varepsilon(u_\varepsilon(t_2), A_\varepsilon(t_2)) - G_\varepsilon(u_\varepsilon(t_1), A_\varepsilon(t_1)) \\ &\quad + W_\varepsilon(a(t_1), d) - W_\varepsilon(a(t_2), d) + D_\varepsilon(t_1) \\ &= -\alpha_\varepsilon \int_{t_1}^{t_2} \int_\Omega |\partial_\Phi u|^2 - \sigma_\varepsilon \int_{t_1}^{t_2} \int_\Omega |E|^2 + \int_{t_1}^{t_2} \alpha_0 \pi \sum_{j=1}^n |\dot{a}_j|^2 + D_\varepsilon(t_1) \end{aligned}$$

Applying the vortex mobility bound (2.8) yields, dropping the electric field term,

$$\begin{aligned} D_\varepsilon(t_2) - D_\varepsilon(t_1) &\leq -\pi\alpha_0 \int_{t_1}^{t_2} \sum_{j=1}^n \left(|\dot{\xi}_j|^2 - |\dot{a}_j|^2 \right) + o_\varepsilon(1) \\ &= -\pi\alpha_0 \int_{t_1}^{t_2} \sum_{j=1}^n (\dot{\xi}_j - \dot{a}_j) \cdot (\dot{\xi}_j + \dot{a}_j) + o_\varepsilon(1) \\ &= -\pi\alpha_0 \int_{t_1}^{t_2} \sum_{j=1}^n \dot{\eta}_j \cdot (\dot{\eta}_j + 2\dot{a}_j) + o_\varepsilon(1) \\ &\leq C \sup_{[t_1,t_2]} |\dot{a}(t)| \int_{t_1}^{t_2} |\dot{\eta}| + o_\varepsilon(1). \end{aligned}$$

From the ODE (3.5), it follows that $|\dot{a}(t)|$ can be controlled by $|\nabla_a W(a,d)|$. Using (5.21), this can in turn be controlled by a function of ρ_a, the inter-vortex distance, h_{ex} and n.

From the definition of η, it follows that $\eta(t)$ satisfies the system of ODEs

$$\alpha_0 \dot{\eta}_j + \beta_0 d_j \dot{\eta}_j^\perp = \alpha_0 \dot{\xi}_j + \beta_0 d_j \dot{\xi}_j^\perp + \frac{1}{\pi} \nabla_{a_j} W(a(t), d)). \tag{3.6}$$

The right-hand side of (3.6) is $\mathscr{R}_j$. Taking the Euclidean norm, we obtain

$$\sqrt{\alpha_0^2 + \beta_0^2} |\dot{\eta}_j| = |\mathscr{R}_j| \tag{3.7}$$

and now $\left|\frac{d}{dt}|\eta|\right| = \left|\frac{d}{dt}\eta\right|$, so the growth of $|\eta|$ can be controlled similarly, leading to (3.3). □

We also set

$$\tilde{\mathscr{R}}_j = \alpha_0 \dot{\xi}_j + \beta_0 d_j \dot{\xi}_j^\perp + \frac{1}{\pi} \nabla_{a_j} W(\xi(t), d)) \tag{3.8}$$

so

$$\left|\mathscr{R}_j - \tilde{\mathscr{R}}_j\right| = \left| \frac{1}{\pi} \nabla_{a_j} W(\xi(t), d)) - \frac{1}{\pi} \nabla_{a_j} W(a(t), d)) \right| \leq \frac{C}{\rho_{a(t)}^2} |\eta(t)| \tag{3.9}$$

In particular it suffices to control $\tilde{\mathscr{R}}_j$ in terms of D_ε.

In the following sequence of propositions, we establish a control over the term $\mathscr{R}_j$ in terms of D_ε and $|\eta|$. This will allow us to later apply Gronwall's inequality.

Proposition 3.2. *Let* $r_0 < \frac{\rho_{a^0}}{4}$ *and* $\delta > 0$ *such that* $\sup_{[0,\delta]} |\eta(t)| + \delta \sup_{[0,\delta]} |\dot{a}| \leq \frac{r_0}{4}$ *(e.g.* $\delta < C r_0^2$, $C = C(a^0)$*).*

Then for $0 < t \leq \delta$,

$$\int_{\Omega_{r_0}(a^0)} |\nabla_A u|^2 \leq |\log r_0| + D_\varepsilon + o_\varepsilon(1) \tag{3.10}$$

Proof. From Theorem 4.1 with $s_\varepsilon \to 0$ (by (2.7)) and $t_\varepsilon \to 0$ (by (2.18)), it follows that $\sigma^*(\varepsilon) \to 0$, where σ^* is defined in (4.4). For ε so small that $r_0 > \sigma^* \to 0$, the relation (4.5) now implies for $r = \frac{1}{4} r_0$

$$\int_{\Omega_r(\xi(t))} e_\varepsilon(|u_\varepsilon|) + \frac{1}{4}\left|\frac{j_{A_\varepsilon}(u_\varepsilon)}{|u_\varepsilon|} - j_{A_\star}(u_\star)\right|^2 \le D_\varepsilon + o_\varepsilon(1), \tag{3.11}$$

and so using the decomposition $|\nabla_A u|^2 = \left|\frac{j_A(u)}{|u|}\right|^2 + |\nabla|u||^2$, it follows that

$$\int_{\Omega_r(\xi(t))} |\nabla_{A_\varepsilon} u_\varepsilon|^2 \le C \int_{\Omega_r(\xi(t)} |j_{A_\star}(u_\star)|^2 + CD_\varepsilon + o_\varepsilon(1). \tag{3.12}$$

From (5.15) and (5.20), we obtain

$$\int_{\Omega_r(\xi(t))} |j_{A_\star}(u_\star)|^2 \le C|\log r|, \tag{3.13}$$

and now $\Omega_{r_0(a^0)} \subset \Omega_r(\xi(t)$ for $0 \le t \le \delta$, and the claim follows. □

Proposition 3.3. *Given the same hypotheses as Proposition 3.2, then with* $R_\varepsilon = (\nabla_{A_\varepsilon})^2 u_\varepsilon + \frac{1}{\varepsilon^2} u_\varepsilon (1 - |u_\varepsilon|^2)$ *there holds*

$$\frac{\alpha_\varepsilon \beta}{\alpha_\varepsilon^2 + \beta^2} \int_0^\delta \int_{\Omega_r(a^0)} |(iR_\varepsilon, \nabla_{A_\varepsilon} u_\varepsilon)| \le C\sqrt{\alpha_\varepsilon}\sqrt{\int_0^\delta (D_\varepsilon(t) + |\log r| + C}. \tag{3.14}$$

Proof. The Cauchy-Schwarz inequality implies that

$$\int_0^\delta \int_{\Omega_r(a(0))} |(iR_\varepsilon, \nabla_{A_\varepsilon} u_\varepsilon)| \le \left(\int_0^\delta \int_\Omega |R_\varepsilon|^2\right)^{1/2} \left(\int_0^\delta \int_{\Omega_r(a^0)} |\nabla_{A_\varepsilon} u_\varepsilon|^2\right)^{1/2}. \tag{3.15}$$

From $(\alpha_\varepsilon + i\beta)\partial_{\Phi_\varepsilon} u_\varepsilon = R_\varepsilon$, we see using (2.6) that

$$\int_0^\delta \int_\Omega |R_\varepsilon|^2 \le \frac{1}{\alpha_\varepsilon^2 + \beta^2} \int_0^\delta \int_\Omega |\partial_{\Phi_\varepsilon} u_\varepsilon|^2 \le \frac{C}{\alpha_\varepsilon}. \tag{3.16}$$

For the other term, we use Proposition 3.2 and obtain (3.14). □

Proposition 3.4. *Choosing δ as in Proposition 3.2,*

$$\frac{\alpha_\varepsilon^2}{\alpha_\varepsilon^2 + \beta^2} \int_0^\delta \int_\Omega |(R_\varepsilon, \nabla_{A_\varepsilon} u_\varepsilon)| \to 0 \quad \text{as } \varepsilon \to 0. \tag{3.17}$$

Proof. Using Cauchy-Schwarz and (2.6),

$$\begin{aligned} \int_0^\delta \int_\Omega |(R_\varepsilon, \nabla_{A_\varepsilon} u_\varepsilon)| &\le \left(\int_0^\delta |R_\varepsilon|^2\right)^{1/2} \left(\int_0^\delta |\nabla_{A_\varepsilon} u_\varepsilon|^2\right)^{1/2} \\ &\le \frac{C}{\sqrt{\alpha_\varepsilon}} \sqrt{\delta} |\log \varepsilon|^{1/2} \le C, \end{aligned}$$

and since $\alpha_\varepsilon^2 \to 0$ this implies the claim. □

In the following, we use the symplectic matrix $\mathbb{J}$ defined by (A.21) that satisfies $\mathbb{J}v = -v^\perp$.

Proposition 3.5. *Under the assumptions of Proposition 3.2, there exists a $C > 0$ with the following property: For any $\phi \in W_0^{2,\infty}(\Omega)$ that is affine in $\bigcup B_r(a_j^0)$ and satisfies $\|D^2\phi\|_{L^\infty} \le \frac{C}{r^2}$, the following estimate holds:*

$$\int_0^\delta \left| \int_\Omega \mathbb{J}_{kl}\partial_{km}\phi \left((\partial_m^{A_\varepsilon} u_\varepsilon, \partial_l^{A_\varepsilon} u_\varepsilon) - (j_{A_\star}(u_\star))_m (j_{A_\star}(u_\star))_l \right) \right| \le C \int_0^\delta D_\varepsilon(t) + o_\varepsilon(1), \tag{3.18}$$

where $\partial_m^{A_\varepsilon} u_\varepsilon$ is the m-component of $\nabla_{A_\varepsilon} u_\varepsilon$ and $u_\star = u_\star(\xi(t), d, \cdot)$ and $A_\star$ are defined by (5.1) *and* (1.26) *with vortex positions $\xi(t)$.*

Proof. We drop the subscript ε for convenience and decompose $\nabla_A u \otimes \nabla_A u$ as in [13] and [15].

We note the orthogonal decomposition

$$\nabla_A u = \nabla|u| \frac{u}{|u|} + \frac{j_A(u)}{|u|}\frac{iu}{|u|} \tag{3.19}$$

and obtain

$$\begin{aligned}(\partial_m^A u, \partial_l^A u) - (j_{A_\star}(u_\star))_m (j_{A_\star}(u_\star))_l &= \partial_m|u|\partial_l|u| \\ &+ \left(\frac{j_A(u)}{|u|} - j_{A_\star}(u_\star)\right)_m \left(\frac{j_A(u)}{|u|} - j_{A_\star}(u_\star)\right)_l \\ &+ \left(\frac{j_A(u)}{|u|} - j_{A_\star}(u_\star)\right)_m (j_{A_\star}(u_\star))_l \\ &+ (j_{A_\star}(u_\star))_m \left(\frac{j_A(u)}{|u|} - j_{A_\star}(u_\star)\right)_l .\end{aligned} \tag{3.20}$$

We multiply each of these terms with $\mathbb{J}_{kl}\partial_{km}\phi$, integrate and estimate. In particular, using (3.11), we obtain

$$\int_\Omega \mathbb{J}_{kl}\partial_{km}\phi\partial_m|u|\partial_l|u| \le \frac{C}{r^2}\int_{\Omega_r} |\nabla|u||^2 \le \frac{C}{r^2}\left(D_\varepsilon + o_\varepsilon(1)\right) \tag{3.21}$$

and

$$\int_\Omega \mathbb{J}_{kl}\partial_{km}\phi \left(\frac{j_A(u)}{|u|} - j_{A_\star}(u_\star)\right)_m \left(\frac{j_A(u)}{|u|} - j_{A_\star}(u_\star)\right)_l \le \frac{C}{r^2}\left(D_\varepsilon + o_\varepsilon(1)\right). \tag{3.22}$$

By symmetry, it suffices to treat one of the remaining terms, so we just need to consider

$$\int_\Omega \mathbb{J}_{kl}\partial_{km}\phi \left(\frac{j_A(u)}{|u|} - j_{A_\star}(u_\star)\right)_m (j_{A_\star}(u_\star))_l .$$

We decompose further $\frac{j_A(u)}{|u|} - j_{A_\star}(u_\star) = (j_A(u) - j_{A_\star}(u_\star)) - \left(j_A(u) - \frac{j_A(u)}{|u|}\right)$ and now

$$\left|j_A(u) - \frac{j_A(u)}{|u|}\right| = \left|\frac{j_A(u)}{|u|}(1-|u|)\right| \le |\nabla_A u|\,|1-|u||\,. \tag{3.23}$$

We obtain using Cauchy-Schwarz and (3.12) that

$$\left|\int_\Omega \mathbb{J}_{kl}\partial_{km}\phi\left(j_A(u) - \frac{j_A(u)}{|u|}\right)_m (j_{A_\star}(u_\star))_l\right| \le C(r)\varepsilon G_\varepsilon(u,A) = o_\varepsilon(1). \tag{3.24}$$

Finally, we need to estimate the term containing $(j_A(u) - j_{A_\star}(u_\star))_m\,(j_{A_\star}(u_\star))_l$. We perform a Hodge decomposition of $j_A(u(t)) - j_{A_\star}(u_\star)(t)$ for every time t (note that $j_{A_\star}(u_\star)(t)$ is defined using $\xi(t)$). This means that $j_A(u) - j_{A_\star}(u_\star) = \nabla h_1(t) + \nabla^\perp h_2(t)$, where $h_1(t) \in W^{1,p}(\Omega)$ solves

$$\Delta h_1(t) = \operatorname{div} j_A(u(t)), \quad \nu\cdot\nabla h_1(t) = 0 \text{ on } \partial\Omega \tag{3.25}$$

and $h_2(t) \in W^{1,p}(\Omega)$ solves

$$-\Delta h_2(t) = \operatorname{curl}(j_A(u(t)) - j_{A_\star}(u_\star)(t)). \tag{3.26}$$

Setting $\zeta_m(t) = \mathbb{J}_{kl}\partial_{km}\phi\,(j_{A_\star}(u_\star)(t))_l$, we need to estimate

$$\int_\Omega \zeta(t)\cdot(j_A(u) - j_{A_\star}(u_\star)) = \int_\Omega \zeta(t)\cdot(\nabla h_1(t) + \nabla^\perp h_2(t)). \tag{3.27}$$

From (4.7), we have that $\|j_A(u) - j_{A_\star}(u_\star)\|_{L^{4/3}(\Omega)} \le C$. This implies by L^p theory for the Hodge decomposition that $\|\nabla h_1(t)\|_{L^{4/3}(\Omega)} \le C$ and so also $\|\nabla h_1\|_{L^{4/3}(\Omega\times(0,\delta))} \le C$. We obtain that the sequence of ∇h_1 (depending on ε) is weakly compact in $L^{4/3}(\Omega\times(0,\delta))$. Using Proposition 3.6, the weak limit must be 0. Hence

$$\left|\int_0^\delta\int_\Omega \zeta(t)\cdot\nabla h_1(t)\right| \to 0 \text{ as } \varepsilon\to 0. \tag{3.28}$$

For the remaining term, we use the Γ-convergence statement and the convergence (2.16) so $\operatorname{curl} j_A(u) \to \operatorname{curl} j_{A_\star}(u_\star)$ in $W^{-1,p}$, which implies for every t

$$\int_\Omega \zeta(t)\cdot\nabla^\perp h_2(t) \to 0, \tag{3.29}$$

and now (3.18) follows. □

Proposition 3.6. *The divergence* $\operatorname{div} j_A(u)$ *tends to zero in the dual space of* $W_0^{1,4}(\Omega\times(0,\delta))$. *More precisely, for any* $\chi \in W_0^{1,4}(\Omega\times(0,\delta))$, *there holds*

$$\left|\int_0^\delta\int_\Omega \nabla\chi\cdot j_A(u)\right| \le C\,\|\chi\|_{W^{1,4}(\Omega\times(0,\delta))}\left(\sqrt{\alpha_\varepsilon} + \varepsilon\sqrt{G_\varepsilon}\right) \tag{3.30}$$

Proof. We use the conservation of mass equation (A.23) and estimate for $\phi \in W_0^{1,4}(\Omega \times (0,\delta))$ by Hölder's inequality

$$
\begin{aligned}
\int_0^\delta \int_\Omega \phi \operatorname{div} j_A(u) &= \alpha_\varepsilon \int_0^\delta \int_\Omega (iu, \partial_\Phi u) + \frac{\beta_\varepsilon}{2} \int_0^\delta \int_\Omega \phi \partial_t (|u|^2 - 1) \\
&\le \alpha_\varepsilon \left(\int_0^\delta \int_\Omega |\phi|^4 \right)^{\frac14} \left(\int_0^\delta \int_\Omega |u|^4 \right)^{\frac14} \left(\int_0^\delta \int_\Omega |\partial_\Phi u|^2 \right)^{1/2} \\
&\qquad - \frac{\beta_\varepsilon}{2} \int_0^\delta \int_\Omega \partial_t \phi (|u|^2 - 1) \\
&\le C\sqrt{\alpha_\varepsilon} \|\phi\|_{L^4(\Omega\times(0,\delta))} + C\varepsilon\beta_\varepsilon \sqrt{G_\varepsilon(u,A)} \|\partial_t \phi\|_{L^4(\Omega\times(0,\delta))} \\
&\qquad \le C\|\phi\|_{W^{1,4}(\Omega\times(0,\delta))} \left(\sqrt{\alpha_\varepsilon} + \varepsilon \sqrt{G_\varepsilon} \right),
\end{aligned}
$$

which tends to 0 as $\varepsilon \to 0$. □

Proposition 3.7. *Let* (u, A) *be a solution of* (1.29) *and* (1.30) *and let* $\phi, \psi \in W_0^{2,\infty}(\Omega)$. *With* $R = (\nabla_A)^2 u + \frac{1}{\varepsilon^2} u(1 - |u|^2)$, *the following evolution equation holds:*

$$
\begin{aligned}
\partial_t \int_\Omega (\alpha\psi g_\varepsilon + \beta\phi J_A) &= -\alpha^2 \int_\Omega \psi |\partial_\Phi u|^2 - \alpha\sigma \int_\Omega \psi |E|^2 \\
&+ \frac{\beta^2}{\alpha^2+\beta^2} \int_\Omega \left(\nabla\nabla^\perp \phi : (\nabla_A u \otimes \nabla_A u) - h \nabla\phi \cdot \sigma E \right) \\
&- \frac{\alpha^2}{\alpha^2+\beta^2} \int_\Omega \nabla\psi \cdot (R, \nabla_A u) \\
&+ \frac{\alpha\beta}{\alpha^2+\beta^2} \int_\Omega (\nabla\psi - \nabla^\perp\phi) \cdot (iR, \nabla_A u) \\
&+ \alpha \int_\Omega \nabla^\perp \psi \cdot E(h - h_{ex}) + \frac{\beta}{2} \int_\Omega (1 - |u|^2) \nabla^\perp \phi \cdot E.
\end{aligned}
\tag{3.31}
$$

Proof. This calculation can be found in the appendix. □

Proposition 3.8. *There exists a time* $T_2 > 0$ *such that the conclusions of Theorem 1.1 hold on* $[0, T_2)$.

Proof. Choose $T_2 < \min(\delta, T_0, T_1)$, where δ is as in Proposition 3.2 and T_1 as in Proposition 2.3. Now we use (3.31) with $\psi = \psi_k^{(j)}$, $\phi = \phi_k^{(j)}$, where

$$\psi_k^{(j)} = x_k \chi_r (x - a_j^0) \tag{3.32}$$

$$\phi_k^{(j)} = (x^\perp)_k \chi_r (x - a_j^0) \tag{3.33}$$

for $k = 1, 2$, where $\chi_r(x) = \chi(\frac{x}{r})$ for a fixed function $\chi \in C_0^\infty(\mathbb{R}^2)$ with $\chi \equiv 1$ in $B_1(0)$ and $\chi \equiv 0$ outside $B_2(0)$. Note that $\nabla\psi_k^{(j)} = \nabla^\perp \phi_k^{(j)}$ in $B_r(a_j^0)$, $k = 1, 2$, and that (dropping the sub- and superscripts) $|\nabla\psi| + |\nabla\phi| \le \frac{C}{r}$.

From Lemma 5.1 we obtain with $u_\star$ and $A_\star$ defined using $\xi(t)$,

$$\frac{\beta^2}{\alpha^2+\beta^2}\int_\Omega \nabla\nabla^\perp\phi : (j_{A_\star}(u_\star)\otimes j_{A_\star}(u_\star)) = \left(1-\frac{\alpha^2}{\alpha^2+\beta^2}\right)\nabla^\perp\phi\cdot\nabla_{a_i}W_\Omega(\xi,d;h_{ex}) \tag{3.34}$$

We calculate using $\nabla\psi = \nabla^\perp\phi$ and (3.31),

$$\begin{aligned}\Big[\Big(\partial_t\int_\Omega(\alpha\psi g_\varepsilon+\beta\phi J_A)\Big)&+\nabla\psi\cdot\nabla_{a_i}W_\Omega(\xi,d;h_{ex})\Big]\\ &=\frac{\alpha^2}{\alpha^2+\beta^2}\nabla\psi\cdot\nabla_{a_i}W_\Omega(\xi,d;h_{ex})\\ &+\frac{\beta^2}{\alpha^2+\beta^2}\int_\Omega\Big(\nabla\nabla^\perp\phi:(\nabla_A u\otimes\nabla_A u-j_{A_\star}(u_\star)\otimes j_{A_\star}(u_\star))\Big)\\ &-\frac{\beta^2\sigma}{\alpha^2+\beta^2}\int_\Omega h\nabla\phi\cdot E-\alpha^2\int_\Omega\psi|\partial_\Phi u|^2-\alpha\sigma\int_\Omega\psi|E|^2\\ &-\frac{\alpha^2}{\alpha^2+\beta^2}\int_\Omega\nabla\psi\cdot(R,\nabla_A u)+\frac{\alpha\beta}{\alpha^2+\beta^2}\int_\Omega(\nabla\psi-\nabla^\perp\phi)\cdot(iR,\nabla_A u)\\ &+\alpha\int_\Omega\nabla^\perp\psi\cdot E(h-h_{ex})+\frac{\beta}{2}\int_\Omega(1-|u|^2)\nabla^\perp\phi\cdot E.\end{aligned} \tag{3.35}$$

We take absolute values and integrate in time to get

$$\begin{aligned}\int_0^\delta\Big|\Big(\partial_t\int_\Omega(\alpha\psi g_\varepsilon+\beta\phi J_A)\Big)&+\nabla\psi\cdot\nabla_{a_i}W_\Omega(\xi(t),d;h_{ex})\Big|\\ &\le \mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathrm{V}+\mathrm{VI}+\mathrm{VII}+\mathrm{VIII}+\mathrm{IX},\end{aligned} \tag{3.36}$$

where

$$
\begin{aligned}
\mathrm{I} &= \frac{\alpha^2}{\alpha^2+\beta^2}\int_0^\delta |\nabla\psi\cdot\nabla_{a_i}W_\Omega(\xi(t),d;h_{ex})| \\
\mathrm{II} &= \frac{\beta^2}{\alpha^2+\beta^2}\int_0^\delta \left|\int_\Omega \Big(\nabla\nabla^\perp\phi : (\nabla_A u\otimes\nabla_A u - j_{A_\star}(u_\star)\otimes j_{A_\star}(u_\star))\Big)\right| \\
\mathrm{III} &= \frac{\beta^2\sigma}{\alpha^2+\beta^2}\int_0^\delta \left|\int_\Omega h\nabla\phi\cdot E\right| \\
\mathrm{IV} &= \alpha^2\int_0^\delta \left|\int_\Omega \psi|\partial_\Phi u|^2\right| \\
\mathrm{V} &= \alpha\sigma\int_0^\delta \left|\int_\Omega \psi|E|^2\right| \\
\mathrm{VI} &= \frac{\alpha^2}{\alpha^2+\beta^2}\int_0^\delta \left|\int_\Omega \nabla\psi\cdot(R,\nabla_A u)\right| \\
\mathrm{VII} &= \frac{\alpha\beta}{\alpha^2+\beta^2}\int_0^\delta \left|\int_\Omega (\nabla\psi-\nabla^\perp\phi)\cdot(iR,\nabla_A u)\right| \\
\mathrm{VIII} &= \alpha\int_0^\delta \left|\int_\Omega \nabla^\perp\psi\cdot E(h-h_{ex})\right| \\
\mathrm{IX} &= \frac{\beta}{2}\int_0^\delta \left|\int_\Omega (1-|u|^2)\nabla^\perp\phi\cdot E\right|.
\end{aligned}
$$

As $\alpha=\alpha_\varepsilon\to 0$, it follows easily that $\mathrm{I}\to 0$ as $\varepsilon\to 0$. From Proposition 3.5, we obtain the bound $\mathrm{II}\le C\int_0^\delta D_\varepsilon(t)+o_\varepsilon(1)$. From (2.16), $h=h_\varepsilon$ is uniformly bounded in $L^2(\Omega)$ for all t; we can therefore use (2.6) (recall $\alpha_\varepsilon|\log\varepsilon|\to\alpha_0$) and the Cauchy-Schwarz inequality to bound $\mathrm{III}\le C\sqrt{\sigma_\varepsilon}\to 0$. It also follows immediately from (2.6) that $\mathrm{IV}+\mathrm{V}\le C\alpha\to 0$. It is the result of Proposition 3.4 that $\mathrm{VI}\to 0$ and of Proposition 3.3 that $\mathrm{VII}\to 0$ as long as D_ε stays bounded. The argument for $\mathrm{VIII}\to 0$ is essentially the same as for III. Finally, the bound $(1-|u|^2)^2\le\varepsilon^2 G_\varepsilon(u_\varepsilon,A_\varepsilon;h_{ex})\le K\varepsilon^2|\log\varepsilon|$ and Hölder's inequality show together with (2.6) that $\mathrm{IX}\to 0$. In particular, we see that the right-hand side of (3.36) is bounded by $\int_0^\delta D_\varepsilon(t)+o_\varepsilon(1)$ as $\varepsilon\to 0$.

From Propositions 2.3 and 2.4, we obtain the weak convergence

$$
\partial_t\int_\Omega \alpha\psi g_\varepsilon+\beta\phi J_A \rightharpoonup \pi\partial_t\left(\alpha_0\psi(\xi_j+\beta_0 d_j\phi(\xi_j)\right). \tag{3.37}
$$

Recall that $\psi=\psi_k^{(j)}$, $\phi=\phi_k^{(j)}$ so $\nabla\psi_k^{(j)}(\xi_j)=\nabla^\perp\phi_k^{(j)}(\xi_j)=\delta_{ik}e_i$. for $k=1,2$, the right-hand side of (3.37) just consists of the two components of the vector $(\alpha_0\dot\xi_j+\beta_0 d_j\dot\xi_j^\perp)$. It follows that $\tilde{\mathscr{R}}_j$ as defined in (3.8) satisfies

(using Fatou's lemma)

(3.38)
$$\int_0^\delta |\tilde{\mathscr{R}}_j| \leq \liminf_{\varepsilon\to 0} \int_0^\delta \left| \left(\partial_t \int_\Omega (\alpha\psi g_\varepsilon + \beta\phi J_A) \right) + \nabla\psi \cdot \nabla_{a_i} W_\Omega(\xi(t), d; h_{ex}) \right|$$
$$\leq C \liminf_{\varepsilon\to 0} \int_0^\delta D_\varepsilon(t).$$

From (3.3) and (3.9), we can now estimate for every $t \leq \delta$

$$\limsup_{\varepsilon\to 0} D_\varepsilon(t) + |\eta|(t) \leq C \int_0^t \liminf_{\varepsilon\to 0} D_\varepsilon(s) + C|\eta(s)|ds. \tag{3.39}$$

It now follows from Gronwall's inequality that $D_\varepsilon(t) \to 0$ for $0 \leq t \leq \delta$ and $\eta(t) \equiv 0$, so $\xi(t) = a(t)$. □

Proof of Theorem 1.1. It follows from Proposition 3.8 that the statement of the theorem is valid on some interval $[0, T_2)$. We let $\overline{T}$ denote the maximal time such that the theorem is valid on $[0, \overline{T})$. If $\overline{T} < T_M$, the maximum time of existence of (1.36), then for every $t < \overline{T}$, $D_\varepsilon(t) \to 0$ so (A.24) implies

$$\int_0^t \int_\Omega \alpha_\varepsilon |\partial_\Phi u_\varepsilon|^2 + \sigma_\varepsilon |E_\varepsilon|^2 \to -W(a(t), d; h_{ex}) + W(a(0), d; h_{ex}) \tag{3.40}$$

as $\varepsilon \to 0$. Since $\overline{T} < T_M$, the right-hand side is bounded as $t \to \overline{T}$. It follows that a kinetic energy bound similar to (2.6) holds on $[0, \overline{T}]$ and there exist vortex paths $\xi \in H^1(0, \overline{T}; \mathbb{R}^{2N})$. As $\xi(t) = a(t)$ on $[0, \overline{T})$, it follows by continuity that $\xi(\overline{T}) = a(\overline{T})$.

By the energy decrease, for every $t < \overline{T}$,

$$G_\varepsilon(u_\varepsilon, A_\varepsilon)(\overline{T}) \leq n(\pi|\log\varepsilon| + \gamma_0) + W(a(t), d; h_{ex}) + o(1)$$

so

$$\limsup D_\varepsilon(\overline{T}) \leq W(a(t), d; h_{ex}) - W(a(\overline{T}), d; h_{ex}),$$

and now letting $t \to \overline{T}$ shows $D_\varepsilon(\overline{T}) \to 0$. It follows that Proposition 3.8 can be applied starting at $\overline{T}$, contradicting the maximality of $\overline{T}$. □

4. The Γ-stability result

This section proves the Γ-stability theorem discussed in the introduction and used extensively in the previous arguments. In this and the following section, ε is a **fixed** positive number, and our results include quantitative rates of convergence.

To simplify notation we define

$$e_A(u) = e_\varepsilon(|u|) + \frac{1}{2} \left| \frac{j_A(u)}{|u|} \right|^2 \tag{4.1}$$

so that

$$g_\varepsilon(u, A; h_{ex}) = e_A(u) + \frac{1}{2} \left| \operatorname{curl} A - h_{ex} \right|^2.$$

The main result of this section is

Theorem 4.1. *Let Ω be a bounded, open simply connected subset of $\mathbb{R}^2$ with C^1 boundary. Then there exists absolute constants C and K_1 such that for any $A \in H^1(\Omega;\mathbb{R}^2)$, if there exists a $p > 2$ such that*

$$\|A - A_\star\|_{L^p(\Omega)} \le t_\varepsilon \tag{4.2}$$

and for any $u \in H^1(\Omega;\mathbb{C})$, if there exist $n \ge 0$, $\xi = (\xi_1, \dots, \xi_n) \in \Omega^{n*}$ and $d \in \{\pm 1\}^n$

$$\left\| J(u) - \sum_{j=1}^{n} \pi d_j \delta_{\xi_j} \right\|_{\dot{W}^{-1,1}(\Omega)} \le s_\varepsilon \tag{4.3}$$

for some $s_\varepsilon \in [\varepsilon\sqrt{\ln(\rho_\xi/\varepsilon)}, \frac{4}{K_1}\left(n\rho_\xi^{-1} + h_{ex}\right)^{-1}]$, and if

$$4 s_\varepsilon \le \sigma^* \le \frac{1}{K_1}\left(n\rho_\xi^{-1} + h_{ex}\right)^{-1},$$

where

$$\sigma^* = 4\left(\frac{s_\varepsilon}{n^3 G_\varepsilon + n^3 h_{ex}^2 + n^5 \rho_a^{-2}}\right)^{\frac{1}{\gamma+1}} \tag{4.4}$$

for any fixed $\gamma < 1$, then

$$\begin{aligned} \int_{\Omega_{\sigma^*}(\xi)} e_\varepsilon(|u|) + \frac{1}{4}\left|\frac{j_A(u)}{|u|} - j_{A_\star}(u_\star)\right|^2 + \frac{1}{2}\int_\Omega |\operatorname{curl} A - \operatorname{curl} A_\star|^2\,dx \\ \le D_\Omega^\varepsilon(u, A; \xi, h_{ex}) + \mathbf{error1} \end{aligned} \tag{4.5}$$

for $u_\star = u_\star(\cdot;\xi)$ as defined in (5.1), where

$$\begin{aligned} \mathbf{error1} \le{}& C s_\varepsilon^{\frac{\gamma}{\gamma+1}}\left(n^4 G_\varepsilon + n^4 h_{ex}^2 + n^6 \rho_a^{-2}\right)^{\frac{1}{\gamma+1}} \\ &+ C h_{ex}\left(s_\varepsilon + \varepsilon G_\varepsilon\right) + C n \varepsilon^\gamma \left(G_\varepsilon^3 + h_{ex}^4\right) + t_\varepsilon\left(n\rho_a^{-1} + h_{ex}\right) \end{aligned} \tag{4.6}$$

and where C depends on γ. Finally,

$$\|j_A(u) - j_{A_\star}(u_\star)\|_{L^{4/3}(\Omega)} \le C\sqrt{D_\Omega^\varepsilon(u, A; \xi, h_{ex})} + \mathbf{error2} \tag{4.7}$$

with

$$\begin{aligned} \mathbf{error2} \le{}& \sqrt{\mathbf{error1}} + C\varepsilon^{1/2} G_\varepsilon^{3/4} \\ &+ C n^{1/4}\sqrt{\sigma^*}\left(G_\varepsilon^{1/2} + n^{3/2} + n^{1/2}\sigma^* h_{ex}\right). \end{aligned}$$

Remark 4.2. Inequality (4.5) comes from studying, more or less, the integral $\int_{\Omega_{\sigma^*}(\xi)} g_\varepsilon(u/u_\star, A - A_\star; 0)$. We note that it would be possible to establish estimates of $\|j_A(u) - j_{A_\star}(u_\star)\|_{L^p(\Omega)}$ for $1 \le p < 2$, in the spirit of (4.7).

Remark 4.3. If $h_{ex} \leq C|\log \varepsilon|$, $\rho_\xi \geq \frac{C}{|\log \varepsilon|}$, and $n \leq C|\log \varepsilon|$ then (4.4) holds and the errors are algebraically small in ε so long as $s_\varepsilon = \varepsilon^\gamma$ and $t_\varepsilon = \varepsilon^\delta$ for some $\gamma, \delta \in (0,1)$. Such bounds on s_ε can be inferred from the localization result, Theorem 3 in [13]. The corresponding bound on t_ε would need to come from other considerations on A and $A_\star$.

Remark 4.4. Although the statement of Theorem 4.1 requires control on $J(u)$ instead of the gauge invariant $J_A(u)$, Lemma 4.5 below shows that these quantities are in fact close to each other. The results above can be adjusted by replacing s_ε with $s_\varepsilon + C\varepsilon\left(G_\varepsilon + h_{ex}^2\right)$.

The proof of Theorem 4.1 rests on the identity

$$\begin{aligned}(4.8)\qquad &\int_{\Omega_\sigma(\xi)} e_\varepsilon(|u|) + \frac{1}{2}\left|\frac{j_A(u)}{|u|} - j_A(u_\star(\xi))\right|^2 + \frac{1}{2}\int_\Omega |\operatorname{curl} A - \operatorname{curl} A_\star|^2 \\ &\quad = G_\varepsilon(u, A; h_{ex}) - W_\Omega^\varepsilon(\xi, d; h_{ex}) + \text{lower order terms.}\end{aligned}$$

The above follows from using the fact that $|\nabla_{A_\star} u_\star| = |j_{A_\star}(u_\star)|$. The main hypotheses of the theorem are that both $\|J_A(u) - \pi \sum d_i \delta_{\xi_i}\|_{\dot{W}^{-1,1}}$ and $\|A - A_\star\|_{L^p}$ are small for some $p > 2$, and under these hypotheses we wish to bound the left-hand side of (4.8) by the surplus energy $D_\Omega^\varepsilon(u, A; \xi, h_{ex}) \approx \int_\Omega [g_\varepsilon(u, A) - g_\varepsilon(u_\star^\varepsilon(\xi), A)]$. We rewrite the first term on the right-hand side as a sum of the surplus energy and contributions from balls $B_\sigma(\xi_i)$:

$$\begin{aligned}&\int_{\Omega_\sigma(\xi)} [e_A(u) - e_{A_\star}(u_\star(\xi))]\ dx + \frac{1}{2}\int_\Omega |h - h_{ex}|^2 - |h_\star - h_{ex}|^2 \\ &\qquad \approx D_\Omega^\varepsilon(u, A; \xi, h_{ex}) + \sum_i \left[I(\sigma, \varepsilon) - \int_{B_\sigma(\xi_i)} e_A(u)\ dx\right].\end{aligned}$$

The integrals over the balls $B_\sigma(\xi_i)$ are shown to be small using results from [12], which require the hypothesis on $\|J(u) - \pi \sum d_i \delta_{\xi_i}\|_{\dot{W}^{-1,1}}$.

We include the next lemma in order to relate the gauged Jacobian J_A to the classical Jacobian J:

Lemma 4.5. Suppose both A and $A_\star$ satisfy the Coulomb gauge and $u_\star \in \mathbb{S}^1$ then

$$\begin{aligned}(4.9)\qquad &\left|\|J_A(u) - J(u_*)\|_{\dot{W}^{-1,1}(\Omega)} - \|J(u) - J(u_*)\|_{\dot{W}^{-1,1}(\Omega)}\right| \\ &\qquad \leq C\varepsilon\left(G_\varepsilon + h_{ex}^2\right)\end{aligned}$$

Proof. Note that $J_A(u) - J(u_*) = J(u) - \frac{1}{2}\operatorname{curl}\left(|u|^2 A\right)\right) + \frac{1}{2}\operatorname{curl} A - J(u_*)$ then for any $\phi \in W_0^{1,\infty}(\Omega)$ with $\|\nabla\phi\|_{L^\infty} \leq 1$:

$$\begin{aligned}\int \phi\left(J(u) - J(u_*)\right) &= \int \phi\left(J_A(u) - J(u_*)\right) + \frac{1}{2}\int \phi \operatorname{curl}\left(A\left(|u|^2 - 1\right)\right) \\ &\leq \int \phi\left(J_A(u) - J(u_*)\right) + \frac{1}{2}\int \nabla\phi \times \left(A\left(|u|^2 - 1\right)\right) \\ &\leq \int \phi\left(J_A(u) - J(u_*)\right) + \frac{1}{2}\|\nabla\phi\|_{L^\infty}\|A\|_{L^2}\left\||u|^2 - 1\right\|_{L^2} \\ &\leq \|J_A(u) - J(u_*)\|_{\dot{W}^{-1,1}(\Omega)} + \frac{\varepsilon}{2}\|A\|_{L^2} G_\varepsilon^{1/2}.\end{aligned}$$

Taking the sup over such ϕ yields a $\dot{W}^{-1,1}$ estimate. By (4.13) and Sobolev embedding we have $\|A\|_{L^2} \leq \|A\|_{H^1} \leq C\left(G_\varepsilon + h_{ex}^2\right)^{1/2}$. Therefore, the last term is bounded by $C\varepsilon\left(G_\varepsilon + h_{ex}^2\right)$. Plugging into the inequality above yields one side of (4.9). The other side follows from a similar calculation. □

In order to simplify the functionals and extract the essential terms, we examine $e_{A-A_\star}(u/u_\star)$:

$$\begin{aligned}e_{A-A_\star}(u/u_\star) &= e_\varepsilon(|u|) + \frac{1}{2}\left|\frac{j_A(u)}{|u|} - |u| j_{A_\star}(u_\star)\right|^2 \\ &= e_\varepsilon(|u|) + \frac{1}{2}\left|\frac{j_A(u)}{|u|}\right|^2 - \frac{1}{2}\left|j_{A_\star}(u_\star)\right|^2 \\ &\quad + \left|j_{A_\star}(u_\star)\right|^2 - j_A(u)\cdot j_{A_\star}(u_\star) + \frac{1}{2}\left(|u|^2 - 1\right)\left|j_A(u_\star))\right|^2 \\ &= e_A(u) - e_{A_\star}(u_\star) \\ &\quad + \left|j_{A_\star}(u_\star)\right|^2 - j_A(u)\cdot j_{A_\star}(u_\star) + \frac{1}{2}\left(|u|^2 - 1\right)\left|j_A(u_\star))\right|^2.\end{aligned}$$

On the other hand

$$\begin{aligned}&\frac{1}{2}\left|\frac{j_A(u)}{|u|} - |u| j_{A_\star}(u_\star))\right|^2 \\ &= \frac{1}{2}\left|\frac{j_A(u)}{|u|}\right|^2 - \frac{j_A(u)}{|u|}\cdot j_{A_\star}(u_\star) + \frac{1}{2}\left|j_{A_\star}(u_\star)\right|^2 \\ &\quad + \frac{j_A(u)}{|u|}\cdot j_{A_\star}(u_\star)\left(|u| - 1\right) + \frac{1}{2}\left|j_A(u_\star)\right|^2\left(|u|^2 - 1\right) \\ &= \frac{1}{2}\left|\frac{j_A(u)}{|u|} - j_{A_\star}(u_\star)\right|^2 + \frac{j_A(u)}{|u|}\cdot j_{A_\star}(u_\star)\left(|u| - 1\right) + \frac{1}{2}\left|j_A(u_\star)\right|^2\left(|u|^2 - 1\right).\end{aligned}$$

As a result,

$$
\begin{aligned}
e_\varepsilon(|u|)+&\frac{1}{2}\left|\frac{j_A(u)}{|u|}-j_A(u_\star)\right|^2 \\
&= e_A(u)-e_{A_\star}(u_\star)+\frac{j_A(u)}{|u|}\cdot j_{A_\star}(u_\star)(|u|-1) \\
&\quad + |j_{A_\star}(u_\star)|^2 - j_A(u)\cdot j_{A_\star}(u_\star).
\end{aligned}
\tag{4.10}
$$

If we set $j_{A_\star}(u_\star) = j(u_\star) - A_\star = -\nabla\times(G-\Xi_\star)$ then we are led to the definition:

$$\Upsilon = G - \Xi_\star$$

where G and $\Xi_\star$ are defined in Section 5. (In the usual terminology, one uses $\xi_\star$ for the Δ^{-1} of a solution to the London equation instead of $\Xi_\star$; however we use $\xi_\star$ for the limit of the ξ_ε in the dynamic part of this article). The second term on the right-hand side of (4.8) is approximately (suppressing the dependence on ξ)

$$
\begin{aligned}
\int_{\Omega_\sigma} j_{A_\star}(u_\star)\cdot(j_{A_\star}(u_\star)-j_A(u)) &= \int_{\Omega_\sigma}(\nabla\times\Upsilon)\cdot(j_A(u_\star)-j_A(u)) \\
&\approx \int_\Omega \tilde\Upsilon\ \nabla\times(j_{A_\star}(u_\star)-j_A(u)),
\end{aligned}
$$

Here Υ is defined in (5.2), and $\tilde\Upsilon$ is a modification of Υ obtained by modifying Ω_σ slightly (so that Υ is constant on each component of $\partial\Omega_\sigma$) and then setting $\tilde\Upsilon$ equal to the suitable constant on each component of $\Omega\setminus\Omega_\sigma$. The right-hand side is then controlled using the assumed bounds on $\|J(u)-\pi\sum d_i\delta_{\xi_i}\|_{\dot W^{-1,1}} = \frac{1}{2}\|\nabla\times(j(u)-j(u_\star))\|_{\dot W^{-1,1}}$; note that this hypothesis turns out to be extremely natural at this point.

Next note that

$$
\begin{aligned}
\frac{1}{2}&\left|\operatorname{curl}A-\operatorname{curl}A_\star\right|^2 \\
&= \frac{1}{2}\left[|\operatorname{curl}A-h_{ex}|^2-|\operatorname{curl}A_\star-h_{ex}|^2\right] \\
&\quad + |\operatorname{curl}A_\star-h_{ex}|^2-(\operatorname{curl}A-h_{ex})(\operatorname{curl}A_\star-h_{ex}) \\
&= \frac{1}{2}\left[|\operatorname{curl}A-h_{ex}|^2-|\operatorname{curl}A_\star-h_{ex}|^2\right] \\
&\quad + (\operatorname{curl}A_\star-h_{ex})(\operatorname{curl}A_\star-\operatorname{curl}A)
\end{aligned}
\tag{4.11}
$$

We will use (4.10) and (4.11) to prove the Gamma stability result.

The proof of the theorem uses the following lemma from [12]. Note that it is essentially the $n=1$ case of Theorem 4.1 on a ball with a single vortex at the center, except that the positive terms on the left-hand side of (4.5) are missing, as well as the gauge field.

Lemma 4.6. There exists an absolute constant $C > 0$ such that if $u \in H^1(B_\sigma; \mathbb{C})$ satisfies
$$\|J(u) \pm \pi\delta_0\|_{\dot W^{-1,1}(B_\sigma)} \le \frac{\sigma}{4},$$
then
$$\begin{aligned} 0 \le \int_{B_\sigma} & e_\varepsilon(u) - I(\sigma,\varepsilon) \\ & + \ C\frac{\varepsilon}{\sigma}\sqrt{\ln\frac{\sigma}{\varepsilon}} + \frac{C}{\sigma}\|J(u) \pm \pi\delta_0\|_{\dot W^{-1,1}(B_\sigma)}. \end{aligned} \tag{4.12}$$

This is Theorem 5 in [12]; the statement there appears slightly different, but the two versions are easily seen to be equivalent using Lemma 5.3.

We start with some embedding estimates that will be used throughout.

Lemma 4.7. Given $G_\varepsilon = G_\varepsilon(u, A; h_{ex})$ defined above and A satisfying the Coulomb gauge, we have
$$\|A\|_{H^1(\Omega)} \le C\left(G_\varepsilon + h_{ex}^2\right)^{1/2} \tag{4.13}$$
$$\left\|1 - |u|^2\right\|_{L^p(\Omega)} \le C\varepsilon^\alpha\, G_\varepsilon^{3/2} \tag{4.14}$$
for any $\alpha < \frac{2}{p}$ with $2 \le p < +\infty$. Here C depends on α and p.

Proof. By the Coulomb gauge there exists a potential Ξ such that $A = \nabla^\perp\Xi$ and $\Xi = 0$ on $\partial\Omega$. Therefore, $-\Delta\Xi = \operatorname{curl} A = \operatorname{curl} A - h_{ex} + h_{ex}$ and $\|\Delta\Xi\|_{L^2} \le \|\operatorname{curl} A - h_{ex}\|_{L^2} + h_{ex}|\Omega|^{1/2} \le G_\varepsilon^{1/2} + h_{ex}|\Omega|^{1/2} \le C\left(G_\varepsilon + h_{ex}^2\right)^{1/2}$. Therefore, $\|\Xi\|_{H_0^2(\Omega)} \le C\left(G_\varepsilon + h_{ex}^2\right)^{1/2}$. This implies (4.13).

Next, since $\|1 - |u|\|_{L^2} \le \left\|1 - |u|^2\right\|_{L^2} \le C\varepsilon G_\varepsilon^{1/2}$ and $\||u|\|_{H^1} \le CG_\varepsilon^{1/2}$ then by interpolation $\|1 - |u|\|_{L^p} \le C\varepsilon^\gamma G_\varepsilon$ for any $\frac{2}{p} > \gamma$, where C depends on p and γ. This implies that $\left\|1 - |u|^2\right\|_{L^p} \le \|1 - |u|\|_{L^r}\|1 + |u|\|_{L^s} \le C\varepsilon^\gamma G_\varepsilon^{3/2}$ by Hölder and Sobolev embedding, where again $\frac{2}{p} > \gamma$. □

For the time being we assume one additional lemma, the proof of which is given below, and we use it to complete the

proof of Theorem 4.1. **Step 1**. We first rewrite $W_\Omega^\varepsilon(\xi, d; h_{ex})$ using facts about the renormalized energy that are collected in Section 5. Recall from (5.10) that W_Ω^ε is defined by $W_\Omega^\varepsilon(\xi, d; h_{ex}) = W_\Omega(\xi, d; h_{ex}) + n(\gamma_0 + \pi\ln\frac{1}{\varepsilon})$, where γ_0 is defined in (5.9). Hence by Lemma 5.6,
$$\begin{aligned} W_\Omega^\varepsilon(\xi, d; h_{ex}) = \int_{\Omega_\sigma} e_{A_\star}(u_\star)\,dx + \frac{1}{2}\int_\Omega |h_\star - h_{ex}|^2\,dx + n(\gamma + \pi\ln\frac{\sigma}{\varepsilon}) \\ + O((\frac{n\sigma}{\rho_\xi})^2) + O\left(n\sigma\left(n\rho_\xi^{-1} + h_{ex}\right)^2\right) \end{aligned}$$

for any $0 \le \sigma \le \rho_\xi$. Thus, recalling the formula (5.12) for the excess energy on a ball,

$$
\begin{aligned}
D^\varepsilon_\Omega(u, A; \xi, h_{ex}) &= \int_\Omega g_\varepsilon(u, A; h_{ex}) - W^\varepsilon_\Omega(\xi; h_{ex}) \\
&= \int_{\Omega_\sigma(\xi)} [e_A(u) - e_{A_\star}(u_\star)]dx \\
&\quad + \frac{1}{2}\int_\Omega |\operatorname{curl} A - h_{ex}|^2 - |\operatorname{curl} A_\star - h_{ex}|^2 \\
&\quad + \sum_{i=1}^n \left[\int_{B_\sigma(\xi_i)} e_A(u)\ dx - I(\sigma,\varepsilon)\right] \\
&\quad + O((\frac{n\sigma}{\rho_\xi})^2) + O\left(n\sigma\left(n\rho_\xi^{-1} + h_{ex}\right)^2\right).
\end{aligned} \tag{4.15}
$$

Step 2. We claim that the assumptions imply

$$
\begin{aligned}
&\sum_j \left[\int_{B_\sigma(\xi_j)} e_A(u)\ dx - I(\sigma,\varepsilon)\right] \\
&\qquad \ge -C\frac{n}{\sigma}s_\varepsilon - Cn\sigma^\gamma\left(G_\varepsilon + h_{ex}^2\right) - Cn\varepsilon^\gamma\left(G_\varepsilon^3 + h_{ex}^4\right).
\end{aligned} \tag{4.16}
$$

The contributions from $B_\sigma(\xi_i)$, $i = 1, \dots, n$ are estimated using Lemma 4.6. However, since we are estimating $e_A(u)$ instead of $e_\varepsilon(u)$, as in [13], we need to tear apart the functional. Without loss of generality assume the center of the ball is the origin then $\int_{B_\sigma} e_A(u)\ dx - I(\sigma,\varepsilon) = \int_{B_\sigma} e_\varepsilon(u)\ dx - I(\sigma,\varepsilon) - \int_{B_\sigma} j_A(u)\cdot A + \frac{1}{2}A^2|u|^2\,dx$. We estimate the last integral; for any $\gamma < 1$ we have

$$
\begin{aligned}
&\left\|j_A(u)\cdot A + \frac{1}{2}A^2|u|^2\right\|_{L^1(B_\sigma)} \\
&\quad \le \left\|\frac{j_A(u)}{|u|}\cdot A\right\|_{L^1(B_\sigma)} + \frac{1}{2}\left\|A^2\right\|_{L^1(B_\sigma)} \\
&\qquad + \left\|\frac{j_A(u)}{|u|}\cdot A\,(|u|-1)\right\|_{L^1(B_\sigma)} + \frac{1}{2}\left\|A^2(|u|^2-1)\right\|_{L^1(B_\sigma)} \\
&\quad \le C\left\|\nabla_A u\right\|_{L^2}\left\|A\right\|_{L^p}\sigma^{1-\frac{2}{p}} + C\left\|A\right\|_{L^4}^2\sigma \\
&\qquad + \left\|\nabla_A u\right\|_{L^2}\left\|A\right\|_{L^p}\left\|1-|u|\right\|_{L^{\frac{2p}{p-2}}} + \left\|A\right\|_{L^4}^2\left\|1-|u|^2\right\|_{L^2} \\
&\quad \le C\left(G_\varepsilon + h_{ex}^2\right)\sigma^\gamma + C\left(G_\varepsilon + h_{ex}^2\right)\sigma \\
&\qquad + C\left(G_\varepsilon^3 + h_{ex}^2\right)\varepsilon^\gamma + C\left(G_\varepsilon + h_{ex}^2\right)\sigma + C\varepsilon\left(G_\varepsilon^{3/2} + h_{ex}^4\right) \\
&\quad \le C\left(G_\varepsilon^3 + h_{ex}^4\right)\varepsilon^\gamma + C\left(G_\varepsilon + h_{ex}^2\right)\sigma^\gamma,
\end{aligned}
$$

where C depends on γ. Hence,

$$\begin{aligned}&\sum_j \left[\int_{B_\sigma(\xi_j)} e_A(u)\ dx - I(\sigma,\varepsilon)\right]\\ &\quad\geq \sum_j \left[\int_{B_\sigma(\xi_j)} e_\varepsilon(u)\ dx - I(\sigma,\varepsilon)\right]\\ &\quad - Cn\left(G_\varepsilon^3 + h_{ex}^4\right)\varepsilon^\gamma - Cn\left(G_\varepsilon + h_{ex}^2\right)\sigma^\gamma.\end{aligned} \tag{4.17}$$

Note that the definitions of the norms imply that $\|J(u)-\pi\delta_{\xi_i}\|_{\dot W^{-1,1}(U_\sigma(\xi_i))} \leq s_\varepsilon$, so the hypotheses of this lemma are satisfied whenever $4s_\varepsilon \leq \sigma$. We may thus apply that lemma to deduce that

$$\int_{B_\sigma(\xi_j)} e_\varepsilon(u)\ dx - I(\sigma,\varepsilon) \geq -\frac{C}{\sigma}\left(\varepsilon\sqrt{\ln\frac{\sigma}{\varepsilon}} + s_\varepsilon\right) \tag{4.18}$$

for each i. Combining the estimates above yield (4.16), since $s_\varepsilon \geq \varepsilon\sqrt{\ln\frac{\rho_\xi}{\varepsilon}}$.

Step 3. We give a lower bound for $\int_{\Omega_\sigma(\xi)}[e_A(u) - e_{A_\star}(u_\star)]dx$ in Lemma 4.8 below; valid for all $\sigma \leq \frac{1}{K_1}\left(n\rho_\xi^{-1} + h_{ex}\right)^{-1}$, with K_1 being fixed in the course of the proof of this lemma. The contributions from $B_\sigma(\xi_i)$, $i = 1,\dots,n$ are estimated by (4.16).

Assembling these estimates (4.21), (4.11), (4.16), and (4.15), simplifying, and using our assumption that $s_\varepsilon \geq \varepsilon\sqrt{\ln\rho_\xi/\varepsilon}$ yields

$$\begin{aligned}&\int_{\Omega_\sigma(\xi)} e_\varepsilon(|u|) + \frac{1}{4}\left|\frac{j_A(u)}{|u|} - j_{A_\star}(u_\star)\right|^2 + \frac{1}{2}\int_\Omega |\operatorname{curl} A - \operatorname{curl} A_\star|^2\\ &\leq \int_\Omega g_\varepsilon(u, A; h_{ex})\ dx - W_\Omega^\varepsilon(\xi, d; h_{ex}) + \mathbf{error1}\end{aligned}$$

where

$$\begin{aligned}\mathbf{error1} &\leq Cs_\varepsilon\left(n\sigma^{-1} + h_{ex}\right) + C\varepsilon\left(n\sigma^{-1} + h_{ex}\right)G_\varepsilon\\ &+ Cn^2\sigma^3\left(n\rho_\xi^{-1} + h_{ex}\right)\left(n\sigma^{-1} + h_{ex}\right)^2\\ &+ C\frac{n}{\sigma}s_\varepsilon + Cn\sigma^\gamma\left(G_\varepsilon + h_{ex}^2\right)\\ &+ C\frac{n^2\sigma^2}{\rho_\xi^2} + n\sigma\left(n\rho_\xi^{-1} + h_{ex}\right) + t_\varepsilon\left(n\rho_\xi^{-1} + h_{ex}\right).\end{aligned}$$

We simplify several terms of the error. Note that

$$\begin{aligned}&n^2\sigma^3\left(n\rho_\xi^{-1} + h_{ex}\right)\left(n\sigma^{-1} + h_{ex}\right)^2\\ &\quad\leq Cn^2\sigma^3\left(n\rho_\xi^{-1} + h_{ex}\right)\left(n^2\sigma^{-2} + h_{ex}^2\right)\\ &\quad\leq Cn^4\sigma\left(n\rho_\xi^{-1} + h_{ex}\right),\end{aligned}$$

since $\sigma^2 h_{ex}^2 \le C\left(n\rho_\xi^{-1} + h_{ex}\right)^{-2} h_{ex}^2 \le C$. Next note that $n^2\sigma^2\rho_\xi^{-2} \le n\rho_\xi^{-1}\sigma$ by a similar argument. Finally since $\gamma < 1$ then $\varepsilon^\gamma > \varepsilon$ and $\sigma^\gamma \ge \sigma$, so

$$\begin{aligned}
\mathbf{error1} \le\ & C\frac{n}{\sigma}s_\varepsilon + Cn\sigma^\gamma\left(G_\varepsilon + h_{ex}^2\right) + Cn^4\sigma\left(n\rho_\xi^{-1} + h_{ex}\right) \\
& + Cn\rho_\xi^{-1}\sigma + Cn\sigma\left(n\rho_\xi^{-1} + h_{ex}\right)^2 \\
& + Cs_\varepsilon h_{ex} + C\varepsilon h_{ex} G_\varepsilon + Cn\varepsilon^\gamma\left(G_\varepsilon^3 + h_{ex}^4\right) + t_\varepsilon\left(n\rho_\xi^{-1} + h_{ex}\right) \\
\le\ & C\frac{n}{\sigma}s_\varepsilon + Cn^4\sigma^\gamma\left(G_\varepsilon + h_{ex}^2 + n^2\rho_\xi^{-2}\right) \\
& + Ch_{ex}\left(s_\varepsilon + \varepsilon G_\varepsilon\right) + Cn\varepsilon^\gamma\left(G_\varepsilon^3 + h_{ex}^4\right) + t_\varepsilon\left(n\rho_\xi^{-1} + h_{ex}\right).
\end{aligned}$$

Optimizing $\sigma = \sigma^*$, defined by

$$\sigma^* = 4\left(\frac{s_\varepsilon}{n^3 G_\varepsilon + n^3 h_{ex}^2 + n^5\rho_\xi^{-2}}\right)^{\frac{1}{\gamma+1}},$$

yields the following error estimate

$$\begin{aligned}
\mathbf{error1} \le\ & Cs_\varepsilon^{\frac{\gamma}{\gamma+1}}\left(n^4 G_\varepsilon + n^4 h_{ex}^2 + n^6\rho_\xi^{-2}\right)^{\frac{1}{\gamma+1}} \\
& + Ch_{ex}\left(s_\varepsilon + \varepsilon G_\varepsilon\right) + Cn\varepsilon^\alpha\left(G_\varepsilon^3 + h_{ex}^4\right) + t_\varepsilon\left(n\rho_a^{-1} + h_{ex}\right)
\end{aligned}$$

where C depends on γ, for any $\alpha < 1$. We arrive at (4.5).

Step 4. The remaining conclusion (4.7) is essentially a corollary of (4.5) and is proved as follows. First note that

$$\begin{aligned}
& \|j_A(u) - j_{A_\star}(u_\star)\|_{L^{4/3}(\Omega)} \\
& \le \|j_A(u) - \frac{j_A(u)}{|u|}\|_{L^{4/3}(\Omega)} + \|\frac{j_A(u)}{|u|} - j_{A_\star}(u_\star)\|_{L^{4/3}(\Omega)} \qquad (4.19) \\
& = T_1 + T_2.
\end{aligned}$$

The first term is easily estimated:

$$\begin{aligned}
T_1 & \le \|\,|\nabla_A u|\ |1 - |u|\,|\,\|_{L^{4/3}(\Omega)} \le \|\nabla_A u\|_{L^2(\Omega)}\ \|1 - |u|\|_{L^4(\Omega)} \\
& \le G_\varepsilon^{1/2}(\varepsilon^2 G_\varepsilon)^{1/4}.
\end{aligned}$$

As for the second term, note that

$$T_2 \le \|\frac{j_A(u)}{|u|} - j_{A_\star}(u_\star)\|_{L^{4/3}(\Omega_{\sigma^*}(\xi))} + \|\frac{j_A(u)}{|u|} - j_{A_\star}(u_\star)\|_{L^{4/3}(\cup_i B_{\sigma^*}(\xi_i))},$$

for σ^* as in Theorem 4.1. By Hölder's inequality,

$$\|\frac{j_A(u)}{|u|} - j_{A_\star}(u_\star)\|_{L^{4/3}(\Omega_{\sigma^*}(\xi))} \le C\|\frac{j_A(u)}{|u|} - j_{A_\star}(u_\star)\|_{L^2(\Omega_{\sigma^*}(\xi))}, \qquad (4.20)$$

and the right-hand side is estimated in (4.5), so we move on by observing that

$$\|\frac{j_A(u)}{|u|} - j_{A_\star}(u_\star)\|_{L^{4/3}(\cup_i B_{\sigma^*}(\xi_i))} \le \|\frac{j_A(u)}{|u|}\|_{L^{4/3}(\cup_i B_{\sigma^*}(\xi_i))} + \|j_{A_\star}(u_\star)\|_{L^{4/3}(\cup_i B_{\sigma^*}(\xi_i))}.$$

Both terms are easily handled. First, by Hölder's inequality,

$$\begin{aligned}\|\frac{j_A(u)}{|u|}\|_{L^{4/3}(\cup_i B_{\sigma^*}(\xi_i))} &\le (\pi n {\sigma^*}^2)^{1/4} \|\frac{j_A(u)}{|u|}\|_{L^2(\cup_i B_{\sigma^*}(\xi_i))} \\ &\le C n^{1/4} \sqrt{\sigma^*} (G_\varepsilon)^{1/2}.\end{aligned}$$

Second, using (5.20) and the coarea formula we compute

$$\begin{aligned}\|j_{A_\star}(u_\star)\|_{L^{4/3}(\cup_i B_{\sigma^*}(\xi_i))} &\le C \left(\sum_i \int_0^{\sigma^*} (\frac{n}{r} + h_{ex})^{4/3} r\, dr \right)^{3/4} \\ &\le C n^{3/4} \sqrt{\sigma^*}\, (n + \sigma^* h_{ex}).\end{aligned}$$

We obtain (4.7) by combining (4.19), (4.20), (4.5) and the other estimates above, and then recalling the definition of σ^*. ☐

We finish the proof by giving the lower bound for $g_\varepsilon(u, A; h_{ex})$ used above.

Lemma 4.8. Let $\Omega \subset \mathbb{R}^2$ be bounded, open, and simply connected, with $\partial\Omega$ of class C^1, and let $u \in H^1(\Omega; \mathbb{C})$, $\xi \in \Omega^{n*}$ and $d \in \{\pm 1\}^n$ satisfy (4.3). Then there exist constants C and K_1, depending only on Ω, such that for any $\sigma \in (0, \frac{1}{K_1}\left(n\rho_\xi^{-1} + h_{ex}\right)^{-1})$,

$$\begin{aligned}&\int_{\Omega_\sigma(\xi)} e_\varepsilon(|u|) + \frac{1}{4}\left|\frac{j_A(u)}{|u|} - j_{A_\star}(u_\star)\right|^2 dx \\ &\quad \le \int_{\Omega_\sigma(\alpha)} [e_A(u) - e_{A_\star}(u_\star)] + C s_\varepsilon \left(n\sigma^{-1} + h_{ex}\right) \\ &\qquad + C\varepsilon \left(n\sigma^{-1} + h_{ex}\right) G_\varepsilon + C n^2 \sigma^3 \left(n\rho_\xi^{-1} + h_{ex}\right) \left(n\sigma^{-1} + h_{ex}\right)^2\end{aligned} \tag{4.21}$$

for $u_\star = u_\star(\cdot; \xi, d)$ as defined in (5.1).

Proof. **Step 1**. Assume that u, ξ, d satisfy (4.3), and let $\sigma > 0$ be such that

$$\sigma \le \frac{1}{K_1}\left(n\rho_\xi^{-1} + h_{ex}\right)^{-1} \tag{4.22}$$

for K_1 to be fixed below. Throughout the proof of this lemma, C will denote a constant that may depend on Ω but is independent of all other parameters. We write Ω_σ for $\Omega_\sigma(\xi)$ and Υ for $\Upsilon(\cdot; \xi, d)$ as defined in (5.2). In particular since $j_{A_\star}(u_\star) = j(u_\star) - A_\star = -\nabla \times (G - \Xi_\star)$ then we define

$$\Upsilon = G - \Xi_\star.$$

Step 2. In Step 3 below we will verify that when (4.22) holds, there exists a set $\tilde{\Omega}_\sigma \subset \Omega_\sigma$ such that

$$|\Omega_\sigma \setminus \tilde{\Omega}_\sigma| \le C n \sigma^3 \left(n\rho_\xi^{-1} + h_{ex} \right), \tag{4.23}$$

and a function $\tilde{\Upsilon}_\sigma$ of the form

$$\tilde{\Upsilon}_\sigma = \begin{cases} \Upsilon & \text{in } \tilde{\Omega}_\sigma, \\ \text{constant} & \text{on each connected component of } \Omega \setminus \tilde{\Omega}_\sigma, \end{cases} \tag{4.24}$$

such that $\tilde{\Upsilon}_\sigma \in W^{1,\infty}(\Omega)$ (in particular $\tilde{\Upsilon}_\sigma$ is continuous across $\Omega \cap \partial\tilde{\Omega}_\sigma$), with

$$\tilde{\chi}\, j_{A_\star}(u_\star) = \nabla \times \tilde{\Upsilon}_\sigma. \tag{4.25}$$

Here $\tilde{\chi}$ is the characteristic function of $\tilde{\Omega}_\sigma$.

For now we assume the existence of $\tilde{\Omega}_\sigma(\xi)$, $\tilde{\Upsilon}_\sigma$ as described above, and we use them to prove that

(4.26)
$$\begin{aligned} &\left| \int_{\Omega_\sigma} |j_{A_\star}(u_\star)|^2 - \frac{j_A(u)}{|u|} \cdot j_{A_\star}(u_\star)\, dx \right| \\ &\quad \le \frac{1}{4} \int_{\Omega_\sigma} \left| \frac{j_A(u)}{|u|} - j_{A_\star}(u_\star) \right|^2 dx + C s_\varepsilon \left(n\sigma^{-1} + h_{ex} \right) \\ &\qquad + C\varepsilon \left(n\sigma^{-1} + h_{ex} \right) G_\varepsilon + C n^2 \sigma^3 \left(n\sigma^{-1} + h_{ex} \right)^2 \left(n\rho_\xi^{-1} + h_{ex} \right). \end{aligned}$$

Note that in view of (4.8), this immediately implies the conclusion of the lemma.

To prove (4.26), we write $\int_{\Omega_\sigma} |j_{A_\star}(u_\star)|^2 - \frac{j_A(u)}{|u|} \cdot j_{A_\star}(u_\star)\, dx = A_1 + A_2 + A_3$, where,

$$A_1 = \int_{\tilde{\Omega}_\sigma} j_{A_\star}(u_\star) \cdot (j_{A_\star}(u_\star) - j_A(u))\, dx,$$

$$A_2 = \int_{\tilde{\Omega}_\sigma} j_{A_\star}(u_\star) \cdot \frac{j_A(u)}{|u|}(|u| - 1)\, dx,$$

and

$$A_3 = \int_{\Omega_\sigma \setminus \tilde{\Omega}_\sigma} j_{A_\star}(u_\star) \cdot (j_{A_\star}(u_\star) - \frac{j_A(u)}{|u|})\, dx.$$

We analyze these terms in turn. First, using (4.25),

$$A_1 = \int_\Omega \nabla \times \tilde{\Upsilon}_\sigma \cdot (j_{A_\star}(u_\star) - j_A(u)).$$

Since $\tilde{\Upsilon}_\sigma = G - \xi_\star = 0$ on $\partial\Omega$, we can integrate by parts and use (4.24) to find that

$$A_1 = 2 \int_\Omega \tilde{\Upsilon}_\sigma (\pi \sum_{i=1}^n d_i \delta_{\xi_i} - J_A(u)).$$

So

$$A_1 \le C\|\nabla\tilde{\Upsilon}_\sigma\|_{L^\infty(\Omega)}\ \|J_A(u)-\pi\sum d_i\delta_{\xi_i}\|_{\dot{W}^{-1,1}(\Omega)}\ \le Cs_\varepsilon\left(\frac{n}{\sigma}+h_{ex}\right)$$

by (5.20), since $\|\nabla\tilde{\Upsilon}_\sigma\|_{L^\infty(\Omega)}=\|j_{A_\star}(u_\star)\|_{L^\infty(\tilde{\Omega}_\sigma)}$.

Next, since $(1-|u|)\le|1-|u|^2|$, Cauchy-Schwarz implies

$$A_2\ \le\ \|j_{A_\star}(u_\star)\|_{L^\infty(\tilde{\Omega}_\sigma)}\ \left\|\frac{j_A(u)}{|u|}\right\|_2\ \|1-|u|^2\|_2.$$

and using (5.20) again we get an estimate of the first term, leading to

$$A_2\le C\varepsilon\left(n\sigma^{-1}+h_{ex}\right)G_\varepsilon(u,A). \tag{4.27}$$

Finally, (4.23) and (5.20) imply that

$$\begin{aligned}A_3 &\le \int_{\Omega_\sigma\setminus\tilde{\Omega}_\sigma}|j_{A_\star}(u_\star)|^2+\frac14\int_{\Omega_\sigma\setminus\tilde{\Omega}_\sigma}\left|j_{A_\star}(u_\star)-\frac{j_A(u)}{|u|}\right|^2\\ &\le\ Cn^2\sigma^3\left(n\rho_\xi^{-1}+h_{ex}\right)\left(n\sigma^{-1}+h_{ex}\right)^2+\frac14\int_{\Omega_\sigma\setminus\tilde{\Omega}_\sigma}\left|j_{A_\star}(u_\star)-\frac{j_A(u)}{|u|}\right|^2.\end{aligned}$$

Step 3. To complete the proof, we construct the set $\tilde{\Omega}_\sigma$ used in Step 2 above. We introduce some notation: First, for $i=1,\dots n$ and $\sigma\le\rho_\xi/2$, we define

$$\ell_i(\sigma)=\min_{|x-\xi_i|=\sigma}\Upsilon(x)\text{ if }d_i<0,\qquad \ell_i(\sigma)=\max_{|x-\xi_i|=\sigma}\Upsilon(x)\text{ if }d_i>0. \tag{4.28}$$

We write $R:=\sigma(1+K_1\sigma\left(n\rho_\xi^{-1}+h_{ex}\right))$, where K_1 is the constant in (4.22), and we define

$$\begin{aligned}\tilde{B}_{i,\sigma}&=\{x\in B_R(\xi_i):\Upsilon(x)\ge\ell_i(r)\}\ \text{ if } d_i<0,\\ \tilde{B}_{i,\sigma}&=\{x\in B_R(\xi_i):\Upsilon(x)\le\ell_i(r)\}\ \text{ if } d_i>0.\end{aligned} \tag{4.29}$$

We fix K_1 to be large enough that

$$\begin{cases} R\le 2\sigma\le\frac12\left(n\rho_\xi^{-1}+h_{ex}\right)^{-1}\le\rho_\xi\\[2ex] \ln\left(1+K_1\sigma\left(n\rho_\xi^{-1}+h_{ex}\right)\right)>C\sigma\left(n\rho_\xi^{-1}+h_{ex}\right)\end{cases} \tag{4.30}$$

whenever (4.22) holds. In fact it is enough to take $K_1=40C$ where C is the constant in estimates (5.15). Finally we define

$$\tilde{\Omega}_\sigma(\xi)=\Omega\setminus\left(\cup_i\tilde{B}_{i,\sigma}\right) \tag{4.31}$$

and

$$\tilde{\Upsilon}_\sigma(x):=\begin{cases}\Upsilon(x) & \text{if } x\in\tilde{\Omega}_\sigma\\ \ell_i(\sigma) & \text{if } x\in\tilde{B}_{i,\sigma}.\end{cases} \tag{4.32}$$

We now verify that the required properties hold. First, as a consequence of Remark 5.5 (which appears immediately after Lemma 5.4) we infer that $B_\sigma(\xi_i) \subset \tilde{B}_{i,\sigma}$ for all i, and hence that $\tilde{\Omega}_\sigma \subset \Omega_\sigma$ as claimed.

Second, it is clear that

$$|\Omega_\sigma \setminus \tilde{\Omega}_\sigma| \leq \sum_{i=1}^{n} |B_R(\xi_i) \setminus B_\sigma(\xi_i)| \leq Cn\sigma^3 \left(n\rho_\xi^{-1} + h_{ex}\right).$$

Finally, it is obvious that (4.25) holds almost everywhere, so we only need to verify that $\tilde{\Upsilon}_\sigma$ is continuous across $\Omega \cap \partial\tilde{\Omega}_\sigma$ and consequently globally Lipschitz. For concreteness, consider the case $d_i = -1$. Then it suffices to verify that $\Upsilon(x) \leq \ell_i(\sigma)$ for $x \in \partial B_R(\xi_i)$.

We use the notation $H_i(x) = G(x) - d_i \ln|x - \xi_i|$. Recall from (4.30) that $R \leq \rho_\xi$, and so (5.22) implies that $|\nabla H_i| \leq \frac{2n}{\rho_\xi}$ in $B_R(\xi_i)$. Fix a point $x_0 \in \partial B_\sigma(\xi)$ at which $\Upsilon(x_0) = \ell_i(\sigma)$. For any $y \in \partial B_R(\xi_i)$

$$\begin{aligned}
\Upsilon(y) &= \ell_i(\sigma) + \Upsilon(y) - \Upsilon(x_0) \\
&= \ell_i(\sigma) + G(y) - G(x_0) - \Xi(y) + \Xi(x_0) \\
&= \ell_i(\sigma) + H_i(y) - H_i(x_0) - \Xi(y) + \Xi(x_0) + \ln\frac{\sigma}{R}.
\end{aligned}$$

Also, $|x_0 - y| \leq R + \sigma < 4\sigma$, so

$$|H_i(y) - H_i(x_0)| \leq 4\sigma \|\nabla H_i\|_{L^\infty(B_R(\xi_i))} \leq 8\sigma \frac{n}{\rho_\xi}$$

and

$$|\Xi_i(y) - \Xi_i(x_0)| \leq 4\sigma \|\nabla \Xi\|_{L^\infty(B_R(\xi_i))} \leq C\sigma\left(n\rho_\alpha^{-1} + h_{ex}\right)$$

and $\ln\frac{\sigma}{R} = -\ln(1 + K_1\left(n\sigma\rho_\xi^{-1} + h_{ex}\right))$, so it follows from (4.30) that $H_i(y) - \Xi_\star(y) < \ell_i(\sigma)$ as required. This completes the proof. □

5. Canonical harmonic map, renormalized energy, and surplus energy

In this final section we prove a number of new estimates on the canonical harmonic map and the renormalized energy. These estimates are crucially used in the proof of the Theorem 4.1.

Here we just give the definitions of these quantities, mostly following Bethuel, Brezis and Hélein [2] and Bethuel and Rivière [3]. Given $a \in \Omega^{n*}$ and $d \in \mathbb{Z}^n$, the canonical harmonic map $u_\star \in W^{1,1}(\Omega; S^1)$ with singularities at points $a = (a_1, \ldots, a_n)$ of degree $d = (d_1, \ldots d_n)$ and natural boundary condition satisfies

$$\nabla \cdot j(u_\star) = 0, \qquad \nabla \times j(u_\star) = 2\pi \sum d_i \delta_{a_i}, \tag{5.1}$$

and $\nu \cdot j(u_\star) = 0$ on $\partial\Omega$. The first equation in (5.1) states that $u_\star$ is a harmonic map into S^1, and the second equation specifies the positions and degrees of the singularities. These conditions uniquely determine $j(u_\star)$. In addition, $j(u_\star)$ determines $u_\star$ up to a constant phase; see [2] Chapter 1. We will sometimes write $u_\star(\cdot; a)$, but more often we do not explicitly indicate the dependence of $u_\star$ on a, d, and we *never* indicate in our notation the dependence of $u_\star$ on the domain Ω.

Let G solve

$$\Delta G = 2\pi \sum_{i=1}^{n} d_i \delta_{a_i} \quad \text{in } \Omega, \quad G = 0 \text{ on } \partial\Omega, \tag{5.2}$$

then $j(u_\star) = -\nabla \times G$. Note also that if we define $H(\cdot; y)$ for $y \in \Omega$ as the solution of

$$\Delta_x H(\cdot, y) = 0 \quad \text{in } \Omega, \quad H(x, y) = -\ln|x - y| \text{ for } x \in \partial\Omega, y \in \Omega \tag{5.3}$$

then

$$G(x; a) = \sum_{i=1}^{n} d_i[\ln|x - a_i| + H(x, a_i)]. \tag{5.4}$$

We next define the limiting London equation:

$$-\Delta h_\star + h_\star = 2\pi \sum d_i \delta_{a_i} \tag{5.5}$$

and $h_\star = h_{ex}$ on $\partial\Omega$. Since $A_\star$ satisfies the Coulomb gauge, we define $\Xi_\star$ such that $-\Delta\Xi_\star = h_\star$ with $\Xi_\star = 0$ on $\partial\Omega$. Then $A_\star = \operatorname{curl}\Xi_\star$ is the limiting magnetic field potential.

Following Bethuel-Riviere [3], see also Spirn [28], we define the renormalized energy $W_\Omega(a, d, h_{ex})$ by

$$\begin{aligned} W_\Omega(a, d, h_{ex}) = \lim_{r\to 0} &\left(\int_{\Omega_r(a)} \frac{1}{2} |\nabla_{A_\star} u_\star|^2 dx - n\pi \ln \frac{1}{r} \right) \\ &+ \int_\Omega \frac{1}{2} |h_\star - h_{ex}|^2 , \end{aligned} \tag{5.6}$$

and we recall from [28] that

$$\begin{aligned} W_\Omega(a, d; h_{ex}) = &-\pi \sum_{i\neq j} d_i d_j \log|a_i - a_j| - \pi \sum_{i,j} d_i d_j H(a_i, a_j) \\ &+ \pi \sum_{j=1}^{d} d_j \Xi_\star(a_j) - \frac{1}{2} \int_{\partial\Omega} h_{ex} \partial_n \Xi_\star + \frac{1}{2} h_{ex}^2 |\Omega| . \end{aligned} \tag{5.7}$$

We give a proof of the equivalence of (5.7) and (5.6), with estimates of the rate of convergence of the right-hand side of (5.6), in Lemma 5.6, in Section 5.

Next, we recall from [2] the notation

$$I(r, \varepsilon) := \inf \left\{ \int_{U_r} e_\varepsilon(u) \; ; \; u \in H^1(B_r; \mathbb{C}), \; u = e^{i\theta} \text{ on } \partial B_r \right\}, \tag{5.8}$$

and we define

$$\gamma_0 = \lim_{r\to\infty} (I(r, \varepsilon) - \pi \ln \frac{r}{\varepsilon}). \tag{5.9}$$

It is known that γ_0 exists, is finite and is independent of ε. Moreover, in Lemma 5.3 we prove that $\gamma_0 - (I(r,\varepsilon) - \pi \ln \frac{r}{\varepsilon}) = O((\varepsilon/r)^2)$. For $a \in \Omega^{n*}$ and $d \in \{\pm 1\}^n$ we will write

$$W_\Omega^\varepsilon(a, d; h_{ex}) := n(\gamma_0 + \pi \ln \frac{1}{\varepsilon}) + W_\Omega(a, d; h_{ex}). \tag{5.10}$$

Like the renormalized energy, W_Ω^ε depends on the domain and the prescribed boundary conditions (here Neumann) in a way that is not explicitly indicated in the notation. Informally, $W_\Omega^\varepsilon(a, d; h_{ex})$ provides an approximate lower bound for the energy E_ε of a function with vortices of degree d_i near a_i, $i = 1, \ldots, n$. This is made precise in Theorem 4.1, see Section 4. This lower bound is very close to sharp; this follows from Lemma 5.8 (see Section 5), in which we construct, for given $a \in \Omega^{n*}$ and $d \in \{\pm 1\}^n$, a function $u_\star^\varepsilon$ with a vortex of degree d_i at the point a_i, $i = 1, \ldots, n$, and with energy extremely close to $W_\Omega^\varepsilon(a, d; h_{ex})$. As remarked in the Introduction, our results can be seen as, among other things, establishing the dynamic stability of the manifold $\{u_\star^\varepsilon(a, d) : a \in \Omega^{*n}\} \subset H^1(\Omega)$.

We will also use the notation

$$\begin{aligned} D_\Omega^\varepsilon(u, A; \alpha, h_{ex}) &:= \int_\Omega g_\varepsilon(u, A; h_{ex})\, dx - W_\Omega^\varepsilon(a, d; h_{ex}) \\ &\approx \int_\Omega [g_\varepsilon(u, A; h_{ex}) - g_\varepsilon(u_\star^\varepsilon(a, d), A; h_{ex})]\, dx. \end{aligned} \tag{5.11}$$

We refer to this quantity as the *surplus energy*; the terminology is justified again by Theorem 4.1. This quantity is only meaningful when $\|J(u) - \pi \sum d_i \delta_{a_i}\|_{\dot W^{-1,1}(\Omega)}$ is small.

We remark that one can check that for a single vortex of degree ± 1 at the center of a ball of radius r, the associated renormalized energy is $\pi \ln r$, and so the associated surplus energy is

$$\int_{U_r(a)} e_\varepsilon(u)\, dx - (\pi \ln \frac{r}{\varepsilon} + \gamma_0). \tag{5.12}$$

This quantity appears in the statement of a number of results.

Next we give a series of lemmas concerning the canonical harmonic map and the renormalized energy. At the end of the section we construct maps that are close to energetically optimal, for a fixed ε and prescribed configuration of vortices. This construction proves in particular that it is possible to find initial data satisfying the hypotheses of Theorem 1.1.

We start with a characterization of the gradient of the renormalized energy.

Lemma 5.1. Let $\xi \in \Omega^{n*}$ and $d \in \{\pm 1\}$ then the canonical harmonic map $u_\star = u_\star(\cdot; \xi, d)$ and the renormalized energy $W_\Omega(\xi, d)$ satisfy

(5.13)

$$\int \mathbb{J}_{kl} \partial_{x_k x_m} \eta \left(j_{A_\star}(u_\star)\right)_m \left(j_{A_\star}(u_\star)\right)_l = \sum_{j=1}^n d_j \mathbb{J}_{km} \partial_m \eta(\xi_j) \left(\nabla_{\xi_j} W_\Omega(\xi, d)\right)_k$$

where $\eta \in C^2(\Omega)$ and $\nabla^2 \eta$ has support in a neighborhood of the ξ_j's.

Proof. The proof of this lemma can be found in [28]. □

Lemma 5.2. Let $-\Delta h_\star + h_\star = 2\pi \sum d_j \delta_{a_j}$ with $h_\star = h_{ex}$ on $\partial\Omega$ then for any $r \leq \rho_a$

$$\|h_\star\|_{L^\infty(\Omega \setminus \cup B_r(a_j))} \leq \frac{C\,(n + h_{ex})}{r}, \tag{5.14}$$

Furthermore, for any $p < 2$

$$\begin{aligned} \|h_\star\|_{W^{1,p}(\Omega)} &\leq C\left(n\rho_a^{-1} + h_{ex}\right) \\ \|A_\star\|_{W^{2,p}(\Omega)} &\leq C\left(n\rho_a^{-1} + h_{ex}\right) \\ \|\Xi_\star\|_{W^{3,p}(\Omega)} &\leq C\left(n\rho_a^{-1} + h_{ex}\right) \end{aligned} \tag{5.15}$$

where C depends only on Ω and p.

Proof. We use the explicit form of the fundamental solution to the London equation, along with maximum principles, to arrive at our estimates. First we decompose the Helmholtz operator. Let $h_\star = h_1 + h_2$ where

$$\begin{aligned} -\Delta h_1 + h_1 &= 0 \text{ in } \Omega \\ h_1 &= h_{ex} \text{ on } \partial\Omega \end{aligned}$$

and

$$\begin{aligned} -\Delta h_2 + h_2 &= 2\pi \sum d_j \delta_{a_j} \text{ in } \Omega \\ h_2 &= 0 \text{ on } \partial\Omega. \end{aligned}$$

Next we aim to prove $W^{1,p}$ estimates on $h_\star = \Delta \Xi_\star$. In particular we have that $h_\star = h_1 + h_2$ with $-\Delta h_2 + h_2 = 2\pi \sum d_j \delta_{a_j}$ and $h_2 = 0$ on the boundary. We can further decompose $h_2 = \sum h_2^k$ where $-\Delta h_2^k + h_2^k = 2\pi d_k \delta_{a_k}$. Since the fundamental solution of $I - \Delta$ is the Bessel function $K_0(r)$ we write

$$h_2^k = d_k K_0(|x - a_k|) + \Phi_k(x)$$

where Φ_k solves

$$\begin{aligned} -\Delta \Phi_k + \Phi_k &= 0 \text{ in } \Omega \\ \Phi_k(y) &= -d_k K_0(|x - y|) \text{ on } \partial\Omega. \end{aligned}$$

Therefore, differentiating we find that on the boundary, $\nabla \Phi_k = -d_k K_0'(|x - a_j|)\frac{x - a_j}{|x - a_j|} = d_k K_1(|x - a_k|)\frac{x - a_k}{|x - a_k|}$ by the standard identity for Bessel functions $K_0'(r) = -K_1(r)$, where K_1 behaves asymptotically like

$$K_1(r) = \begin{cases} \frac{1}{r} & \text{for } r \ll 1 \\ \sqrt{\frac{\pi}{2r}} e^{-r} & \text{for } r \gg 1 \end{cases}, \tag{5.16}$$

see for example [1].

By a straightforward calculation one sees that $|\nabla \Phi_k(x)|$ satisfies

$$(I - \Delta)\,|\nabla \Phi_k| \leq 0$$

in Ω with $|\nabla\Phi_k| \leq 2\pi K_1(\mathrm{dist}(a_k,\partial\Omega))$ on $\partial\Omega$. Then by the maximum principle

$$|\nabla\Phi_k(x)| \leq 2\pi K_1(\mathrm{dist}(a_k,\partial\Omega)) \tag{5.17}$$

for all $x \in \Omega$. In particular $\|\nabla\Phi_k\|_{L^p(\Omega)} \leq C\rho_a^{-1}$.

Since $|\nabla K_0(|x-a_j|)|$ and $|\nabla\Phi_k|$ have the same asymptotic behavior then for any $r \leq \rho_a$ we see that

$$\left\|\nabla h_2^k\right\|_{L^\infty(\Omega\setminus B_r(a_k))} \leq \frac{C}{r}$$

and so $\left\|\nabla h_2^k\right\|_{L^p(\Omega)} \leq C$ for any $p < 2$. Summing over k yields

$$\|\nabla h_2\|_{L^p(\Omega)} \leq Cn\rho_a^{-1}. \tag{5.18}$$

Finally, from standard elliptic estimates

$$\begin{aligned}
\int_\Omega |\Delta h_1|^2 &= \int_\Omega h_1\Delta h_1 \\
&= \int_{\partial\Omega} h_1\partial_n h_1 - \int_\Omega |\nabla h_1|^2 \\
&= -\int_\Omega |\nabla h_1|^2 + h_{ex}\int_\Omega \Delta h_1 \\
&\leq -\int_\Omega |\nabla h_1|^2 + \frac{h_{ex}^2}{2} + \frac{1}{2}\int_\Omega |\Delta h_1|^2 .
\end{aligned}$$

Therefore,

$$\|h_1\|_{H_0^2} \leq Ch_{ex}. \tag{5.19}$$

In particular $\|\nabla h_1\|_{L^p(\Omega)} \leq Ch_{ex}$ by (5.19) and Hölder inequality.

Combining our results together yields $\|h_\star\|_{W^{1,p}(\Omega)} \leq C_p\max\{n\rho_a^{-1}, h_{ex}\}$ for each $p < 2$, and hence by standard elliptic estimates $\|\Xi_\star\|_{W^{3,p}(\Omega)} \leq C_p\max\{n\rho_a^{-1}, h_{ex}\}$. □

Recall from (5.9) the constant $\gamma_0 := \lim_{r\to\infty}[I(r,\varepsilon) - \pi\ln r/\varepsilon]$. The following lemma establishes the rate at which the right-hand side converges.

Lemma 5.3 ([12], Lemma 16)**.** $\left|\gamma_0 - (I(r,\varepsilon) - \pi\ln\frac{r}{\varepsilon})\right| \leq C(\frac{\varepsilon}{r})^2$.

Next we estimate the derivatives of the canonical harmonic map and renormalized energy.

Lemma 5.4. There exists absolute constants C such that for every bounded, open $\Omega \subset \mathbb{R}^2$, $a \in \Omega^{n*}$ and $d \in \{\pm1\}^n$, the renormalized energy $W_\Omega(a,d)$, canonical harmonic map $u_\star(\cdot,;a,d)$ and its potential $G(\cdot;a,d)$ as defined in (5.2) satisfy

$$\|j(u_\star)\|_{L^\infty(\Omega_r(a))} = \|\nabla G\|_{L^\infty(\Omega_r(a))} \leq \frac{2n}{r} \tag{5.20}$$

for all $r \leq \rho_a$, and

$$|\nabla_i W_\Omega(a,d)| \leq C\left(n\rho_a^{-1} + h_{ex}\right) \tag{5.21}$$

for every $i, j \in \{1, \dots, n\}$. Finally, for $H_i(x) := G(x) - d_i \ln|x - a_i|$,

$$\|\nabla H_i\|_{L^\infty(B_{\rho_a}(a_i))} \ \leq \ \frac{2n}{\rho_a}. \tag{5.22}$$

Remark 5.5. For every i, it follows from (5.22) that G cannot have any critical points in $\{x : 0 < |x - a_i| < \rho_a/2n\}$.

Proof. In view of the definitions (5.4), (5.6), the conclusions all follow from the estimates

$$|\nabla_x H(x,y)| \leq \frac{1}{\operatorname{dist}(y,\partial\Omega)}, \qquad |\nabla_x^2 H(x,y)| \leq \frac{C}{\operatorname{dist}(y,\partial\Omega)^2}.$$

By differentiating the definition (5.3) of the auxiliary function H, we find that

$$-\Delta_x H_{x_i}(\cdot,y) = 0 \ \text{ in } \Omega, \quad H_{x_i}(x,y) = -\frac{x_i - y_i}{|x-y|^2} \text{ for } x \in \partial\Omega, y \in \Omega \tag{5.23}$$

for $i = 1, 2$. It follows that $-\Delta|\nabla H| \leq 0$ in Ω, and so the maximum principle implies that

$$|\nabla H(x;y)| \leq \operatorname{dist}(y,\partial\Omega)^{-1}$$

for all $x \in \Omega, y \in \Omega$. A similar argument shows that $|\nabla_x^2 H(x,y)| \leq \frac{C}{\operatorname{dist}(y,\partial\Omega)^2}$. □

We next estimate the rate of convergence of the limit used in (5.6) to define the renormalized energy.

Lemma 5.6. There exists an absolute constant C such that

$$\begin{aligned} W_\Omega(a,d) + n\pi \ln\frac{1}{r} - \left[\int_{\Omega_r(a)} \frac{1}{2}|\nabla_{A_\star} u_\star|^2 \ + \int_\Omega \frac{1}{2}\left|\operatorname{curl} A_\star - h_{ex}\right|^2\right] \\ \leq \ Cn^3\left(\frac{r}{\rho_a}\right)^2 + Crn\left(n\rho_a^{-1} + h_{ex}\right)^2 \end{aligned} \tag{5.24}$$

for all bounded, open $\Omega \subset \mathbb{R}^2$, all $n \geq 1$, $a \in \Omega^{n*}$, $d \in \{\pm 1\}^n$, and $r < \rho_a$. Here we assume that $\operatorname{diam}(\Omega)$ is bounded.

Proof. We will write Ω_r as shorthand for $\Omega_r(a)$. We define H_i as in Lemma 5.4 and compute

$$\begin{aligned} \frac{1}{2}\int_{\Omega_r} |\nabla u_\star|^2 \, dx &= \frac{1}{2}\int_{\Omega_r} |\nabla G|^2 \, dx \\ &= \frac{1}{2}\int_{\partial\Omega_r} \frac{1}{2} G\nabla_\nu G \, d\mathcal{H}^1 \\ &= -\frac{1}{2}\sum_i \left[\int_{\partial B_r(a_i)} (H_i + d_i \ln r)(\nabla_\nu H_i + \frac{d_i}{r}) \, d\mathcal{H}^1\right]. \end{aligned}$$

The sign changes because the outward normal to $\partial\Omega_r$ is the inward normal to ∂B_r and vice versa. Using the mean value property of harmonic functions and integrating by parts again in the terms involving $H\nabla_\nu H$, we get

$$\frac{1}{2}\int_{\Omega_r} |\nabla u_\star|^2 \, dx = \sum_i \pi(d_i^2 \ln\frac{1}{r} - d_i\, H_i(a_i)) - \frac{1}{2}\int_{B_r(a_i)} |\nabla H_i|^2 \, dx. \tag{5.25}$$

Next we estimate the part of the energy that interacts with the electromagnetic field.

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega_r} |\nabla_{A_\star} u_\star|^2 + \frac{1}{2}\int_\Omega |\operatorname{curl} A_\star - h_{ex}|^2 \\
&= \int_{\Omega_r} \frac{1}{2}|\nabla u_\star|^2 - A_\star\cdot j(u_\star) + \frac{1}{2}|A_\star|^2 + \frac{1}{2}\int_\Omega |\operatorname{curl} A_\star - h_{ex}|^2 .
\end{aligned}$$

Since $A_\star \cdot j(u_\star) = -\nabla^\perp \Xi_\star \cdot \nabla^\perp G = -\operatorname{div}(\Xi_\star \nabla G) + \Xi_\star \Delta G = -\operatorname{div}(\Xi_\star\nabla G)$ in Ω_r. Therefore,

$$\int_{\Omega_r} -A_\star\cdot j(u_\star) = -\int_{\Omega_r} \operatorname{div}(\Xi_\star\nabla G) = -\sum_j \int_{\partial B_r(a_j)} \Xi_\star \partial_n G d\ell.$$

Since $\partial_n G|_{\partial B_r(a_j)} = -\frac{d_j}{|x-a_j|} + d_j \partial_n H_j(x)$ and

$$\begin{aligned}
\left|\int_{\partial B_r(a_j)} \frac{d_j \Xi_\star(s)}{|x-a_j|} d\ell - 2\pi d_j \Xi_\star(a_j)\right| &\le \left|\int_{\partial B_1(0)} \int_{s=0}^1 \frac{d}{ds}\Xi_\star(a_j + rs\omega) d\omega\right| \\
&\le Cr \left\|\nabla \Xi_\star\right\|_{L^\infty(B_r(a_j))} \\
&\le Cr\left(n\rho_a^{-1} + h_{ex}\right)
\end{aligned}$$

and

$$\begin{aligned}
\left|\int_{\partial B_r(a_j)} \Xi_\star d_j \partial_n H_j(x) d\ell\right| &\le Crn\rho_a^{-1} \left\|\Xi_\star\right\|_{L^\infty(\cup_j B_r(a_j))} \\
&\le Crn\rho_a^{-1}\left(n\rho_a^{-1} + h_{ex}\right),
\end{aligned}$$

then

$$\left| -\int_{\Omega_r} A_\star \cdot j(u_\star) - \sum_j 2\pi d_j \Xi_\star(a_j)\right| \le Cnr\left(n\rho_a^{-1} + h_{ex}\right)\left(1 + n\rho_a^{-1}\right). \tag{5.26}$$

Furthermore,

$$\int_{\cup B_r(a_j)} |\nabla \Xi_\star|^2 \le Cnr^2 \left\|\nabla\Xi_\star\right\|^2_{L^\infty(\Omega)} \le Cr^2 n\left(n\rho_a^{-1} + h_{ex}\right)^2. \tag{5.27}$$

When $|d_i| = 1$ for all i, (5.6) implies that

$$W_\Omega(a, d;) = -\pi \sum d_i H_i(a_i) + \sum 2\pi d_j \Xi(a_j) + \frac{1}{2}\int_\Omega |\nabla \Xi_\star|^2 + |\Delta\Xi_\star - h_{ex}|^2,$$

and so

$$
\begin{aligned}
W_\Omega(a,d) + n\pi\ln\frac{1}{r} &- \int_{\Omega_r}\frac{1}{2}|\nabla_{A_\star}u_\star|^2\,dx - \int_\Omega \frac{1}{2}\,|h_\star - h_{ex}|^2 \\
&= \frac{1}{2}\sum_i\int_{B_r(a_i)}|\nabla H_i|^2\,dx + \frac{1}{2}\sum_i\int_{B_r(a_i)}|\nabla \Xi_\star|^2\,dx \\
&\quad + \left|-\int_{\Omega_r} A_\star\cdot j(u_\star)dx - 2\pi\sum d_j\Xi_\star(a_j)\right| \\
&\le Cn^3\left(\frac{r}{\rho_a}\right)^2 + Cr^2 n\left(n\rho_a^{-1}+h_{ex}\right)^2 \\
&\quad + Crn\left(n\rho_a^{-1}+h_{ex}\right)\left(1+n\rho_a^{-1}\right)
\end{aligned} \tag{5.28}
$$

by (5.22). Since $\mathrm{diam}(\Omega) < +\infty$ and $n \ge 1$ then we can bounded $r \le C$ and $1 + n\rho_a^{-1} \le C\left(h_{ex} + n\rho_a^{-1}\right)$. □

The next lemma gives a rough estimate of how the renormalized energy scales with the number of vortices.

Lemma 5.7. For a smooth, bounded domain $\Omega \subset \mathbb{R}^n$, if $a \in \Omega^{n^*}$ and $d \in \{\pm 1\}^n$, then

$$
W_\Omega(a,d;h_{ex}) \le C(n^3 + \frac{n^2}{\rho_a^2} + h_{ex}^2). \tag{5.29}
$$

Proof. Let $u_\star = u_\star(\cdot, a, d)$. For $r < \rho_a$ we compute

$$
\int_{\Omega_r}|\nabla u_\star|^2 = \int_{\Omega_{\rho_a}}|\nabla u_\star|^2 + \sum_{i=1}^n \int_{B_{\rho_a}\setminus B_r(a_i)}\left|\nabla H_i + d_i\frac{x-a_i}{|x-a_i|^2}\right|^2\,dx
$$

using the notation of Lemma 5.4. From (5.20) we estimate

$$
\int_{\Omega_{\rho_a}}|\nabla u_\star|^2\,dx \le \frac{Cn^2}{\rho_a^2}.
$$

Next, since H_i is harmonic in $B_{\rho_a}(a_i)$,

$$
\int_{B_{\rho_a}\setminus B_r(a_i)}\nabla H_i\cdot\frac{x-a_i}{|x-a_i|^2}dx = \int_r^{\rho_a}\frac{1}{s}\int_{\partial B_s(a_i)}\nu\cdot\nabla H_i\,\mathcal{H}^1(dx)\,ds = 0,
$$

so using (5.22) we check that for $i = 1, \ldots, n$,

$$
\begin{aligned}
\int_{B_{\rho_a}\setminus B_r(a_i)}\left|\nabla H_i + d_i\frac{x-a_i}{|x-a_i|^2}\right|^2\,dx &= \int_{B_{\rho_a}\setminus B_r(a_i)}|\nabla H_i|^2 + \left|\frac{x-a_i}{|x-a_i|^2}\right|^2\,dx \\
&\le\ Cn^2 + 2\pi\ln\frac{\rho_a}{r}.
\end{aligned}
$$

Next note that

$$
\left|\sum_j \pi\Xi_\star(a_j)\right| \le \pi n\,\|\Xi_\star\|_{L^\infty} \le Cn\left(n\rho_a^{-1}+h_{ex}\right)
$$

and

$$\left|\int_\Omega h_{ex}h_\star\right| \le h_{ex}\,|\Omega|^{1/2}\,\|h_\star\|_{L^2(\Omega)} \le Ch_{ex}\left(n\rho_a^{-1}+h_{ex}\right)$$

Combining these estimates and recalling the characterization of $W_\Omega(a,d;h_{ex})$ in (5.6), we deduce (5.29). □

We conclude this section by constructing maps that are close to energetically optimal for fixed ε and configuration of vortices a, d. Recall the definition

$$I(r,\varepsilon) = \inf\left\{\int_{U_r} e_\varepsilon(u)\ ;\ u\in H^1(B_r;\mathbb{C}),\ u=e^{i\theta}\text{ on }\partial B_r\right\}.$$

It is known that the infimum on the right-hand side of the above definition is attained, and moreover the minimizer $u_{\varepsilon,r}$ has the form

$$u_{\varepsilon,r}(x) = f_{\varepsilon,r}(|x|)\frac{x}{r} \tag{5.30}$$

for an increasing function $f_{\varepsilon,r} : [0,\infty) \to [0,1]$ such that $f_{\varepsilon,r}(0) = 0$ and $f_{\varepsilon,r}(r) = 1$. One can easily check that $f_{\varepsilon,r}(s) = f_{\lambda\varepsilon,\lambda r}(\lambda s)$ for all $\lambda > 0$, and hence $I(r,\varepsilon) = I(r/\varepsilon, 1)$ for all r, ε.

We will use the notation

$$u_\star^{r,\varepsilon}(x;a,d) = u_\star(x;a,d)\prod_{i=1}^d f_{\varepsilon,r}(|x-a_i|). \tag{5.31}$$

For $r \le C\rho_a$, this yields a map with vortex configuration a, d and with nearly optimal energy. We will usually write simply $u_\star^{r,\varepsilon}$ when no confusion can result.

Lemma 5.8. For any $a \in \Omega^{n*}$ and $d \in \{\pm1\}^n$ and for $r \le \rho_a$, the map $u_\star^{r,\varepsilon}(\cdot;a,d)$ constructed above satisfies

$$\begin{aligned}\int_\Omega g_\varepsilon(u_\star^{r,\varepsilon}, A_\star; h_{ex})\,dx \le W_\Omega^\varepsilon(a,d;h_{ex}) + Cn(\frac{\varepsilon}{r})^2\\ + Cnr\left(n\rho_a^{-1}+h_{ex}\right)\left(1+rn\rho_a^{-1}\right)\end{aligned} \tag{5.32}$$

and

$$\|j(u_\star^{r,\varepsilon})\|_{L^p(B_r(a_j))} \le Cr^{\frac{2-p}{p}} + Cr^{2/p}n\rho_a^{-1} \tag{5.33}$$

for $1 \le p < 2$, and

$$\|J_{A_\star}(u_\star^{r,\varepsilon}) - \pi\sum_{i=1}^n d_i\delta_{a_i}\|_{\dot W^{-1,1}(\Omega)} \le Cn\varepsilon(1+\varepsilon\frac{n^3}{\rho_a^2}+\varepsilon r^2 h_{ex}). \tag{5.34}$$

The proof will show that

$$\|J(u_\star^{r,\varepsilon}) - \pi\sum_{i=1}^n d_i\delta_{a_i}\|_{\dot W^{-1,1}(\Omega)} \le Cn\varepsilon \tag{5.35}$$

if $r \leq \rho_a/cn$ for a suitable constant c. Throughout the body of this paper we refer to a function $u_\star^\varepsilon(a,d)$. We define

$$u_\star^\varepsilon(a,d) := u_\star^{r_\star,\varepsilon}(a,d) \quad \text{for } r_\star := \frac{\rho_a}{cn},$$

so that in particular (5.35) holds for $u_\star^\varepsilon(a,d)$.

Proof. 1. To prove (5.32), note that

$$\begin{aligned}
\int_\Omega g_\varepsilon(u_\star^{r,\varepsilon}, A_\star; h_{ex})\, dx = &\int_{\Omega_r} \frac{1}{2}\left|\nabla u_\star\right|^2 - A_\star \cdot j(u_\star) \\
&+ \int_\Omega \frac{1}{2} A_\star^2 + \frac{1}{2}\left|\operatorname{curl} A_\star - h_{ex}\right|^2 \\
&+ \sum_i \int_{B_r(a_i)} e_\varepsilon(u_\star^{r,\varepsilon})\, dx. \\
&+ \sum_i \int_{B_r(a_i)} \frac{1}{2}\left(f_{\varepsilon,r}^2 - 1\right) A_\star^2 - f_{\varepsilon,r}^2 A_\star \cdot j(u_\star).
\end{aligned} \tag{5.36}$$

In $B_r(a_i) = B_r$, $u_\star^{r,\varepsilon} = f_{\varepsilon,r}(|x-a_i|)u_\star(x)$ and so we compute that

$$e_\varepsilon(u_\star^{r,\varepsilon}) \;=\; \frac{1}{2}(|\nabla f_{\varepsilon,r}|^2 + f_{\varepsilon,r}^2|\nabla u_\star|^2) + \frac{1}{4\varepsilon^2}(f_{\varepsilon,r}^2 - 1)^2.$$

Writing $f_{\varepsilon,r}^2|\nabla u_\star|^2 = f_{\varepsilon,r}^2|\nabla H_i + d_i \frac{x-a_i}{|x-a_i|^2}|^2$ as in the proof of Lemma 5.7, we find as before that the cross-terms integrate to 0, using the radial symmetry of $f_{\varepsilon,r}$. Thus

$$\begin{aligned}
&\int_{B_r(a_i)} e_\varepsilon(u_\star^{r,\varepsilon}) dx \\
&= \int_{B_r(a_i)} \left[e_\varepsilon(f_{\varepsilon,r}) + \frac{2f_{\varepsilon,r}^2}{|x-a_i|^2} \right] dx + \int_{B_r(a_i)} \frac{1}{2} f_{\varepsilon,r}^2 |\nabla H_i|^2\, dx.
\end{aligned} \tag{5.37}$$

The first integral on the right-hand side is exactly $I(r,\varepsilon)$, by the definition of $f_{\varepsilon,r}$. Next since $f_{\varepsilon,r} \leq 1$ then

$$\begin{aligned}
\|j(u_\star^{r,\varepsilon})\|_{L^1(B_r(a_j))} &\leq \|\nabla \log|x-a_j| + \nabla H_j\|_{L^1(B_r(a_j))} \\
&\leq Cr\left(1 + rn\rho_a^{-1}\right)
\end{aligned}$$

by the bound on ∇H_i. This implies

$$\begin{aligned}
\left| \sum_j \int_{B_r(a_j)} A_\star \cdot j(u_\star^{r,\varepsilon}) \right| &\leq \|A_\star\|_{L^\infty(\Omega)} \sum_j \|j(u_\star)\|_{L^1(B_r(a_j))} \\
&\leq Crn\left(n\rho_a^{-1} + h_{ex}\right)\left(1 + rn\rho_a^{-1}\right).
\end{aligned} \tag{5.38}$$

So combining (5.36) with (5.37), and recalling (5.28) we deduce that

$$\begin{aligned}\int_\Omega g_\varepsilon(u_\star^{r,\varepsilon}, A_\star; h_{ex})\, dx = \;& W_\Omega(a,d;h_{ex}) + n(\pi \ln \frac{1}{r} + I(r,\varepsilon)) \\ & + \frac{1}{2}\sum_i \int_{B_r(a_i)} (f_{\varepsilon,r}^2 - 1)|\nabla H_i|^2 + (f_{\varepsilon,r}^2 - 1)A_\star^2 \\ & + \left| -\int_\Omega A_\star \cdot j(u_\star^{r,\varepsilon}) - 2\pi \sum_j \Xi(a_j) \right| .\end{aligned}$$

The first integrals on the right-hand side are all negative, and by using (5.3), (5.26), and (5.38) we find that

$$\int_\Omega e_\varepsilon(u_\star^{r,\varepsilon})\, dx \;\le\; W_\Omega^\varepsilon(a,d) + O\left(n(\frac{\varepsilon}{r})^2\right) + Crn\left(n\rho_a^{-1} + h_{ex}\right)\left(1 + rn\rho_a^{-1}\right).$$

2. Without loss of generality assume that the vortex ball is centered at the origin. From the construction of $u_\star^{r,\varepsilon}$ we see

$$\begin{aligned}\|j(u_\star^{r,\varepsilon})\|_{L^p(B_r)} &= \| |u_\star^{r,\varepsilon}|\, j(u_\star)\|_{L^p(B_r)} \\ &= \left\| |f_\star^{r,\varepsilon}|^2\, \nabla \times G \right\|_{L^p(B_r)} \\ &\le \left\| \nabla \times \left(\frac{1}{\pi} \ln|x| + H_j \right) \right\|_{L^p(B_r)} \\ &\le C\left(\int_0^r s^{1-p} ds \right)^{1/p} + Cn\rho_a^{-1} r^{2/p} \\ &\le C r^{\frac{2-p}{p}} + Cn\rho_a^{-1} r^{2/p}.\end{aligned}$$

3. From [13] we have

$$\|J(u_\star^{r,\varepsilon}) - \pi d_i \delta_{a_i}\|_{W^{-1,1}(\Omega)} \le Cn\varepsilon(1 + \varepsilon \frac{n^3}{\rho_a^2}), \tag{5.39}$$

and on the other hand we note from the proof of Lemma 4.5 that

$$\begin{aligned}\|J_A(u) - J(u_\star)\|_{\dot W^{-1,1}} &\le \|J(u) - J(u_\star)\|_{\dot W^{-1,1}} + \varepsilon \|A_\star\|_{L^2} \|1 - |u_\star^{r,\varepsilon}|\|_{L^2} \\ &\le \|J(u) - J(u_\star)\|_{\dot W^{-1,1}} + \varepsilon^2 \left(n\rho_a^{-1} + h_{ex}\right) nr^2\end{aligned}$$

which implies the bound. □

Appendix A. Some identities and conservation laws

A.1. **Commutator identities.** We collect some identities.

Lemma A.1. If $u, v \in H^1(U;\mathbb{C})$ for some domain $U \subset \mathbb{R}^2$ then

$$\nabla(u,v) = (\nabla_A u, v) + (u, \nabla_A v). \tag{A.1}$$

For $u, v \in H^1(I;\mathbb{C})$ for some interval $I \subset \mathbb{R}$,

$$\partial_t(u,v) = (\partial_\Phi u, v) + (u, \partial_\Phi v). \tag{A.2}$$

Proof. We have, setting $\partial_k^A = \partial_k - iA_k$,

$$\begin{aligned}\partial_k(u,v) &= (\partial_k u, v) + (u, \partial_k v)\\ &= (\partial_k u, v) + (u, \partial_k v) + (-iA_k u, v) + (iA_k u, v)\\ &= (\partial_k u, v) + (-iA_k u, v) + (u, \partial_k v) + (u, -iA_k v)\\ &- (\partial_k^A u, v) + (u, \partial_k^A v).\end{aligned}$$

The proof for (A.2) is essentially the same. □

Lemma A.2. The commutator $[\partial_\Phi, \nabla_A] = \partial_\Phi \nabla_A - \nabla_A \partial_\Phi$ can be written as

$$[\partial_\Phi, \nabla_A] = iE \tag{A.3}$$

and the commutator between two covariant derivatives is

$$[\partial_i^A, \partial_j^A] = -i(\partial_i A_j - \partial_j A_i). \tag{A.4}$$

In particular,

$$[\partial_1^A, \partial_2^A] = -i \operatorname{curl} A = -ih. \tag{A.5}$$

Proof. This is just a calculation:

$$\begin{aligned}(\partial_\Phi \nabla_A - \nabla_A \partial_\Phi)u &= (\partial_t + i\Phi)(\nabla - iA)u - (\nabla - iA)(\partial_t + i\Phi)u\\ &= \partial_t \nabla u + i\Phi\nabla u + A\Phi u - i\partial_t(Au)\\ &\quad - \nabla\partial_t u - i\nabla(\Phi u) - A\Phi u + iA\partial_t u\\ &= -i(\nabla\Phi + \partial_t A)u.\end{aligned}$$

and

$$\begin{aligned}(\partial_i^A \partial_j^A - \partial_j^A \partial_i^A)u &= (\partial_i - iA_i)(\partial_j - iA_j)u - (\partial_j - iA_j)(\partial_i - iA_i)u\\ &= \partial_i\partial_j u - iA_i\partial_j u - i\partial_i(A_j u) + A_i A_j\\ &\quad - (\partial_j\partial_i u - iA_j\partial_i u - i\partial_j(A_i u) + A_j A_i)\\ &= -iA_i\partial_j u - iA_j\partial_i u - iu\partial_i A_j\\ &\quad - (-iA_j\partial_i u - iA_i\partial_j u - iu\partial_j A_i)\\ &= -iu(\partial_i A_j - \partial_j A_i).\end{aligned}$$

□

A.2. **Static identities.** To simplify our notation, we define some more quantities.

The following quantities contain the gauged time derivative ∂_Φ: the *charge* q,

$$q(u) = (iu, \partial_\Phi u)$$

and the *"supermomentum"* p given by

$$p(u) = (\partial_\Phi u, \nabla_A u).$$

We collect some identities for these quantities by simple calculation:

$$\operatorname{div} p = (\nabla_A \partial_\Phi u, \nabla_A u) + (\partial_\Phi u, (\nabla_A)^2 u)$$

so after using the commutator relation (A.3),

$$\operatorname{div} p = (\partial_\Phi \nabla_A u, \nabla_A u) + (\partial_\Phi u, (\nabla_A)^2 u) - E \cdot j_A. \tag{A.6}$$

The time-derivative of the energy density is given by

$$\partial_t g_\varepsilon = (\partial_\Phi \nabla_A u, \nabla_A u) - \frac{1}{\varepsilon^2}(1-|u|^2)(\partial_\Phi u, u) + \frac{1}{2}\partial_t |h - h_{ex}|^2$$

so (A.6) shows

$$\partial_t g_\varepsilon = \operatorname{div} p - (\partial_\Phi u, (\nabla_A)^2 u + \frac{1}{\varepsilon^2}(1-|u|^2)u) + \partial_t \frac{1}{2}|h-h_{ex}|^2 + E \cdot j_A \tag{A.7}$$

or

$$\partial_t g_\varepsilon = \operatorname{div} p - (\partial_\Phi u, R) + \partial_t \frac{1}{2}|h-h_{ex}|^2 + E \cdot j_A. \tag{A.8}$$

For the negative L^2 gradient of the energy with respect to u, we use the abbreviation R, so

$$R = -\nabla_{L^2} G_\varepsilon = (\nabla_A)^2 u + \frac{1}{\varepsilon^2} u (1-|u|^2). \tag{A.9}$$

The time-derivative of the supercurrent, $\partial_t j_A$, can be calculated as follows:

$$\partial_t (iu, \nabla_A u) = (i\partial_\Phi u, \nabla_A u) + (iu, \partial_\Phi \nabla_A u) \tag{A.10}$$

so after using (A.3),

$$\partial_t j_A = (i\partial_\Phi u, \nabla_A u) + (iu, \nabla_A \partial_\Phi u) + |u|^2 E. \tag{A.11}$$

For the gradient of the charge density we obtain

$$\nabla q = \nabla (iu, \partial_\Phi u) = (i\nabla_A u, \partial_\Phi u) + (iu, \nabla_A \partial_\Phi u) \tag{A.12}$$

so combining (A.11) and (A.12) yields

$$\partial_t j_A = 2(i\partial_\Phi u, \nabla_A u) + \nabla q + |u|^2 E \tag{A.13}$$

and

$$\partial_t J_A = \operatorname{curl}(i\partial_\Phi u, \nabla_A u) + \frac{1}{2}\operatorname{curl}(\partial_t A + |u|^2 E). \tag{A.14}$$

Using $\partial_t A = -E - \nabla\Phi$, we rewrite this as

$$\partial_t J_A = \operatorname{curl}(i\partial_\Phi u, \nabla_A u) - \frac{1}{2}\operatorname{curl}((1-|u|^2)E). \tag{A.15}$$

We define the *stress-energy tensor* T as

$$\mathsf{T}_{ij}(u, A) := (\partial_i^A u, \partial_j^A u) - \frac{1}{2}\delta_{ij}\left(|\nabla_A u|^2 - |h|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2\right). \tag{A.16}$$

Proposition A.3. *We have the following identities relating the stress-energy tensor to other quantities:*

$$\operatorname{div} \mathsf{T}(u, A) = (R, \nabla_A u) + h(-j_A^\perp + \nabla h) \tag{A.17}$$

$$\operatorname{curl}\operatorname{div} \mathsf{T} = \operatorname{curl}\operatorname{div}(\nabla_A u \otimes \nabla_A u) = \operatorname{curl}(R, \nabla_A u) - \operatorname{curl}(j_A^\perp h) \tag{A.18}$$

In particular, the stress-energy tensor is divergence free for solutions of (1.29)–(1.30).

Proof. We calculate

$$
\begin{aligned}
(\operatorname{div} \mathsf{T})_j = \partial_i \mathsf{T}_{ij} &= \big((\partial_i^A)^2 u, \partial_j^A u\big) + \big(\partial_i^A u, \partial_i^A \partial_j^A u\big) \\
&\quad - \partial_j \frac{1}{2}\left(\partial_i^A u, \partial_i^A u\right) + \partial_j \frac{1}{2}|h|^2 - \partial_j \frac{1}{4\varepsilon}^2 (1-|u|^2)^2 \\
= \bigg(((\nabla_A)^2 u + \frac{1}{\varepsilon^2} u(1-|u|^2), \partial_j^A u\bigg) &+ \big((\partial_i^A \partial_j^A - \partial_j^A \partial_i^A) u, \partial_i^A u\big) + \partial_j \frac{1}{2}|h|^2.
\end{aligned}
$$

Now by the commutator relation (A.5) and using the shortcut (A.9),

$$
(\operatorname{div} \mathsf{T})_j = (R, \partial_j^A u) - \big(i(\partial_i A_j - \partial_j A_i) u, \partial_i^A u\big) + \partial_j \frac{1}{2}|h|^2. \tag{A.19}
$$

Now

$$
\partial_i A_j - \partial_j A_i = h \mathbb{J}_{ij}, \tag{A.20}
$$

where $\mathbb{J} = (\mathbb{J}_{ij})$ is the symplectic matrix

$$
\mathbb{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{A.21}
$$

so for any vector $v \in \mathbb{R}^2$, $\mathbb{J}v = -v^{\perp}$. In particular, using $\mathbb{J}_{ij} = -\mathbb{J}ji$,

$$
\operatorname{div} \mathsf{T} = (R, \nabla_A u) - h(iu, \nabla_A u)^{\perp} + \frac{1}{2}\nabla |h|^2, \tag{A.22}
$$

so (A.17) follows. To show the first equality in (A.18), it suffices to check (A.16) and note that the gradient terms disappear when taking the curl. ☐

A.3. **Conservation laws.** In this section we derive some conservation laws for solutions of the equation (1.29).

Testing the equation (1.29) with iu, we obtain

$$
\alpha(iu, \partial_\Phi u) + \beta(iu, i\partial_\Phi u) = (iu, (\nabla_A)^2 u).
$$

We note that $\operatorname{div} j_A(u) = (iu, (\nabla_A)^2 u)$ and $\partial_t \frac{1}{2}(|u|^2 - 1) = (u, \partial_t u)$ so

$$
\operatorname{div} j_A = \alpha q + \beta \partial_t \frac{1}{2}(|u|^2 - 1). \tag{A.23}
$$

From (A.8) the energy evolution gives us

$$
\partial_t G_\varepsilon(u_\varepsilon, A_\varepsilon) = -\alpha \int_\Omega |\partial_\Phi u|^2 + \partial_t \frac{1}{2}|h - h_{ex}|^2 + E \cdot j_A.
$$

We note that, using (1.18) and (1.19),

$$
\partial_t \frac{1}{2}|h - h_{ex}|^2 = (h - h_{ex})\partial_t h = -(h - h_{ex}) \operatorname{curl} E,
$$

and now we can express j_A using (1.30),

$$
\begin{aligned}
\partial_t \frac{1}{2}|h - h_{ex}|^2 + E \cdot j_A &= -(h - h_{ex}) \operatorname{curl} E - E \cdot \nabla^{\perp} h - \sigma |E|^2 \\
&= -\sigma |E|^2 - \operatorname{curl}(E(h - h_{ex})).
\end{aligned}
$$

We combine this into

$$\partial_t g_\varepsilon(u_\varepsilon, A_\varepsilon) = -\alpha|\partial_\Phi u|^2 - \sigma|E|^2 + \operatorname{div} p - \operatorname{curl}(E(h - h_{ex})),$$

and using the boundary conditions we obtain

$$\partial_t G_\varepsilon(u_\varepsilon, A_\varepsilon) = -\alpha \int_\Omega |\partial_\Phi u|^2 - \sigma \int_\Omega |E|^2. \tag{A.24}$$

For completeness the divergence of (1.30) is

$$\operatorname{div} \sigma E = -\operatorname{div} j_A = -\alpha q - \partial_t \frac{1}{2}(|u|^2 - 1),$$

and the curl of (1.30) is

$$\operatorname{curl} \sigma E = -\Delta h + h - 2J_A(u). \tag{A.25}$$

We now state a result that combines motion laws for energy density and Jacobian. Using (1.29),

$$\partial_\Phi u = \frac{\alpha - i\beta}{\alpha^2 + \beta^2} R,$$

so (A.8) gives

$$\begin{aligned}\partial_t g_\varepsilon = \frac{\alpha}{\alpha^2+\beta^2} \operatorname{div}(R, \nabla_A u) - \frac{\beta}{\alpha^2+\beta^2} \operatorname{div}(iR, \nabla_A u) \\ - \alpha|\partial_\Phi u|^2 - \sigma|E|^2 - \operatorname{curl}(E(h - h_{ex})).\end{aligned}$$

Similarly, (A.15) yields

$$\partial_t J_A = \frac{\beta}{\alpha^2+\beta^2} \operatorname{curl}(R, \nabla_A u) + \frac{\alpha}{\alpha^2+\beta^2} \operatorname{curl}(iR, \nabla_A u) - \frac{1}{2} \operatorname{curl}((1 - |u|^2 E),$$

so we obtain for test functions ψ and ϕ the evolution

$$\begin{aligned}\partial_t \int_\Omega (\alpha\psi g_\varepsilon + \beta\phi J_A) = {} & -\alpha^2 \int_\Omega \psi|\partial_\Phi u|^2 - \alpha\sigma \int_\Omega \psi|E|^2 \\ & - \frac{1}{\alpha^2+\beta^2} \int_\Omega \left(\alpha^2 \nabla\psi + \beta^2 \nabla^\perp\phi\right) \cdot (R, \nabla_A u) \\ & + \frac{\alpha\beta}{\alpha^2+\beta^2} \int_\Omega (\nabla\psi - \nabla^\perp\phi) \cdot (iR, \nabla_A u) \\ & + \alpha \int_\Omega \nabla^\perp\psi \cdot E(h - h_{ex}) + \frac{\beta}{2} \int_\Omega (1 - |u|^2)\nabla^\perp\phi \cdot E.\end{aligned}$$

Using (A.18), we can also write

$$-\int_\Omega \nabla^\perp\phi \cdot (R, \nabla_A u) = \int_\Omega \nabla\nabla^\perp\phi : (\nabla_A u \otimes \nabla_A u) - h\nabla^\perp\phi \cdot \left(j_A^\perp - \nabla h\right), \tag{A.26}$$

and using (1.30), $j_A^\perp - \nabla h = \sigma E^\perp$, so we can rewrite the only term contributing to the motion law in the $\varepsilon \to 0$ limit as

$$-\int_\Omega \nabla^\perp\phi \cdot (R, \nabla_A u) = \int_\Omega \nabla\nabla^\perp\phi : (\nabla_A u \otimes \nabla_A u) - \sigma h \nabla\phi \cdot E, \tag{A.27}$$

as in the statement of Proposition 3.7.

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