GINZBURG-LANDAU VORTICES DRIVEN BY THE LANDAU-LIFSHITZ-GILBERT EQUATION

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Abstract. A simplified model for the energy of the magnetization of a thin ferromagnetic film gives rise to a version of the theory of Ginzburg-Landau vortices for sphere-valued maps. In particular we have the development of vortices as a certain parameter tends to 0. The dynamics of the magnetization is ruled by the Landau-Lifshitz-Gilbert equation, which combines characteristic properties of a nonlinear Schrödinger equation and a gradient flow. This paper studies the motion of the vortex centers under this evolution equation.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be an open set. For $\epsilon > 0$, we study the functionals

$$E_\epsilon(\boldsymbol{m}) = \frac{1}{2}\int_\Omega \left(|\nabla \boldsymbol{m}|^2 + \frac{m_3^2}{\epsilon^2} \right) dx,$$

defined on the space $H^1(\Omega; S^2)$, which consists of all maps $\boldsymbol{m}$ in the Sobolev space $H^1(\Omega; \mathbb{R}^3)$ satisfying the constraint $|\boldsymbol{m}| = 1$ almost everywhere. We interpret this functional as a model for the energy associated to the magnetization of a ferromagnetic sample in the shape of a thin film. It is explained in Section 7 how the model is obtained from the theory of micromagnetics.

The dynamics of the magnetization is described by the Landau-Lifshitz-Gilbert equation. For the energy functional E_ϵ, this equation takes the form

$$(1) \quad \frac{\partial \boldsymbol{m}}{\partial t} + \boldsymbol{m} \times \left(\Delta \boldsymbol{m} - \frac{m_3}{\epsilon^2}\boldsymbol{e}_3 \right) + \alpha_\epsilon \boldsymbol{m} \times \left(\boldsymbol{m} \times \left(\Delta \boldsymbol{m} - \frac{m_3}{\epsilon^2}\boldsymbol{e}_3 \right)\right) = 0 \quad \text{in } \Omega \times (0,\infty),$$

where $\boldsymbol{e}_3 = (0,0,1)$, the symbol $\times$ denotes the vector product in $\mathbb{R}^3$, and $\alpha_\epsilon > 0$ is a constant. Note that the expression $\Delta \boldsymbol{m} - \frac{m_3}{\epsilon^2}\boldsymbol{e}_3$ in this equation is minus the L^2-gradient of E_ϵ. Writing

$$\boldsymbol{f}_\epsilon(\boldsymbol{m}) = \Delta \boldsymbol{m} + |\nabla \boldsymbol{m}|^2 \boldsymbol{m} - \frac{1}{\epsilon^2}(m_3 \boldsymbol{e}_3 - m_3^2 \boldsymbol{m})$$

for its orthogonal projection onto the tangent space of the unit sphere S^2, we can rewrite the Landau-Lifshitz-Gilbert equation as

$$(2) \quad \frac{\partial \boldsymbol{m}}{\partial t} = \alpha_\epsilon \boldsymbol{f}_\epsilon(\boldsymbol{m}) - \boldsymbol{m} \times \boldsymbol{f}_\epsilon(\boldsymbol{m}) \quad \text{in } \Omega \times (0,\infty).$$

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More details about this equation are given in Section 7.

In this paper we study the asymptotic behavior of equation (2) as ϵ tends to 0. In this context, we can first observe a similarity between the functionals E_ϵ and the Ginzburg-Landau functionals

$$F_\epsilon(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{4\epsilon^2}(1 - |u|^2)^2 \right) dx, \quad u \in H^1(\Omega; \mathbb{C}),$$

which have a certain relevance in superconductivity and the theory of superfluids and which have been studied in great detail since the seminal work of Bethuel, Brezis, and Hélein [4]. Indeed, in both cases we have the sum of the Dirichlet energy and a lower order term, and the latter can be thought of as a penalizing term that favors values of $\boldsymbol{m}$ or u close to a unit circle. Moreover, it was shown by Hang and Lin [16] that some of the tools from the theory of Ginzburg-Landau vortices can be used for the study of minimizers of E_ϵ. The characteristic feature of the theory is the development of vortices in the limit and the concentration of the energy at their centers.

The analog of the Landau-Lifshitz-Gilbert equation in the classical Ginzburg-Landau theory is the equation

$$\frac{\partial u}{\partial t} = (\alpha_\epsilon - i) \left(\Delta u + \frac{1}{\epsilon^2}(1 - |u|^2)u \right) \quad \text{in } \Omega \times (0, \infty).$$

When we study the asymptotic behavior of this equation as $\epsilon \searrow 0$, then we obtain a time-dependent set of vortices in the limit, and it is natural to ask how they move. The answer to this question will of course depend on the values of α_ϵ. Perhaps the most interesting case is when α_ϵ decreases logarithmically with ϵ. Henceforth we consider only the case

$$\alpha_\epsilon = \frac{1}{\log \frac{1}{\epsilon}}.$$

In this situation, the problem for the functionals F_ϵ has been studied independently by Miot [30] and by the authors [25], and a motion law for the vortices was obtained in the form of a system of ordinary differential equations. In this paper we derive a similar result for equation (2). Although it is related to a study of conservation laws by Lin and Shatah [28] for the undamped problem, this is the first rigorous result of this type for the Landau-Lifshitz-Gilbert equation.

Before we can give a rigorous statement of the main result, we need to describe the conditions under which we study the equation. It is also convenient to introduce some notation at this point.

Throughout the paper, we assume that Ω is bounded and simply connected, and its boundary $\partial\Omega$ is smooth. We fix a smooth map $\boldsymbol{g} : \partial\Omega \to S^1 \times \{0\}$, which we will use to prescribe the boundary data. Since $\partial\Omega$ can be identified with S^1, the map $\boldsymbol{g}$ has a well-defined degree d, and we assume that $d \geq 1$.

It will often be necessary to decompose a given map $\boldsymbol{m} : \Omega \to S^2$ into its projection onto $\mathbb{R}^2 \times \{0\}$ and its third component. In this case, the notation $\boldsymbol{m} = (m, m_3)$ is useful, where $m = (m_1, m_2)$. We consider a number of quantities associated to $\boldsymbol{m}$, among them the energy density

$$e_\epsilon(\boldsymbol{m}) = \frac{1}{2}|\nabla \boldsymbol{m}|^2 + \frac{m_3^2}{2\epsilon^2}$$

and the magnetic vorticity

$$\omega(\boldsymbol{m}) = \left\langle \boldsymbol{m}, \frac{\partial \boldsymbol{m}}{\partial x_1} \times \frac{\partial \boldsymbol{m}}{\partial x_2} \right\rangle.$$

The notation $\langle \cdot\,, \cdot \rangle$ stands for the scalar product in $\mathbb{R}^3$. We write $(\cdot\,, \cdot)$ for the scalar product in $\mathbb{R}^2$ when it is identified with the subspace $\mathbb{R}^2 \times \{0\}$ of the target space $\mathbb{R}^3$, whereas we use a dot for the scalar product in the domain. Whenever it is convenient, we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$. Thus the next quantity that we define,

$$j(m) = (im, \nabla m),$$

could also be represented as the third component of $\boldsymbol{m} \times \nabla \boldsymbol{m}$. Finally, we consider the Jacobian

$$J(m) = \frac{1}{2} \operatorname{curl} j(m) = \frac{\partial m_1}{\partial x_1}\frac{\partial m_2}{\partial x_2} - \frac{\partial m_1}{\partial x_2}\frac{\partial m_2}{\partial x_1}.$$

In the classical theory of Ginzburg-Landau vortices (for the functionals F_ϵ), the corresponding energy density or the Jacobian $J(u)$ can be used to identify the vortices. In the situation that we study here, the vorticity $\omega(\boldsymbol{m})$ takes the place of the Jacobian. Nevertheless, it is still useful to consider $J(m)$, especially when we work with the projection of $\boldsymbol{m}$ onto $\mathbb{R}^2 \times \{0\}$.

Note that the vorticity $\omega(\boldsymbol{m})$ is also a geometric quantity; indeed it can be regarded as the Jacobian of $\boldsymbol{m}$ as a map into S^2. Let $\mathcal{H}^2$ denote the two-dimensional Hausdorff measure and for $\boldsymbol{y} \in S^2$, let $N(\boldsymbol{y}; \boldsymbol{m})$ be the multiplicity function that counts the number of points in the preimage $\boldsymbol{m}^{-1}(\{\boldsymbol{y}\})$, with a positive or negative sign depending on whether $\boldsymbol{m}$ preserves or reverses the orientation near $\boldsymbol{y}$. Then we have

$$\int_\Omega \omega(\boldsymbol{m})\, dx = \int_{S^2} N(\boldsymbol{y}; \boldsymbol{m})\, d\mathcal{H}^2(\boldsymbol{y})$$

by the area formula. If $\boldsymbol{m}$ is smooth, then up to a multiple of 4π this number depends only on the boundary values of $\boldsymbol{m}$ by standard results from degree theory.

We are interested in a family of maps $\boldsymbol{m}_\epsilon \in H^1(\Omega; S^2)$ that satisfy the Dirichlet boundary conditions $\boldsymbol{m}_\epsilon|_{\partial\Omega} = \boldsymbol{g}$. Moreover, we fix a constant C_0 and we require that

$$E_\epsilon(\boldsymbol{m}_\epsilon) \leq \pi d \log \frac{1}{\epsilon} + C_0.$$

In view of the standard estimates in the theory of Ginzburg-Landau vortices [4, 16], this means that the energy of $\boldsymbol{m}_\epsilon$ differs from the least possible energy for these boundary data at most by a fixed constant. The reasons for working with Dirichlet boundary data are mostly technical, although there is some justification for this on physical grounds (discussed in more detail in Section 7). The advantage of Dirichlet boundary conditions is that they permit rigorous results without too many technicalities. We expect, however, that many of our tools will be useful under different conditions, even though it will be necessary to combine them with further arguments.

Under the above conditions, we typically expect, at least for some subsequence, a convergence of the renormalized energy densities $\alpha_\epsilon e_\epsilon(\boldsymbol{m}_\epsilon)$ and the vorticities $\omega(\boldsymbol{m}_\epsilon)$ to a weighted sum of Dirac measures located at the vortex centers. Given a point $a \in \Omega$, we write δ_a for the Dirac measure centered at a. If we have a set of d points $a_1, \ldots, a_d \in \Omega$, then we write $a = (a_1, \ldots, a_d) \in \Omega^d$ and we use the notation

$$\delta_a = \sum_{\ell=1}^{d} \delta_{a_\ell}.$$

Then the typical convergence for the energy density is

$$\alpha_\epsilon e_\epsilon(\boldsymbol{m}_\epsilon) \to \pi \delta_a,$$

and for the vorticity we expect something similar, for example

$$\omega(\boldsymbol{m}_\epsilon) \to 2\pi \delta_a,$$

in the sense of distributions in Ω, where $a \in \Omega^d$ satisfies $a_k \neq a_\ell$ for $k \neq \ell$. Thus we have vortices at $a_1, \ldots, a_d$, each of degree 1. We do not claim that the limits always exists, but it is useful to imagine this situation as a motivation for the next definitions. We will make some rigorous statements about the convergence of $\alpha_\epsilon e_\epsilon(\boldsymbol{m}_\epsilon)$ and $\omega(\boldsymbol{m}_\epsilon)$ later.

When we examine how the energy behaves as $\epsilon \searrow 0$, it is convenient to split it into several parts. The vortices give rise to an energy of $\pi d \log \frac{1}{\epsilon}$. If we define

$$I_\epsilon = \inf \left\{ \int_{B_1(0)} e_\epsilon(\boldsymbol{m})\, dx : \ \boldsymbol{m}(x) = (x, 0) \text{ on } \partial B_1(0) \right\},$$

then it is easily verified that the quantity $I_\epsilon - \pi \log \frac{1}{\epsilon}$ is non-decreasing in ϵ, and therefore the limit

$$\gamma = \lim_{\epsilon \searrow 0} \left(I_\epsilon - \pi \log \frac{1}{\epsilon} \right)$$

exists. This is the energy of a vortex core, and as we have d vortices, we obtain the energy $d\gamma$. The vortex positions give rise to the so-called reduced or renormalized energy, given by a function W that depends only on $a_1, \ldots, a_d$. In order to define W, we consider the unique map $\boldsymbol{m}_*(\,\cdot\,; a) : \Omega \backslash \{a_1, \ldots, a_d\} \to S^1 \times \{0\}$ such that

$$m_*(z; a) = e^{i\theta(z)} \prod_{\ell=1}^{d} \frac{z - a_\ell}{|z - a_\ell|} \tag{3}$$

for a function $\theta \in C^\infty(\overline{\Omega})$ with $\Delta\theta = 0$ in Ω and such that $\boldsymbol{m}_*|_{\partial\Omega} = \boldsymbol{g}$. We define

$$\Omega_r(a) = \Omega \backslash \bigcup_{\ell=1}^{d} \overline{B_r(a_\ell)}$$

and

$$W(a) = \lim_{r \searrow 0} \left(\frac{1}{2} \int_{\Omega_r(a)} |\nabla \boldsymbol{m}_*(x; a)|^2\, dx - \pi d \log \frac{1}{r} \right).$$

The sum of these three energy contributions,

$$W_\epsilon(a) = \pi d \log \frac{1}{\epsilon} + d\gamma + W(a),$$

gives asymptotically the energy that is necessary to develop vortices at $a_1, \dots, a_d$, as we will see later. Accordingly, the difference between the actual energy and this quantity,

$$D_\epsilon(\boldsymbol{m}; a) = E_\epsilon(\boldsymbol{m}) - W_\epsilon(a),$$

is called the energy excess.

We now consider equation (2) with Dirichlet boundary conditions given by $\boldsymbol{g}$ and with initial data $\boldsymbol{m}_\epsilon^0 \in C^\infty(\overline{\Omega}; S^2)$ such that $\boldsymbol{m}_\epsilon^0|_{\partial\Omega} = \boldsymbol{g}$ and

$$E_\epsilon(\boldsymbol{m}_\epsilon^0) \le \pi d \log \frac{1}{\epsilon} + C_0.$$

Moreover, we assume that there exists an $a^0 \in \Omega^d$ with $a_k^0 \neq a_\ell^0$ for $k \neq \ell$ such that

$$\alpha_\epsilon e_\epsilon(\boldsymbol{m}_\epsilon^0) \to \pi \delta_{a^0} \quad \text{and} \quad \omega(\boldsymbol{m}_\epsilon^0) \to 2\pi \delta_{a^0}$$

in the sense of distributions, and finally,

$$\lim_{\epsilon \searrow 0} D_\epsilon(\boldsymbol{m}_\epsilon^0; a^0) = 0.$$

The last condition means that the initial data have just enough energy to develop the vortices at the points $a_1^0, \dots, a_d^0$.

We claim that the system of ordinary differential equations

$$\pi(1+2i) a_\ell' = -\frac{\partial}{\partial a_\ell} W(a), \quad \ell = 1, \dots, d, \tag{4}$$

describes the motion of the vortices. It follows from the theory of Ginzburg-Landau vortices [4] that $W(a) \to \infty$ if either $|a_k - a_\ell| \to 0$ for $k \neq \ell$ or $\operatorname{dist}(a_\ell, \partial\Omega) \to 0$. Since W is a Lyapunov function for (4), a solution of the initial value problem with $a(0) = a^0$ exists for all times.

Theorem 1.1. *There exists a number $\epsilon_0 > 0$ such that for every $\epsilon \in (0, \epsilon_0]$, there is a smooth solution $\boldsymbol{m}_\epsilon \in C^\infty(\overline{\Omega} \times [0,T); S^2)$ of (2) with $\boldsymbol{m}_\epsilon(\,\cdot\,, 0) = \boldsymbol{m}_\epsilon^0$ and $\boldsymbol{m}_\epsilon(\,\cdot\,, t)|_{\partial\Omega} = \boldsymbol{g}$ for every $t \ge 0$. Let $a \in C^\infty([0,\infty); \Omega^d)$ be the solution of (4) with $a(0) = a^0$. Then for every $t \in [0, \infty)$,*

$$\alpha_\epsilon e_\epsilon(\boldsymbol{m}_\epsilon(\,\cdot\,, t)) \to \pi \delta_{a(t)} \quad \text{and} \quad \omega(\boldsymbol{m}_\epsilon(\,\cdot\,, t)) \to 2\pi \delta_{a(t)}$$

as $\epsilon \searrow 0$ in the sense of distributions.

The proof of Theorem 1.1 follows the same strategy as the proofs for the corresponding results for F_ϵ in the aforementioned papers [30, 25]. These in turn make use of certain ideas developed originally for the gradient flow

$$\frac{\partial u}{\partial t} = \alpha_\epsilon \left(\Delta u + \frac{1}{\epsilon^2}(1 - |u|^2) u \right)$$

and the Gross-Pitaevsky equation

$$\frac{\partial u}{\partial t} = -i \left(\Delta u + \frac{1}{\epsilon^2}(1 - |u|^2) u \right),$$

in particular from Sandier and Serfaty [38, 37], Colliander and Jerrard [7, 8], Lin and Xin [29], and Jerrard and Spirn [19, 20], in addition to the standard tools from the theory of Ginzburg-Landau vortices. Equation (2), however, requires some further analysis, owing mostly to the different geometry of the target (S^2 instead of $\mathbb{C}$). In particular, we have to overcome two difficulties.

1. The curvature of S^2 gives rise to the nonlinear term $|\nabla \boldsymbol{m}|^2 \boldsymbol{m}$ in the equation. As a consequence, solutions of (2) are not necessarily smooth, even for a fixed positive ϵ. It is only due to the vanishing energy excess that we obtain smooth solutions in Theorem 1.1.
2. The vorticity $\omega(\boldsymbol{m})$ is not globally a curl, in contrast to the Jacobian $J(m)$. From the technical point of view, this means that it is more difficult to control the vorticity. But this is in fact a sign of a deeper issue that will be discussed in more detail in Section 4.

In detail the proof is organized as follows. In Section 2 we prove several estimates on the blowup of the energy density for maps $\boldsymbol{m}$ with $\boldsymbol{m} = \boldsymbol{g}$ on $\partial\Omega$, assuming

$$E_\epsilon(\boldsymbol{m}) \leq \pi d \log \frac{1}{\epsilon} + C_0.$$

The penalization term $\frac{m_3^2}{\epsilon^2}$ in the energy induces the S^2 vector field $\boldsymbol{m}$ to be mostly planar. Due to the topological constraint on the boundary, this leads to the creation of d discrete vortices in the planar direction, each of which carries $\pi \log \frac{1}{\epsilon}$ energy. In particular in Theorem 2.1 we prove estimates that lead almost immediately to a subsequence with $\alpha_{\epsilon_k} e_{\epsilon_k}(\boldsymbol{m}_{\epsilon_k}) \to \pi \delta_a$, which generalizes the result of Hang and Lin [16] to non-minimizing sequences. We prove this by embedding the S^2-valued map into $\mathbb{R}^3$ and studying the gradient flow of the relaxed energy density

$$e_\epsilon(\boldsymbol{m}) = \frac{1}{2} |\nabla \boldsymbol{m}|^2 + \frac{m_3^2}{2\epsilon^2} + \frac{(1 - |\boldsymbol{m}|^2)^2}{4\epsilon^2}.$$

We can then use machinery from Ginzburg-Landau theory and the harmonic map heat flow to achieve lower bounds.

Section 3 establishes the compactness and limiting behavior of the vorticity $\omega(\boldsymbol{m})$. In order to prove this we first define a modified vorticity, similar to the modified Jacobian of Alberti, Baldo and Orlandi [1], which is more concentrated at the site of "bubbling" of hemispheres and full spheres. The vorticity represents the local area element of the covering of S^2. Since $\boldsymbol{m}$ is mostly planar, the limiting vorticity will be concentrated at certain points and there is a quantization related to the topological degree. At the vortices we have naturally a covering of a hemisphere, which corresponds to a concentration of the vorticity of 2π. We are able to rule out the formation of extra bubbles, due to the well-preparedness of the initial data, since each full cover of S^2 carries energy on the order of at least 4π. The vorticity is then shown to converge to $2\pi\delta_a$.

Section 4 proves that for sequences of solutions to the Landau-Lifshitz-Gilbert equations, there exists a set of d curves $a_\ell(t) \in H^1([0,T];\Omega)$ such that the vorticity and the energy concentrate along $a_\ell(t)$. We also prove a crucial kinetic energy lower bound, which shows that the kinetic energy of the vortex paths is bounded by the kinetic energy of $\boldsymbol{m}$ in the limit, namely

$$\pi \sum_{\ell=1}^{d} \int_0^T \left|a_\ell'\right|^2 \, dt \leq \liminf_{k\to\infty} \left(\alpha_{\epsilon_k} \int_0^T \int_\Omega \left| \frac{\partial \boldsymbol{m}_{\epsilon_k}}{\partial t} \right|^2 \, dx\, dt \right).$$

Section 5 contains the last technical tool needed for the proof. In particular we establish a measure of the strong H^1-convergence of $\boldsymbol{m}$ to $\boldsymbol{m}_* = \boldsymbol{m}_*(\,\cdot\,;a)$ away from the vortices. In particular we show that

$$\int_{\Omega_r(a)} \left(\tilde{e}_\epsilon(\boldsymbol{m}) + \frac{1}{4} \left| \frac{j(m)}{|m|} - j(m_*) \right|^2 \right) dx \leq D_\epsilon(\boldsymbol{m}; a) + o(1)$$

as $\epsilon \searrow 0$, where

$$\tilde{e}(\boldsymbol{m}) = \frac{1}{2} |\nabla |m||^2 + \frac{1}{2} |\nabla m_3|^2 + \frac{m_3^2}{2\epsilon^2}.$$

The energy excess can be controlled by energy conservation and the well-preparedness of the initial data.

In order to understand where the vortices move as time evolves, we look at the conservation laws. We have the conservation of vorticity

$$\frac{\partial}{\partial t} \omega(\boldsymbol{m}) = \operatorname{curl} \langle \boldsymbol{f}_\epsilon(\boldsymbol{m}), \nabla \boldsymbol{m} \rangle + \alpha_\epsilon \operatorname{curl} \langle \boldsymbol{m} \times \boldsymbol{f}_\epsilon(\boldsymbol{m}), \nabla \boldsymbol{m} \rangle \tag{5}$$

and the conservation of energy

$$\frac{\partial}{\partial t} e_\epsilon(\boldsymbol{m}) = \alpha_\epsilon \operatorname{div} \langle \boldsymbol{f}_\epsilon(\boldsymbol{m}), \nabla \boldsymbol{m} \rangle - \operatorname{div} \langle \boldsymbol{m} \times \boldsymbol{f}_\epsilon(\boldsymbol{m}), \nabla \boldsymbol{m} \rangle - \alpha_\epsilon |\boldsymbol{f}_\epsilon(\boldsymbol{m})|^2. \tag{6}$$

Since both $\alpha_\epsilon e_\epsilon(\boldsymbol{m}_\epsilon)$ and $\omega(\boldsymbol{m}_\epsilon)$ converge to delta functions, we can track the motion of the concentrations by testing against either measure with a test function $\varphi(x) = x\chi(x)$, where χ is a smoothed step function at the vortex center. Both (5) and (6) contain a difficult term $\langle \boldsymbol{m} \times \boldsymbol{f}_\epsilon(\boldsymbol{m}), \nabla \boldsymbol{m} \rangle$. We circumvent this difficulty by considering a combination of both conservation laws simultaneously and thereby achieving a partial cancellation of the difficult terms. We then compare the vortex paths a with the solutions $\hat{a}$ of the system of ordinary differential equations (4). We can use the difference $|a - \hat{a}|$ to control the growth of $D_\epsilon(\boldsymbol{m}_\epsilon; a)$. On the other hand, using the estimates from Section 5 we obtain a bound for $|a' - \hat{a}'|$ in terms of the energy excess. An argument involving the Gronwall lemma then implies that a and $\hat{a}$ coincide.

The final section is independent of the proof of Theorem 1.1. It gives some explanation of the underlying physical model and interpretations of the results.

2. Estimates for the energy

We first derive some tools that are independent of the Landau-Lifshitz-Gilbert equation. Instead, they rely mostly on the energy bound

$$E_\epsilon(\boldsymbol{m}) \leq \pi d \log \frac{1}{\epsilon} + C_0 \tag{7}$$

and in some cases on an L^2-control of $\boldsymbol{f}_\epsilon(\boldsymbol{m})$ as well, such as

$$\int_\Omega |\boldsymbol{f}_\epsilon(\boldsymbol{m})|^2 \, dx \leq \frac{C_0}{\epsilon}. \tag{8}$$

For $a \in \Omega^d$, we use the notation

$$\rho(a) = \min\left\{\min_{k\neq\ell}|a_k - a_\ell|, \frac{1}{2}\min_{\ell=1,\dots,d}\operatorname{dist}(a_\ell, \partial\Omega)\right\}.$$

We have the following estimates.

Theorem 2.1. *There exist two constants C and $\rho_0 > 0$, and for every $R \in (0, \rho_0]$ there exists a constant Γ such that the following holds true. Let $\epsilon \in (0,1]$ and $\boldsymbol{m} \in H^1(\Omega; S^2)$ with $\boldsymbol{m}|_{\partial\Omega} = \boldsymbol{g}$ and such that (7) is satisfied. Then*

$$\int_\Omega \left(|\nabla m_3|^2 + \frac{m_3^2}{\epsilon^2}\right) dx \leq C. \tag{9}$$

There exists an $a \in \Omega^d$ with $\rho(a) \geq \rho_0$ such that

$$\int_{\Omega_R(a)} e_\epsilon(\boldsymbol{m})\, dx \leq \Gamma \tag{10}$$

and

$$\left|\int_{B_R(a_\ell)} e_\epsilon(\boldsymbol{m})\, dx - \pi \log\frac{1}{\epsilon}\right| \leq \Gamma \tag{11}$$

for $\ell = 1, \dots, d$. Moreover, if (8) is satisfied, then

$$\int_\Omega m_3^2 |\nabla \boldsymbol{m}|^2\, dx \leq C. \tag{12}$$

The key for the proof of these inequalities is an idea due to Lin [27]. The strategy is to improve the properties of $\boldsymbol{m}$ using a gradient flow. We first extend E_ϵ to $H^1(\Omega; \mathbb{R}^3)$ by setting

$$e_\epsilon(\boldsymbol{m}) = \frac{1}{2}|\nabla \boldsymbol{m}|^2 + \frac{m_3^2}{2\epsilon^2} + \frac{(1 - |\boldsymbol{m}|^2)^2}{4\epsilon^2}$$

and

$$E_\epsilon(\boldsymbol{m}) = \int_\Omega e_\epsilon(\boldsymbol{m})\, dx.$$

Then

$$\boldsymbol{f}_\epsilon^{\#}(\boldsymbol{m}) = \Delta \boldsymbol{m} - \frac{1}{\epsilon^2}(m_3 \boldsymbol{e}_3 - (1 - |\boldsymbol{m}|^2)\boldsymbol{m})$$

is minus the L^2-gradient of E_ϵ in $H^1(\Omega; \mathbb{R}^3)$.

Lemma 2.2. *There exists a constant C such that for any $\boldsymbol{m} \in H^1(\Omega; S^2)$ satisfying $\boldsymbol{m}|_{\partial\Omega} = \boldsymbol{g}$ and inequality (7), there exists a map $\boldsymbol{M} \in C^\infty(\overline{\Omega}; \mathbb{R}^3)$ with $E_\epsilon(\boldsymbol{M}) \leq E_\epsilon(\boldsymbol{m})$ and $|\nabla \boldsymbol{M}| \leq C/\epsilon$ in Ω, such that*

$$\left\|\boldsymbol{f}_\epsilon^{\#}(\boldsymbol{M})\right\|_{L^2(\Omega)} \leq C, \tag{13}$$

and for every $\eta \in C^\infty(\overline{\Omega})$,

$$\int_\Omega \eta^2 (e_\epsilon(\boldsymbol{m}) - e_\epsilon(\boldsymbol{M}))\, dx \leq C\|\eta\|_{C^1(\overline{\Omega})}\left(\|\eta\|_{C^1(\overline{\Omega})} + \left(\int_\Omega \eta^2 e_\epsilon(\boldsymbol{M})\, dx\right)^{1/2}\right). \tag{14}$$

Proof. Consider the gradient flow for E_ϵ in $H^1(\Omega;\mathbb{R}^3)$, which is given by the equation

$$\frac{\partial \tilde{\boldsymbol{m}}}{\partial t} = \boldsymbol{f}_\epsilon^\#(\tilde{\boldsymbol{m}}) \quad \text{in } \Omega \times (0,1),$$

with boundary conditions

$$\begin{aligned} \tilde{\boldsymbol{m}} &= \boldsymbol{m} \quad \text{on } \Omega \times \{0\}, \\ \tilde{\boldsymbol{m}} &= \boldsymbol{g} \quad \text{on } \partial\Omega \times (0,1). \end{aligned}$$

This problem has a solution $\tilde{\boldsymbol{m}} \in C^\infty(\overline{\Omega} \times (0,1];\mathbb{R}^3)$ with $|\nabla \tilde{\boldsymbol{m}}| \leq C_1/\epsilon$ in $\overline{\Omega} \times [\frac{1}{2}, 1]$ for a constant C_1 that depends only on Ω and $\boldsymbol{g}$. Moreover,

$$E_\epsilon(\tilde{\boldsymbol{m}}(\,\cdot\,,\tau)) + \int_0^\tau \int_\Omega \left|\frac{\partial \tilde{\boldsymbol{m}}}{\partial t}\right|^2 dx\,dt = E_\epsilon(\boldsymbol{m})$$

for every $\tau \in (0,1)$. Multiplying the third component of $\boldsymbol{f}_\epsilon^\#(\tilde{\boldsymbol{m}})$ with $\tilde{m}_3$, we derive the inequality

$$\frac{\partial}{\partial t}\tilde{m}_3^2 \leq \Delta \tilde{m}_3^2 + \frac{2}{\epsilon^2}(1 - |\tilde{\boldsymbol{m}}|^2)\tilde{m}_3^2.$$

As $1 - |\tilde{\boldsymbol{m}}|^2 \leq 0$ when $\tilde{m}_3^2 \geq 1$, the maximum principle implies $\tilde{m}_3^2 \leq 1$ in $\overline{\Omega} \times [0,1]$. Hence

$$e_\epsilon(\tilde{\boldsymbol{m}}) \geq \frac{1}{2}|\nabla \tilde{m}|^2 + \frac{1}{2\epsilon^2}\tilde{m}_3^2 + \frac{1}{4\epsilon^2}(1 - |\tilde{m}|^2 - \tilde{m}_3^2)^2 \geq \frac{1}{2}|\nabla \tilde{m}|^2 + \frac{1}{8\epsilon^2}(1 - |\tilde{m}|^2)^2.$$

Thus by the theory of Ginzburg-Landau vortices [4, Chapter III], there exists a constant $C_2 = C_2(\Omega, \boldsymbol{g})$ with

$$E_\epsilon(\tilde{\boldsymbol{m}}(\,\cdot\,,t)) \geq \pi d \log\frac{1}{\epsilon} - C_2.$$

We conclude in particular that

$$\int_0^1 \int_\Omega \left|\frac{\partial \tilde{\boldsymbol{m}}}{\partial t}\right|^2 dx\,dt \leq C_0 + C_2.$$

Therefore, there exists a number $t_0 \in [\frac{1}{2}, 1]$ such that the map $\boldsymbol{M}(x) = \tilde{\boldsymbol{m}}(x, t_0)$ satisfies (13).

For $\eta \in C^\infty(\overline{\Omega})$, we have

$$\frac{d}{dt}\int_{\Omega\times\{t\}} \eta^2 e_\epsilon(\tilde{\boldsymbol{m}})\,dx = -\int_{\Omega\times\{t\}} \left(\eta^2 \left|\frac{\partial \tilde{\boldsymbol{m}}}{\partial t}\right|^2 + 2\eta\nabla\eta \cdot \left\langle \frac{\partial \tilde{\boldsymbol{m}}}{\partial t}, \nabla \tilde{\boldsymbol{m}} \right\rangle\right) dx.$$

Set

$$\phi(t) = \int_{\Omega\times\{t\}} \eta^2 e_\epsilon(\tilde{\boldsymbol{m}})\,dx \quad \text{and} \quad \psi(t) = \int_{\Omega\times\{t\}} \left|\frac{\partial \tilde{\boldsymbol{m}}}{\partial t}\right|^2 dx.$$

Then we have $\|\psi\|_{L^1(0,1)} \leq C_0 + C_2$ and

$$|\phi'(t)| \leq \|\eta\|_{C^1(\overline{\Omega})}^2 \psi(t) + 2\|\eta\|_{C^1(\overline{\Omega})}\sqrt{\psi(t)\phi(t)} \leq 2\|\eta\|_{C^1(\overline{\Omega})}^2 \psi(t) + \phi(t). \tag{15}$$

Using the second estimate in (15), we find a constant C_3, depending only on Ω, $\boldsymbol{g}$, and C_0, such that

$$\phi(t) \leq C_3\left(\|\eta\|_{C^1(\overline{\Omega})}^2 + \phi(t_0)\right)$$

for every $t \in (0,1)$. With the first inequality in (15), we then obtain (14). □

Obviously, the map $\boldsymbol{M}$ constructed here is easier to control than $\boldsymbol{m}$. For example, inequality (13) makes it possible to use an energy quantization result such as the following.

Lemma 2.3. *Let $R \in (0,1]$ and $x_0 \in \Omega$ such that $\Omega \cap B_R(x_0)$ is starshaped. Suppose that $\epsilon \in (0, \frac{R}{2}]$ and $\boldsymbol{m} \in H^1(\Omega;\mathbb{R}^3)$ with*

$$\int_{B_R(x_0)\cap\Omega} e_\epsilon(\boldsymbol{m})\,dx \le \lambda \log \frac{1}{\epsilon}$$

and

$$R^2 \int_{B_R(x_0)\cap\Omega} |\boldsymbol{f}_\epsilon^\#(\boldsymbol{m})|^2\,dx \le \lambda. \tag{16}$$

If λ is sufficiently small, then $|\boldsymbol{m}(x_0)| \ge \frac{3}{4}$ and $|m_3(x_0)| \le \frac{1}{4}$.

Proof. Note that

$$\int_0^R r \int_{B_r(x_0)\cap\Omega} |\boldsymbol{f}_\epsilon^\#(\boldsymbol{m})|^2\,dx\,dr = \frac{1}{2}\int_{B_R(x_0)\cap\Omega} (R^2 - |x-x_0|^2)|\boldsymbol{f}_\epsilon^\#(\boldsymbol{m})|^2\,dx.$$

Hence there exists a radius $r \in [\epsilon, R]$ such that

$$r\int_{\partial B_r(x_0)\cap\Omega} e_\epsilon(\boldsymbol{m})\,d\mathcal{H}^1 \le C_1\lambda$$

and

$$r^2\int_{B_r(x_0)\cap\Omega} |\boldsymbol{f}_\epsilon^\#(\boldsymbol{m})|^2\,dx \le C_1\alpha_\epsilon\lambda$$

for a universal constant C_1.

Let $x_1 \in B_R(x_0)\cap\Omega$ such that $B_R(x_0)\cap\Omega$ is starshaped about x_1. As $\boldsymbol{f}_\epsilon^\#(\boldsymbol{m}) \in L^2(B_R(x_0)\cap\Omega;\mathbb{R}^3)$, we have $\boldsymbol{m} \in H^2_{\mathrm{loc}}(B_R(x_0)\cap\Omega;\mathbb{R}^3)$. Thus we can compute

$$\begin{aligned}\operatorname{div}&\left(e_\epsilon(\boldsymbol{m})(x-x_1) - \langle (x-x_1)\cdot\nabla\boldsymbol{m}, \nabla\boldsymbol{m}\rangle\right)\\ &= \frac{1}{\epsilon^2}\left(m_3^2 + \frac{1}{2}(1-|\boldsymbol{m}|^2)^2\right) - \left\langle (x-x_1)\cdot\nabla\boldsymbol{m}, \boldsymbol{f}_\epsilon^\#(\boldsymbol{m})\right\rangle.\end{aligned}$$

An integration over $B_r(x_0)\cap\Omega$ gives rise to a Pohozaev identity, similarly as in the classical Ginzburg-Landau theory [4, Chapter III]. Eventually we obtain a universal constant C_2 with

$$\int_{B_r(x_0)\cap\Omega} (m_3^2 + (1-|\boldsymbol{m}|^2)^2)\,dx \le C_2\lambda\epsilon^2. \tag{17}$$

Using standard elliptic estimates and (16), and assuming that $\lambda \le 1$, we find a constant c, dependent only on Ω, $\boldsymbol{g}$, and R, such that

$$\operatorname*{osc}_{B_{c\epsilon}(x_0)\cap\Omega} \boldsymbol{m} \le \frac{1}{8}. \tag{18}$$

If λ is sufficiently small, then we immediately obtain the required inequalities from (17) and (18). □

Proof of Theorem 2.1. First note that

$$e_\epsilon(\boldsymbol{m}) = \frac{1}{2}|\nabla m|^2 + \frac{1}{4\epsilon^2}(1-|m|^2) + \frac{1}{2}|\nabla m_3|^2 + \frac{m_3^2}{4\epsilon^2}.$$

We have $1-|m|^2 \geq (1-|m|^2)^2$, and the theory of Ginzburg-Landau vortices [4, Chapter III] gives the inequality

$$\int_\Omega \left(\frac{1}{2}|\nabla m|^2 + \frac{1}{4\epsilon^2}(1-|m|^2)\right) dx \geq \pi d \log\frac{1}{\epsilon} - C_1$$

for a constant $C_1 = C_1(\Omega, \boldsymbol{g})$. Hence we have (9).

Let $\boldsymbol{M}$ be the map from Lemma 2.2. With the help of Lemma 2.3 and with standard arguments, such as in the book by Bethuel, Brezis, and Hélein [4], we see that there exist $a_1, \ldots, a_d$ such that $\rho(a) \geq \rho_0 = \rho_0(\Omega, \boldsymbol{g}, C_0)$ and

$$\int_{\Omega_R(a)} e_\epsilon(\boldsymbol{M})\, dx \leq \pi d \log\frac{1}{R} + C_2$$

for a constant $C_2 = C_2(\Omega, \boldsymbol{g}, C_0)$. Using inequality (14), we can prove (10). Inequality (11) then follows from standard estimates as well.

If we have (8), then we choose a radius $R \in [\rho_0/2, \rho_0]$ such that

$$\int_{\partial B_R(a_\ell)} e_\epsilon(\boldsymbol{m})\, d\mathcal{H}^1 \leq C_5 = C_5(\Omega, \boldsymbol{g}, C_0)$$

for every ℓ. Then we consider the product of the third component of $\boldsymbol{f}_\epsilon(\boldsymbol{m})$ with m_3. An integration by parts gives

$$\int_{B_R(a_\ell)} m_3^2|\nabla \boldsymbol{m}|^2\, dx = \int_{B_R(a_\ell)} m_3 f_{\epsilon 3}(\boldsymbol{m})\, dx + \frac{1}{\epsilon^2}\int_{B_R(a_\ell)} (m_3^2 - m_3^4)\, dx$$
$$+ \int_{B_R(a_\ell)} |\nabla m_3|^2\, dx - \int_{\partial B_R(a_\ell)} m_3 \frac{x-a_\ell}{|x-a_\ell|}\cdot \nabla m_3\, d\mathcal{H}^1.$$

We use the Hölder inequality for the first and the last term on the right-hand side, and then we have estimates for all of the resulting terms. This finally gives (12). □

Next we examine the functional D_ϵ. We want to show that it does in fact give a measure for the energy exceeding the amount that is needed for the development of vortices at the points $a_1, \ldots, a_d$. This is an S^2 target version of a result that is essentially due to Serfaty [40, 41]. We will also derive a more quantitative estimate for D_ϵ in Theorem 5.3 below.

Proposition 2.4. *For $\epsilon_k \searrow 0$, let $\boldsymbol{m}_k \in H^1(\Omega; S^2)$ with $\boldsymbol{m}_k|_{\partial\Omega} = \boldsymbol{g}$ and*

$$\alpha_{\epsilon_k} e_{\epsilon_k}(\boldsymbol{m}_k) \overset{*}{\rightharpoonup} \pi\delta_a$$

weakly in $(C_0^0(\Omega))^*$ for some $a \in \Omega^d$ with $\rho(a) > 0$. Then*

$$\liminf_{k\to\infty} D_{\epsilon_k}(\boldsymbol{m}_{\epsilon_k}; a) \geq 0.$$

In order to see that D_ϵ truly deserves the name "energy excess", we would also have to show that there exists a sequence developing vortices at $a_1, \ldots, a_d$ such that the reverse inequality holds. This can be done by a rather obvious construction (which is similar to the corresponding construction for the classical Ginzburg-Landau functionals [4, Chapter VIII]). Since this is not needed for the proof of Theorem 1.1, we omit the details.

Proof. We may assume that after the choice of a subsequence, there exists a constant C_1 such that

$$E_{\epsilon_k}(\boldsymbol{m}_k) \leq d\pi \log \frac{1}{\epsilon_k} + C_1$$

for every k, for otherwise there is nothing to prove. We first replace the given sequence by a subsequence such that the limit $\lim_{k\to\infty} D_{\epsilon_k}(\boldsymbol{m}_{\epsilon_k}; a)$ exists.

By a straightforward modification of the theory for the harmonic map heat flow due to Struwe [43] and Chang [6], the problem

$$\begin{aligned} \frac{\partial \tilde{\boldsymbol{m}}_k}{\partial t} &= \boldsymbol{f}_{\epsilon_k}(\tilde{\boldsymbol{m}}_k) \quad \text{in } \Omega \times (0,\infty), \\ \tilde{\boldsymbol{m}}_k &= \boldsymbol{m}_k \quad \text{on } \Omega \times \{0\}, \\ \tilde{\boldsymbol{m}}_k &= \boldsymbol{g} \quad \text{on } \partial\Omega \times (0,\infty), \end{aligned}$$

has a weak solution $\tilde{\boldsymbol{m}}_k \in L^\infty(0,\infty; H^1(\Omega; S^2))$ that satisfies the energy inequality

$$\int_0^T \int_\Omega \left| \frac{\partial \tilde{\boldsymbol{m}}_k}{\partial t} \right|^2 dx\, dt + E_{\epsilon_k}(\tilde{\boldsymbol{m}}_k(\,\cdot\,, T)) \leq E_{\epsilon_k}(\boldsymbol{m}_k)$$

for every k. Similarly as in the proof of Lemma 2.2, we conclude that

$$\int_0^\infty \int_\Omega |\boldsymbol{f}_{\epsilon_k}(\tilde{\boldsymbol{m}}_k)|^2 \, dx\, dt \leq C_2$$

for a constant C_2 that depends only on Ω, $\boldsymbol{g}$, and C_1. We can now choose a sequence of numbers $t_k \in (0, \sqrt{|\log \epsilon_k|})$ such that the maps $\hat{\boldsymbol{m}}_k(x) = \tilde{\boldsymbol{m}}(x, t_k)$ satisfy the inequality

$$\int_\Omega |\boldsymbol{f}_{\epsilon_k}(\hat{\boldsymbol{m}}_k)|^2 \, dx \leq C_2 \sqrt{\alpha_{\epsilon_k}}. \tag{19}$$

Using Theorem 2.1, we can select a subsequence such that

$$\alpha_{\epsilon_k} e_{\epsilon_k}(\hat{\boldsymbol{m}}_{\epsilon_k}) \overset{*}{\rightharpoonup} \pi \delta_{\hat{a}}$$

weakly* in $(C_0^0(\Omega))^*$ for some $\hat{a} \in \Omega^d$ with $\rho(\hat{a}) > 0$.

Let $\eta \in W^{1,\infty}(\Omega)$ and set

$$\phi_k(t) = \left(\int_{\Omega \times \{t\}} \eta^2 e_{\epsilon_k}(\tilde{\boldsymbol{m}}_{\epsilon_k}) \, dx \right)^{1/2}.$$

Then we have $\phi_k \in \mathrm{BV}(0,T)$ for every $T > 0$ and we compute

$$\phi_k'(t) \leq \|\nabla \eta\|_{L^\infty(\Omega)} \left(2 \int_{\Omega \times \{t\}} \left| \frac{\partial \tilde{\boldsymbol{m}}_{\epsilon_k}}{\partial t} \right|^2 dx \right)^{1/2}. \tag{20}$$

Let $R > 0$. By Theorem 2.1, we have

$$\limsup_{k\to\infty} \int_{\Omega_R(a)} e_{\epsilon_k}(\boldsymbol{m}_k) \, dx < \infty.$$

Using (20), we see that

$$\limsup_{k\to\infty} \left(\sqrt{\alpha_{\epsilon_k}} \int_{\Omega_R(a)} e_{\epsilon_k}(\hat{\boldsymbol{m}}_k) \, dx \right) < \infty.$$

We conclude that $\hat{a} = a$. Hence we have

$$\limsup_{k\to\infty} \int_{\Omega_R(a)} e_{\epsilon_k}(\hat{\boldsymbol{m}}_k)\,dx < \infty$$

for every $R > 0$. In particular, we may assume that $\hat{\boldsymbol{m}}_k$ converges weakly in $H^1_{\mathrm{loc}}(\Omega\backslash\{a_1,\dots,a_d\})$. Using (19) and the third component of the formula

$$\operatorname{div}(\boldsymbol{m}\times\nabla\boldsymbol{m}) = \boldsymbol{m}\times\boldsymbol{f}_\epsilon(\boldsymbol{m}) + \frac{m_3}{\epsilon^2}\boldsymbol{m}\times\boldsymbol{e}_3,$$

we see that the limit is harmonic, thus it must be $\boldsymbol{m}_*(\,\cdot\,;a)$.

Fix $R > 0$ again. For every $k \in \mathbb{N}$ there exists a radius $r_k \in [R, 2R]$ with

$$\int_{\partial B_{r_k}(a_\ell)} e_{\epsilon_k}(\hat{\boldsymbol{m}}_k)\,d\mathcal{H}^1 \le C_3, \quad \ell = 1,\dots,d,$$

for a constant C_3 that is independent of k. We may assume that $r_k \to r_0$, and then we have

$$\int_{\partial B_{r_k}(a)} |\hat{\boldsymbol{m}}_k - \boldsymbol{m}_*(\,\cdot\,;a)|^2\,d\mathcal{H}^1 \to 0$$

as $k \to \infty$. Recalling that m_* is of the form (3), we conclude that there exists a function $\sigma : (0,\infty) \to \mathbb{R}$ with $\sigma \to 0$ as $r \searrow 0$ such that

$$\liminf_{k\to\infty}\left(\int_{B_{r_k}(a_\ell)} e_{\epsilon_k}(\hat{\boldsymbol{m}}_k)\,dx - I_{\epsilon_k/r_k}\right) \ge \sigma(R), \quad \ell = 1,\dots,d.$$

Clearly we have

$$\liminf_{k\to\infty}\int_{\Omega_{r_k}(a)} e_{\epsilon_k}(\hat{\boldsymbol{m}}_k)\,dx \ge \frac{1}{2}\int_{\Omega_{r_0}(a)} |\nabla\boldsymbol{m}_*(x;a)|^2\,dx.$$

Letting $R \to 0$, we now obtain the desired inequality. □

3. Estimates for the vorticity

The previous results are useful to control the energy, but we also have to find out more about the vorticity. When we work with the energy density, then it is often convenient to identify it with an element of the dual space of $C^0_0(\Omega)$ or possibly $C^0(\overline{\Omega})$. For the vorticity, we use the dual of a smaller space, namely $W^{1,\infty}_0(\Omega)$ or $W^{1,\infty}(\Omega)$. We use the notation $\dot{W}^{-1,1}(\Omega)$ and $W^{-1,1}(\Omega)$, respectively, for these dual spaces. In particular, we have the norms

$$\|\phi\|_{\dot{W}^{-1,1}(\Omega)} = \sup\left\{\phi(u) :\ u \in W^{1,\infty}_0(\Omega) \text{ with } \|\nabla u\|_{L^\infty(\Omega)} \le 1\right\}$$

and

$$\|\psi\|_{W^{-1,1}(\Omega)} = \sup\left\{\psi(u) :\ u \in W^{1,\infty}(\Omega) \text{ with } \|u\|_{W^{1,\infty}(\Omega)} \le 1\right\}.$$

We first prove a compactness property for the vorticity.

Proposition 3.1. *For $\epsilon \in (0,1]$, let $\boldsymbol{m}_\epsilon \in H^1(\Omega; S^2)$ with $\boldsymbol{m}_\epsilon|_{\partial\Omega} = \boldsymbol{g}$ and*

$$E_\epsilon(\boldsymbol{m}_\epsilon) \le \pi d \log\frac{1}{\epsilon} + C_0.$$

Furthermore, suppose that

$$\int_\Omega |\boldsymbol{f}_\epsilon(\boldsymbol{m}_\epsilon)|^2\,dx \le \frac{C_0}{\epsilon}.$$

Then there exist a sequence $\epsilon_k \searrow 0$ and a Radon measure μ on $\overline{\Omega}$ such that $\omega(\boldsymbol{m}_{\epsilon_k}) \to \mu$ in $W^{-1,1}(\Omega)$.

Proof. Let $\chi \in C^\infty(\mathbb{R})$ with $\chi(s) = s$ for $|s| \ge \frac{2}{3}$ and $\chi(s) = 0$ for $|s| \le \frac{1}{3}$, and such that $\chi' \ge 0$. For $\boldsymbol{m} \in H^1(\Omega; S^2)$ we define the modified vorticity $\omega'(\boldsymbol{m})$ as follows. Let $m_3' = \chi(m_3)$ and

$$\boldsymbol{m}' = \left(\sqrt{1-(m_3')^2}\,\frac{m}{|m|}, m_3'\right).$$

Then $\omega'(\boldsymbol{m}) = \omega(\boldsymbol{m}')$. We claim that

$$\omega(\boldsymbol{m}) - \omega'(\boldsymbol{m}) = \operatorname{curl}\left((\chi(m_3)-m_3)\frac{j(m)}{|m|^2}\right). \tag{21}$$

Before we prove this identity, we explain how it implies the statement of the lemma. First we see that there exists a universal constant C_1 such that if $\boldsymbol{m}|_{\partial\Omega} = \boldsymbol{g}$, then

$$\left|\int_\Omega \eta(\omega(\boldsymbol{m}) - \omega'(\boldsymbol{m}))\,dx\right| \le C_1\epsilon\|\nabla\eta\|_{L^\infty(\Omega)}E_\epsilon(\boldsymbol{m}) \tag{22}$$

for any $\eta \in W^{1,\infty}(\Omega)$, using the fact that

$$|\chi(m_3) - m_3| \le |m_3|$$

and $\chi(m_3) = m_3 = 0$ on $\partial\Omega$. On the other hand, we have $\omega'(\boldsymbol{m}) = 0$ at any point where $|m_3| \le \frac{1}{3}$. Hence there exits another universal constant C_2 such that

$$|\omega'(\boldsymbol{m})| \le C_2 m_3^2 |\nabla \boldsymbol{m}|^2.$$

Because of inequality (12) in Theorem 2.1, we have a sequence $\epsilon_k \searrow 0$ and a Radon measure μ on $\overline{\Omega}$ such that $\omega'(\boldsymbol{m}_{\epsilon_k}) \overset{*}{\rightharpoonup} \mu$ weakly* in $(C^0(\overline{\Omega}))^*$. Since $W^{1,\infty}(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$, it follows that the convergence also holds strongly in $W^{-1,1}(\Omega)$, and in view of (22), this is sufficient for the proof of the lemma.

It remains to verify (21). We first assume that $\boldsymbol{m} \in C^\infty(\Omega; S^2)$. Then it suffices to prove the identity in the set $\left\{x \in \Omega :\ |m_3(x)| \le \frac{2}{3}\right\}$, as both sides of (21) vanish in its complement. Locally in this set, we can write $\boldsymbol{m}$ in the form

$$\boldsymbol{m} = (\rho e^{i\phi}, m_3),$$

where $\rho = \sqrt{1-m_3^2}$, and then we calculate

$$\nabla \boldsymbol{m} = \nabla m_3\left(-\frac{m_3}{\rho}e^{i\phi}, 1\right) + \nabla\phi(i\rho e^{i\phi}, 0).$$

Since

$$\left(-\frac{m_3}{\rho}e^{i\phi}, 1\right) \times (i\rho e^{i\phi}, 0) = -\boldsymbol{m},$$

we obtain

$$\omega(\boldsymbol{m}) = -\operatorname{curl}(m_3\nabla\phi) = -\operatorname{curl}\left(m_3\frac{j(m)}{|m|^2}\right).$$

The corresponding calculation for $\boldsymbol{m}'$ then yields (21) for smooth maps.

If we have only $\boldsymbol{m} \in H^1(\Omega; S^2)$, then we use the density of $C^\infty(\Omega; S^2)$ in this space, which has been proved by Schoen and Uhlenbeck [39]. Let $\boldsymbol{m}_\ell \in C^\infty(\Omega; S^2)$ with $\boldsymbol{m}_\ell \to \boldsymbol{m}$ in $H^1(\Omega; \mathbb{R}^3)$. We may assume that for almost every $x \in \Omega$, we have $\boldsymbol{m}_\ell(x) \to \boldsymbol{m}(x)$ and $\nabla \boldsymbol{m}_\ell(x) \to \nabla \boldsymbol{m}(x)$, and furthermore,

$$\|\nabla \boldsymbol{m}_\ell - \nabla \boldsymbol{m}\|^2_{L^2(\Omega)} \leq 2^{-\ell}.$$

Then

$$\psi = 2|\nabla \boldsymbol{m}|^2 + 2\sum_{\ell=1}^\infty |\nabla \boldsymbol{m}_\ell - \nabla \boldsymbol{m}|^2$$

is a function in $L^1(\Omega)$. We have $|\omega(\boldsymbol{m}_\ell)| \leq \psi$ for every ℓ, hence by Lebesgue's convergence theorem,

$$\omega(\boldsymbol{m}_\ell) \to \omega(\boldsymbol{m}) \quad \text{in } L^1(\Omega).$$

Similarly,

$$\omega'(\boldsymbol{m}_\ell) \to \omega'(\boldsymbol{m}) \quad \text{in } L^1(\Omega).$$

The quantities on the right-hand side of (21) are easy to control, so passing to the limit, we conclude that the identity holds for $\boldsymbol{m}$. □

The next lemma will be useful when we want to determine the measure μ from Proposition 3.1 in the situation of Theorem 1.1.

Lemma 3.2. *Suppose that $\boldsymbol{m}_\epsilon \in H^1(B_1(0); S^2)$ and $\boldsymbol{m} \in H^1(B_1(0); S^2)$ satisfy*

$$\limsup_{\epsilon \searrow 0} \left(\alpha_\epsilon \int_{B_{1/2}(0)} e_\epsilon(\boldsymbol{m}_\epsilon)\, dx + \int_{B_1(0)\backslash B_{1/2}(0)} e_\epsilon(\boldsymbol{m}_\epsilon)\, dx \right) < \infty$$

and $\boldsymbol{m}_\epsilon \rightharpoonup \boldsymbol{m}$ weakly in $H^1(B_1(0)\backslash B_{1/2}(0); \mathbb{R}^3)$. Then there exist two sequences $\epsilon_k \searrow 0$ and $r_k \nearrow 1$ such that

$$\lim_{k\to\infty} \int_{B_{r_k}(0)} \omega(\boldsymbol{m}_{\epsilon_k})\, dx = \int_{B_1(0)} \omega(\boldsymbol{m})\, dx + 4\pi L$$

for some $L \in \mathbb{Z} \cup \{-\infty, \infty\}$. If $\boldsymbol{m}_\epsilon \rightharpoonup \boldsymbol{m}$ weakly in $H^1(B_1(0); \mathbb{R}^3)$, then

$$\limsup_{k\to\infty} \int_{B_1(0)} e_{\epsilon_k}(\boldsymbol{m}_{\epsilon_k})\, dx \geq \frac{1}{2}\int_{B_1(0)} |\nabla \boldsymbol{m}|^2\, dx + 4\pi|L|.$$

Proof. We may assume that we have continuous maps, because we can use a density argument similarly as in the preceding proof. Note that the assumptions imply $|\boldsymbol{m}_\epsilon - \boldsymbol{m}|^2 \to 0$ strongly in $W^{1,1}(B_1(0)\backslash B_{1/2}(0); \mathbb{R}^3)$.

Choose $\epsilon_k \searrow 0$ and let

$$\tau_k = \||\boldsymbol{m}_{\epsilon_k} - \boldsymbol{m}|^2\|^{1/2}_{W^{1,1}(B_1(0)\backslash B_{1/2}(0))}.$$

Then for every k there exists a radius $r_k \in [1 - \sqrt{\tau_k}, 1]$ such that

$$\int_{\partial B_{r_k}(0)} \left(|\nabla|\boldsymbol{m}_{\epsilon_k} - \boldsymbol{m}|^2| + |\boldsymbol{m}_{\epsilon_k} - \boldsymbol{m}|^2\right) d\mathcal{H}^1$$
$$\leq \frac{3}{\sqrt{\tau_k}} \int_{B_1(0)\backslash B_{1/2}(0)} \left(|\nabla|\boldsymbol{m}_{\epsilon_k} - \boldsymbol{m}|^2| + |\boldsymbol{m}_{\epsilon_k} - \boldsymbol{m}|^2\right) dx = 3\tau_k^{3/2}$$

and

$$\int_{\partial B_{r_k}(0)} \left(e_{\epsilon_k}(\boldsymbol{m}_{\epsilon_k}) + |\nabla \boldsymbol{m}|^2\right) \, d\mathcal{H}^1 \leq \frac{3}{\sqrt{\tau_k}} \int_{B_1(0)} \left(e_{\epsilon_k}(\boldsymbol{m}_{\epsilon_k}) + |\nabla \boldsymbol{m}|^2\right) \, dx.$$

The first inequality means in particular that

$$\lim_{k\to\infty} \sup_{\partial B_{r_k}(0)} |\boldsymbol{m}_{\epsilon_k} - \boldsymbol{m}| = 0.$$

We define the maps $\hat{\boldsymbol{m}}_k$ on $B_{r_k+\tau_k}(0)$ by $\hat{\boldsymbol{m}}_k(x) = \boldsymbol{m}_{\epsilon_k}(x)$ for $x \in B_{r_k}(0)$ and

$$\hat{\boldsymbol{m}}_k(x) = \frac{1}{\tau_k}\left((|x| - r_k)\boldsymbol{m}\left(\frac{r_k x}{|x|}\right) + (r_k + \tau_k - |x|)\boldsymbol{m}_{\epsilon_k}\left(\frac{r_k x}{|x|}\right)\right)$$

for $x \in B_{r_k+\tau_k}(0)\backslash B_{r_k}(0)$. Then we have

$$\frac{1}{2} \leq |\hat{\boldsymbol{m}}_k| \leq 2$$

for every sufficiently large k, and therefore the maps

$$\tilde{\boldsymbol{m}}_k = \frac{\hat{\boldsymbol{m}}_k}{|\hat{\boldsymbol{m}}_k|}$$

are well-defined. Moreover, we have

$$e_{\epsilon_k}(\tilde{\boldsymbol{m}}_k) \leq C_1 e_{\epsilon_k}(\hat{\boldsymbol{m}}_k)$$

for a universal constant C_1. We can now estimate

$$\begin{aligned}\int_{B_{r_k+\tau_k}(0)\backslash B_{r_k}(0)} e_{\epsilon_k}(\tilde{\boldsymbol{m}}_k) \, dx \leq C_2\tau_k \int_{\partial B_{r_k}(0)} (e_{\epsilon_k}(\boldsymbol{m}_{\epsilon_k}) + |\nabla \boldsymbol{m}|^2) \, d\mathcal{H}^1 \\ + \frac{C_2}{\tau_k} \int_{\partial B_{r_k}(0)} |\boldsymbol{m}_{\epsilon_k} - \boldsymbol{m}|^2 \, d\mathcal{H}^1 \to 0\end{aligned}$$

for another universal constant C_2. Hence

$$\lim_{k\to\infty} \int_{B_{r_k}(0)} \omega(\boldsymbol{m}_{\epsilon_k}) \, dx = \lim_{k\to\infty} \int_{B_{r_k+\tau_k}(0)} \omega(\tilde{\boldsymbol{m}}_k) \, dx$$

once we have chosen a subsequence such that the limit exists. But of course we have

$$\tilde{\boldsymbol{m}}_k((r_k + \tau_k)x) = \boldsymbol{m}(r_k x)$$

for every $x \in \partial B_1(0)$, and therefore there exist $L_k \in \mathbb{Z}$ with

$$\int_{B_{r_k+\tau_k}(0)} \omega(\tilde{\boldsymbol{m}}_k) \, dx = \int_{B_{r_k}(0)} \omega(\boldsymbol{m}) \, dx + 4\pi L_k.$$

The first claim now follows immediately.

To prove the second claim, we note that under the assumption that $\boldsymbol{m}_\epsilon \rightharpoonup \boldsymbol{m}$ weakly in $H^1(B_1(0); \mathbb{R}^3)$, we must have $m_3 = 0$ almost everywhere in $B_1(0)$. For $\beta > 0$, define

$$A_{k\beta} = \{x \in B_{r_k+\tau_k}(0) : \ \tilde{m}_{k3}(x) \geq \beta\}.$$

Then we have

$$\frac{1}{2} \int_{B_1(0)} |\nabla \boldsymbol{m}|^2 \, dx \leq \liminf_{k\to\infty} \int_{\Omega\backslash A_{k\beta}} e_{\epsilon_k}(\tilde{\boldsymbol{m}}_k) \, dx.$$

On the other hand, the area formula gives

$$\frac{1}{|L_k|} \liminf_{\beta \searrow 0} \int_{A_{k\beta}} e_{\epsilon_k}(\tilde{\boldsymbol{m}}_k)\,dx \geq 4\pi$$

uniformly in k. Thus we obtain the required inequality. □

4. The Landau-Lifshitz-Gilbert equation

We now turn our attention to equation (2). In this section, we first discuss a few results concerning the construction of weak solutions and we state some of the properties of these solutions. Then we examine their behavior as $\epsilon \searrow 0$, and in particular we derive some preliminary results on the motion of the vortices.

In the existence and regularity theory for the Landau-Lifshitz-Gilbert equation, most authors use a version of the equation that belongs to the Dirichlet functional. It is not difficult to see, however, that a functional such as E_ϵ permits the same arguments. Many of the ideas used in this theory go back to a paper by Struwe [43] on the gradient flow for the Dirichlet functional on a surface without boundary. His results were extended to domains with boundary by Chang [6]. The same methods were used for the Landau-Lifshitz-Gilbert equation by Guo and Hong [15].

We now summarize the results that are relevant in the context of Theorem 1.1. Let $\boldsymbol{g}$ and $\boldsymbol{m}_\epsilon^0$ be as in the introduction. Then there exist weak solutions

$$\boldsymbol{m}_\epsilon \in \bigcap_{T>0} H^1(\Omega \times (0,T); S^2)$$

of (2) with $\boldsymbol{m}_\epsilon(\,\cdot\,,0) = \boldsymbol{m}_\epsilon^0$ and $\boldsymbol{m}_\epsilon(\,\cdot\,,t)|_{\partial\Omega} = \boldsymbol{g}$ for almost every $t \geq 0$. Moreover, for each ϵ, the map $\boldsymbol{m}_\epsilon$ is smooth away from finitely many singular points. For every $\eta \in C^\infty(\overline{\Omega})$ with $\eta \geq 0$ and for $0 \leq t_1 \leq t_2$, it satisfies the local energy inequality

$$\frac{\alpha_\epsilon}{1+\alpha_\epsilon^2} \int_{t_1}^{t_2} \int_\Omega \eta \left| \frac{\partial \boldsymbol{m}_\epsilon}{\partial t} \right|^2 dx\,dt + \int_{\Omega \times \{t_2\}} \eta e_\epsilon(\boldsymbol{m}_\epsilon)\,dx$$
$$\leq \int_{\Omega \times \{t_1\}} \eta e_\epsilon(\boldsymbol{m}_\epsilon)\,dx - \int_{t_1}^{t_2} \int_\Omega \nabla\eta \cdot \left\langle \frac{\partial \boldsymbol{m}_\epsilon}{\partial t}, \nabla \boldsymbol{m}_\epsilon \right\rangle dx\,dt,$$

in fact we have equality if t_1 and t_2 are not separated by a singular time (i.e., a time where $\boldsymbol{m}_\epsilon$ has one of its finitely many singularities). In particular, $E_\epsilon(\boldsymbol{m}_\epsilon(\,\cdot\,,t))$ is nonincreasing in t.

On the other hand, we have a constant C_1, dependent only on Ω and $\boldsymbol{g}$, such that

$$E_\epsilon(\boldsymbol{m}_\epsilon(\,\cdot\,,t)) \geq \pi d \log \frac{1}{\epsilon} - C_1$$

for every $t > 0$, by similar arguments as in the proof of Theorem 2.1. Therefore,

$$0 \leq E_\epsilon(\boldsymbol{m}_\epsilon^0) - E_\epsilon(\boldsymbol{m}_\epsilon(\,\cdot\,,t)) \leq C_1 + C_0. \tag{23}$$

Using this information, we can improve the local energy inequality as follows. For every $\epsilon > 0$ there exists a function μ_ϵ on $[0,\infty)$ with values in the space of nonnegative Radon measures on Ω, such that

$$\mu_\epsilon(t)(\Omega) \le C_1 + C_0$$

for every t and

$$\frac{\alpha_\epsilon}{1+\alpha_\epsilon^2}\int_{t_1}^{t_2}\int_\Omega \eta \left|\frac{\partial \boldsymbol{m}_\epsilon}{\partial t}\right|^2 dx\,dt + \int_{\Omega\times\{t_2\}} \eta e_\epsilon(\boldsymbol{m}_\epsilon)\,dx + \int_\Omega \eta\,d\mu_\epsilon(t_2) \\ = \int_{\Omega\times\{t_1\}} \eta e_\epsilon(\boldsymbol{m}_\epsilon)\,dx + \int_\Omega \eta\,d\mu_\epsilon(t_1) - \int_{t_1}^{t_2}\int_\Omega \nabla\eta\cdot\left\langle \frac{\partial \boldsymbol{m}_\epsilon}{\partial t}, \nabla \boldsymbol{m}_\epsilon\right\rangle dx\,dt \tag{24}$$

for every $\eta \in C^\infty(\overline{\Omega})$ and for $0 \le t_1 \le t_2$. It also follows that there exists a constant C_2, dependent only on Ω, $\boldsymbol{g}$, and C_0, such that

$$\int_0^\infty \int_\Omega \left|\frac{\partial \boldsymbol{m}_\epsilon}{\partial t}\right|^2 dx\,dt \le C_2 \log\frac{1}{\epsilon}. \tag{25}$$

If (x_0, t_0) is one of the finitely many singular points for $\boldsymbol{m}_\epsilon$, then there exist $x_k \to x_0$, $r_k \searrow 0$, and $t_k \nearrow t_0$ such that the rescaled maps

$$\tilde{\boldsymbol{m}}_k(x) = \boldsymbol{m}_\epsilon(r_k x + x_k, t_k)$$

converge to a nonconstant harmonic map $\tilde{\boldsymbol{m}} : \mathbb{R}^2 \to S^2$ with finite Dirichlet energy. This is usually called a bubble in this context. By results of Sacks and Uhlenbeck [36] and standard results for harmonic maps [11], the limit map satisfies

$$\frac{1}{2}\int_{\mathbb{R}^2} |\nabla \tilde{\boldsymbol{m}}|^2\,dx = 4\pi|L| \quad \text{and} \quad \int_{\mathbb{R}^2} \omega(\tilde{\boldsymbol{m}})\,dx = 4\pi L$$

for some $L \in \mathbb{Z}\backslash\{0\}$. Hence we have

$$E_\epsilon(\boldsymbol{m}_\epsilon(\,\cdot\,, t_0)) + 4\pi|L| \le \liminf_{t\nearrow t_0} E_\epsilon(\boldsymbol{m}_\epsilon(\,\cdot\,, t)).$$

In particular, if we have $T > 0$ such that

$$E_\epsilon(\boldsymbol{m}_\epsilon^0) - E_\epsilon(\boldsymbol{m}_\epsilon(\,\cdot\,, T)) < 4\pi,$$

then $\boldsymbol{m}_\epsilon$ is smooth in $\overline{\Omega}\times[0,T]$. On the other hand, if we do have a singular point (x_0, t_0), then for $r > 0$, the function

$$t \mapsto \int_{B_r(x_0)} \omega(\boldsymbol{m}_\epsilon)\,dx$$

will typically jump by a multiple of 4π at t_0 (although there may be other bubbles that neutralize the effect of $\tilde{\boldsymbol{m}}$). This will be accompanied by a change of the topological degree of $\boldsymbol{m}_\epsilon$ at this time. If we study Ginzburg-Landau vortices, then it is important that we control the degree, hence such observations would be relevant in situations where singularities are possible. Under the conditions of Theorem 1.1, however, we can eventually rule them out.

If we have a smooth solution of (2), then we compute the law for the conservation of energy

$$
\begin{aligned}
\frac{\partial}{\partial t} e_\epsilon(\boldsymbol{m}_\epsilon) &= \operatorname{div} \left\langle \frac{\partial \boldsymbol{m}_\epsilon}{\partial t}, \nabla \boldsymbol{m}_\epsilon \right\rangle - \left\langle \frac{\partial \boldsymbol{m}_\epsilon}{\partial t}, \boldsymbol{f}_\epsilon(\boldsymbol{m}_\epsilon) \right\rangle \\
&= \alpha_\epsilon \operatorname{div} \langle \boldsymbol{f}_\epsilon(\boldsymbol{m}_\epsilon), \nabla \boldsymbol{m}_\epsilon \rangle - \operatorname{div} \langle \boldsymbol{m}_\epsilon \times \boldsymbol{f}_\epsilon(\boldsymbol{m}_\epsilon), \nabla \boldsymbol{m}_\epsilon \rangle \\
&\quad - \alpha_\epsilon |\boldsymbol{f}_\epsilon(\boldsymbol{m}_\epsilon)|^2
\end{aligned} \tag{26}
$$

and the conservation of vorticity

$$
\begin{aligned}
\frac{\partial}{\partial t} \omega(\boldsymbol{m}_\epsilon) &= \operatorname{curl} \left\langle \boldsymbol{m}_\epsilon \times \frac{\partial \boldsymbol{m}_\epsilon}{\partial t}, \nabla \boldsymbol{m}_\epsilon \right\rangle \\
&= \operatorname{curl} \langle \boldsymbol{f}_\epsilon(\boldsymbol{m}_\epsilon), \nabla \boldsymbol{m}_\epsilon \rangle + \alpha_\epsilon \operatorname{curl} \langle \boldsymbol{m}_\epsilon \times \boldsymbol{f}_\epsilon(\boldsymbol{m}_\epsilon), \nabla \boldsymbol{m}_\epsilon \rangle .
\end{aligned} \tag{27}
$$

We use the notation $\nabla \boldsymbol{m} \otimes \nabla \boldsymbol{m}$ for the (2×2)-matrix with entries

$$
\left\langle \frac{\partial \boldsymbol{m}}{\partial x_k}, \frac{\partial \boldsymbol{m}}{\partial x_\ell} \right\rangle .
$$

Then we have

$$
\operatorname{div}(\nabla \boldsymbol{m} \otimes \nabla \boldsymbol{m}) = \langle \boldsymbol{f}_\epsilon(\boldsymbol{m}), \nabla \boldsymbol{m} \rangle + \nabla e_\epsilon(\boldsymbol{m}).
$$

Thus we can write (27) as

$$
\frac{\partial}{\partial t} \omega(\boldsymbol{m}_\epsilon) = \operatorname{curl} \operatorname{div}(\nabla \boldsymbol{m}_\epsilon \otimes \nabla \boldsymbol{m}_\epsilon) + \alpha_\epsilon \operatorname{curl} \langle \boldsymbol{m}_\epsilon \times \boldsymbol{f}_\epsilon(\boldsymbol{m}_\epsilon), \nabla \boldsymbol{m}_\epsilon \rangle . \tag{28}
$$

The conservation of energy (26) is of course equivalent to the energy identity (24) for smooth solutions. A combination of both laws will be crucial for the proof of Theorem 1.1.

Theorem 4.1. *There exist a number $T > 0$, a sequence $\epsilon_k \searrow 0$, and a curve $a \in H^1(0, T; \Omega^d)$ with $a(0) = a^0$, such that for every $t \in (0, T)$,*

$$
\alpha_{\epsilon_k} e_{\epsilon_k}(\boldsymbol{m}_{\epsilon_k}(\,\cdot\,, t)) \overset{*}{\rightharpoonup} \pi \delta_{a(t)}
$$

weakly in $(C_0^0(\Omega))^*$ and*

$$
\omega(\boldsymbol{m}_{\epsilon_k}(\,\cdot\,, t)) \to 2\pi \delta_{a(t)}, \quad J(m_{\epsilon_k}(\,\cdot\,, t)) \to \pi \delta_{a(t)}
$$

in $W^{-1,1}(\Omega)$. Moreover,

$$
\pi \int_{t_1}^{t_2} |a'|^2 \, dt \le \liminf_{k \to \infty} \left(\alpha_{\epsilon_k} \int_{t_1}^{t_2} \int_\Omega \left| \frac{\partial \boldsymbol{m}_{\epsilon_k}}{\partial t} \right|^2 dx \, dt \right) \tag{29}
$$

for all $t_1, t_2 \in (0, T)$ with $t_1 \le t_2$.

Before we prove this result, we state a technical lemma which will play the role of the "equipartition of energy" results proved by other authors in similar contexts. This lemma will be used in the proof of Theorem 4.1 in order to study the behavior of matrices of the form $\nabla \boldsymbol{m} \otimes \nabla \boldsymbol{m}$. In the following we write id for the identity matrix in $\mathbb{R}^{2 \times 2}$.

Lemma 4.2. *Let $\Psi_k \in L^1(\Omega; \mathbb{R}^{2\times 2})$ such that $|\Psi_k| \le \operatorname{tr} \Psi_k$ and*

$$
\operatorname{tr} \Psi_k \overset{*}{\rightharpoonup} \sum_{\ell=1}^{d} b_\ell \delta_{a_\ell}
$$

weakly in* $(C_0^0(\Omega))^*$ *for some* $a \in \Omega^d$ *and* $b_1, \ldots, b_\ell \in [0, \infty)$. *Further suppose that there exist* $\xi_k \in L^1(\Omega; \mathbb{R}^2)$ *with*

$$\limsup_{k \to \infty} \int_\Omega |\xi_k| \, dx \leq \Gamma$$

and such that for all $\phi \in C_0^1(\Omega; \mathbb{R}^2)$,

$$\int_\Omega \left(\frac{1}{2} \operatorname{div} \phi \operatorname{tr} \Psi_k - \nabla\phi \cdot \Psi_k - \phi \cdot \xi_k \right) dx \to 0$$

as $k \to \infty$. *Then*

$$\Psi_k \stackrel{*}{\rightharpoonup} \frac{1}{2} \mathrm{id} \sum_{\ell=1}^d b_\ell \delta_{a_\ell}$$

weakly in* $(C_0^0(\Omega; \mathbb{R}^{2\times 2}))^*$.

Proof. First note that it suffices to prove this convergence for a subsequence, as the limit is determined by the assumptions. We may assume that there exist $\psi_1, \ldots, \psi_\ell \in \mathbb{R}^{2\times 2}$ such that

$$\Psi_k \stackrel{*}{\rightharpoonup} \sum_{\ell=1}^d \psi_\ell \delta_{a_\ell}$$

weakly* in $(C_0^0(\Omega; \mathbb{R}^{2\times 2}))^*$. We may further assume that there exists a Radon measure Ξ on Ω with values in $\mathbb{R}^2$ such that $\xi_k \stackrel{*}{\rightharpoonup} \Xi$ weakly* in $(C_0^0(\Omega; \mathbb{R}^2))^*$. Moreover, we have

$$\sum_{\ell=1}^d \left(\frac{1}{2} \operatorname{div} \phi(a_\ell) b_\ell - \nabla\phi(a_\ell) \cdot \psi_\ell \right) = \int_\Omega \phi \cdot d\Xi$$

for every $\phi \in C_0^1(\Omega; \mathbb{R}^2)$.

We may assume that $a_k \neq a_\ell$ for $k \neq \ell$. Choose an arbitrary $\lambda > 0$ and arbitrary $\Phi_1, \ldots, \Phi_\ell \in \mathbb{R}^{2\times 2}$. Then there exists a $\phi \in C_0^1(\Omega; \mathbb{R}^2)$ such that $\nabla\phi(a_\ell) = \Phi_\ell$ and such that $\sup_\Omega |\phi| \leq \lambda$. Thus

$$\left| \sum_{\ell=1}^d \Phi_\ell \cdot \left(\frac{1}{2} b_\ell \mathrm{id} - \psi_\ell \right) \right| \leq \Gamma\lambda.$$

The claim follows immediately when we let $\lambda \searrow 0$. □

Proof of Theorem 4.1. Choose $T > 0$. Let $\{\eta_n\}_{n\in\mathbb{N}}$ be a sequence that is dense in $C_0^0(\Omega)$, and define

$$\theta_{\epsilon n}(t) = \alpha_\epsilon \int_{\Omega\times\{t\}} \eta_n e_\epsilon(\boldsymbol{m}_\epsilon) \, dx + \alpha_\epsilon \int_\Omega \eta_n d\mu_\epsilon(t).$$

For any fixed n, this is a family of uniformly equicontinuous functions by (23)–(25), therefore there exists a sequence $\epsilon_k \searrow 0$ such that

$$\theta_{\epsilon_k n} \to \theta_n$$

as $k \to \infty$ uniformly in $[0, T)$. With a diagonal sequence argument, we even find a sequence $\epsilon_k \searrow 0$ such that this holds for every n. By Theorem 2.1, we have some $a : [0,T) \to \Omega^d$ such that

$$\theta_n(t) = \pi \sum_{\ell=1}^{d} \eta_n(a_\ell(t)).$$

As the functions θ_n are continuous, the curves a_ℓ must be continuous as well. The convergence of $\alpha_{\epsilon_k} e_{\epsilon_k}(\boldsymbol{m}_{\epsilon_k})$ now follows with a density argument.

Now we use the fact that

$$\int_{\Omega\times\{t\}} (\operatorname{div}\phi\, e_\epsilon(\boldsymbol{m}_\epsilon) - \nabla\phi \cdot (\nabla \boldsymbol{m}_\epsilon \otimes \nabla \boldsymbol{m}_\epsilon) - \phi \cdot \langle f_\epsilon(\boldsymbol{m}_\epsilon), \nabla \boldsymbol{m}_\epsilon\rangle)\, dx = 0$$

for every $\phi \in C_0^1(\Omega;\mathbb{R}^2)$ and almost every $t \in (0,T)$. By Theorem 2.1, we have

$$\lim_{\epsilon\to 0} \left(\frac{\alpha_\epsilon}{\epsilon^2} \int_{\Omega\times\{t\}} m_{\epsilon 3}^2\, dx \right) = 0,$$

hence we can apply Lemma 4.2 to

$$\Psi_k = \alpha_{\epsilon_k} \nabla \boldsymbol{m}_{\epsilon_k}(\,\cdot\,,t) \otimes \nabla \boldsymbol{m}_{\epsilon_k}(\,\cdot\,,t)$$

and

$$\xi_k = \alpha_{\epsilon_k} \left\langle \boldsymbol{f}_{\epsilon_k}(\boldsymbol{m}_{\epsilon_k}(\,\cdot\,,t)), \nabla \boldsymbol{m}_{\epsilon_k}(\,\cdot\,,t) \right\rangle.$$

Using also Lebesgue's convergence theorem, we obtain

$$\alpha_{\epsilon_k} \int_{t_1}^{t_2} \int_\Omega \eta \nabla \boldsymbol{m}_{\epsilon_k} \otimes \nabla \boldsymbol{m}_{\epsilon_k}\, dx\, dt \to \pi \mathrm{id} \sum_{\ell=1}^{d} \int_{t_1}^{t_2} \eta(a_\ell(t))\, dt$$

for every $\eta \in C^0(\overline{\Omega} \times [0,T])$.

Next we want to prove (29). We first replace $\{\epsilon_k\}$ by a subsequence such that the right hand side of the inequality becomes a limit. Suppose that $t_2 - t_1$ is so small that there exists a radius $R > 0$ with

$$a_1(t) \in B_R(a_1(t_1)), \quad a_2(t), \ldots, a_d(t) \notin B_{3R}(a_1(t_1))$$

for $t \in [t_1,t_2]$. Define

$$\sigma_k(t) = \alpha_{\epsilon_k} \int_0^t \int_{B_{2R}(a_1(t_1))} \left| \frac{\partial \boldsymbol{m}_{\epsilon_k}}{\partial t} \right|^2 dx\, d\tau.$$

Then this gives a sequence that is bounded in $\mathrm{BV}(0,T)$ by (25). After selecting a subsequence, we obtain a function $\sigma \in \mathrm{BV}(0,T)$ such that $\sigma_k(t) \to \sigma(t)$ for almost every t.

Let $\chi \in C_0^\infty(B_{2R}(a_1(t_1)))$ with $\chi \equiv 1$ in $B_R(a_1(t_1))$. We test (24) with functions of the form $\eta(x) = \chi(x) b \cdot x$ for $b \in S^1$ and we use the convergence proved earlier, in particular

$$\int_{t_0}^{t_0+h} \int_\Omega \chi^2 |b \cdot \nabla \boldsymbol{m}_{\epsilon_k}|^2\, dx\, dt = \int_{t_0}^{t_0+h} \int_\Omega \chi^2 (b\otimes b) \cdot (\nabla \boldsymbol{m}_{\epsilon_k} \otimes \nabla \boldsymbol{m}_{\epsilon_k})\, dx\, dt \to \pi h$$

for every $t_0 \in (t_1,t_2)$ and every $h \in (t_1 - t_0, t_2 - t_0)$. We obtain

$$\pi |a_1(t_0+h) - a_1(t_0)| \le \sqrt{\pi h |\sigma(t_0+h) - \sigma(t_0)|}.$$

Thus

$$\pi|a_1'(t)|^2 \le |\sigma'(t)|$$

almost everywhere in (t_1, t_2). It is now easy to complete the proof of (29).

To control the Jacobians $J(m_{\epsilon_k})$, we can use the results of Jerrard and Soner [21] or Sandier and Serfaty [38] together with standard arguments.

Only the convergence of $\omega(\boldsymbol{m}_{\epsilon_k}(\,\cdot\,,t))$ remains to be proved. Note first that $W_\epsilon(a)$ is a continuous function. Hence if T is small enough, then we have a number $\kappa > 0$ such that

$$\limsup_{\epsilon\searrow 0} D_\epsilon(\boldsymbol{m}_\epsilon^0; a(t)) \le 4\pi - \kappa$$

for all $t \in [0,T]$. By Proposition 2.4, this inequality implies

$$\limsup_{k\to\infty} \left(E_{\epsilon_k}(\boldsymbol{m}_{\epsilon_k}^0) - E_{\epsilon_k}(\boldsymbol{m}_{\epsilon_k}(\,\cdot\,,t))\right) \le 4\pi - \kappa.$$

Provided that k is large enough, we conclude that $\boldsymbol{m}_{\epsilon_k}$ is smooth in $\overline{\Omega}\times[0,T]$. Moreover, we have

$$\limsup_{k\to\infty} D_{\epsilon_k}(\boldsymbol{m}_{\epsilon_k}(\,\cdot\,,t); a(t)) \le 4\pi - \kappa \tag{30}$$

as well.

Let A_k be the set of all $t \in [0,T)$ such that

$$\int_{\Omega\times\{t\}} \left|\boldsymbol{f}_{\epsilon_k}(\boldsymbol{m}_{\epsilon_k})\right|^2 \, dx > \frac{1}{\epsilon_k}.$$

Then we have $|A_k| \le C_1\epsilon_k \log\frac{1}{\epsilon_k}$ for some constant C_1. Now fix $t_0 \notin \bigcup_{k=1}^\infty A_k$, then by Proposition 3.1, there exists a measure $\mu(t_0)$ on $\overline{\Omega}$ such that

$$\omega(\boldsymbol{m}_{\epsilon_k}(\,\cdot\,,t_0)) \to \mu(t_0) \tag{31}$$

in $W^{-1,1}(\Omega)$. Let $B_{2r}(x_0) \subset \Omega\backslash\{a_1(t_0),\dots,a_d(t_0)\}$. Because of Theorem 2.1, we know that $\boldsymbol{m}_{\epsilon_k}(\,\cdot\,,t_0)$ is uniformly bounded in $H^1(B_r(x_0);\mathbb{R}^3)$. Thus we can apply Lemma 3.2 to a rescaled subsequence. Note, however, that the last inequality in this lemma can hold only for $L = 0$ owing to (30). (Otherwise it would be easy to construct maps that violate the inequality of Proposition 2.4). Hence $\mu(t_0)$ vanishes in $B_r(x_0)$, and we conclude that it is a weighted sum of Dirac measures centered at $a_1(t_0),\dots,a_d(t_0)$. On the other hand, applying Lemma 3.2 again in a similar way as above, but for balls centered at $a_\ell(t_0)$, we see that the weights must be multiples of 2π. That is, there exist $\beta_1(t_0),\dots,\beta_d(t_0) \in \mathbb{Z}$ such that

$$\mu(t_0) = 2\pi\sum_{\ell=1}^d \beta_\ell(t_0)\delta_{a_\ell(t_0)}. \tag{32}$$

If $t_0 \in \bigcup_{k=1}^\infty A_k$, then we proceed as follows. For any k there exists a number $t_k \in [0,T)\backslash A_k$ such that $|t_0 - t_k| \le 2C_1\epsilon_k\log\frac{1}{\epsilon_k}$. For $\eta \in W^{1,\infty}(\Omega)$

we have

$$\int_\Omega \eta(x)(\omega(\boldsymbol{m}_{\epsilon_k}(x,t_k)) - \omega(\boldsymbol{m}_{\epsilon_k}(x,t_0)))\,dx$$
$$= -\int_{t_0}^{t_k}\int_\Omega \nabla^\perp\eta \left\langle \boldsymbol{m}_{\epsilon_k} \times \frac{\partial \boldsymbol{m}_{\epsilon_k}}{\partial t}, \nabla \boldsymbol{m}_{\epsilon_k}\right\rangle dx\,dt,$$

where $\nabla^\perp = (-\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1})$, and the right-hand side is bounded by

$$C_2\sqrt{\epsilon_k}\left(\log\frac{1}{\epsilon_k}\right)^{3/2}$$

for some constant C_2. We have a similar estimate for the energy, and therefore we can draw similar conclusions as above first for $\omega(\boldsymbol{m}_{\epsilon_k}(\,\cdot\,,t_k))$ and then also for $\omega(\boldsymbol{m}_{\epsilon_k}(\,\cdot\,,t_0))$. In other words, in either case we have a measure $\mu(t_0)$ of the form (32) such that (31) holds. We now claim that the functions β_ℓ are continuous. Once this is shown, the proof is complete.

Let $t_0 \in [0,T)$ and choose $R>0$. If T is small enough, then there exists a relatively open interval $I \subset [0,T)$ with $t_0 \in I$ such that $a_\ell(t) \notin \Omega_R(a(t_0))$ for $\ell = 1,\ldots,d$. Theorem 2.1 and (24) then show that

$$\limsup_{k\to\infty}\sup_{t\in I}\int_{\Omega_R(a(t_0))} e_{\epsilon_k}(\boldsymbol{m}_{\epsilon_k}(x,t))\,dx < \infty.$$

If $\eta \in C_0^\infty(\Omega)$ such that $\nabla\eta \equiv 0$ in $B_R(a_\ell(t_0))$ for $\ell = 1,\ldots,d$, then it follows from (28) and (25) that the functions

$$t \mapsto \int_{\Omega\times\{t\}} \eta\omega(\boldsymbol{m}_{\epsilon_k})\,dx$$

are uniformly equicontinuous in I. This implies the claim. □

5. An estimate for the energy excess

For the proof of Theorem 1.1, we also need a somewhat technical result. Theorem 5.3 below is the analog of the Γ-stability results proved by Jerrard and Spirn [20] for a similar purpose. But first we need an S^2-target version of results found in papers by Colliander and Jerrard [8] and by Lin and Xin [29].

Lemma 5.1. *For any $\lambda > 0$ there exist $\epsilon_0 \in (0,1)$ and $\iota > 0$ with the following property. Suppose that $\epsilon \in (0,\epsilon_0]$ and $\boldsymbol{m} \in H^1(B_r(a);S^2)$ with*

$$\|J(m) - \pi\delta_a\|_{\dot W^{-1,1}(B_r(a))} \le \iota r. \tag{33}$$

Then

$$\int_{B_r(a)} e_\epsilon(\boldsymbol{m})\,dx \ge I_{\epsilon/r} - \lambda.$$

Proof. We may assume that $\lambda \le 1$ and $B_r(a) = B_1(0)$. Moreover, it suffices to consider the case where

$$\int_{B_1(0)} e_\epsilon(\boldsymbol{m})\,dx \le \pi\log\frac{1}{\epsilon} + C_1 \tag{34}$$

for some universal constant C_1.

Let $r_0 \in (0, \frac{1}{2})$ be a fixed number, the value of which will be determined later (depending only on λ). We consider the set

$$P_1 = \left\{ \rho \in [r_0, 1] : \ |m| \geq \frac{1}{2} \text{ on } \partial B_\rho(0) \text{ and } \deg(m; \partial B_\rho(0)) = 1 \right\}.$$

Under the conditions (33) and (34), the results of Jerrard and Spirn [19], especially Proposition 4.2 and Lemma 4.3, show that whenever ϵ_0 and ι are small enough, we have

$$\int_{B_{1/2}(0)\setminus B_{r_0}(0)} e_\epsilon(\boldsymbol{m})\, dx \leq \pi \log \frac{1}{2r_0} + C_2$$

and

$$|P_1| \geq 1 - r_0 - C_3\iota - C_3\epsilon \log \frac{1}{\epsilon}$$

for certain universal constants C_2 and C_3.

Now define

$$P_2 = \left\{ \rho \in [r_0, 1/2] : \ \rho \int_{\partial B_\rho(0)} e_\epsilon(\boldsymbol{m})\, d\mathcal{H}^1 \leq \pi + \lambda \right\}.$$

Then we compute

$$\begin{aligned}\pi \log \frac{1}{2r_0} + C_2 &\geq \int_{[r_0,1/2]\setminus P_2} \int_{\partial B_\rho(0)} e_\epsilon(\boldsymbol{m})\, d\mathcal{H}^1\, d\rho \geq (\pi + \lambda) \int_{[r_0,1/2]\setminus P_2} \frac{d\rho}{\rho} \\ &\geq (\pi + \lambda) \int_{r_0 + |P_2|}^{1/2} \frac{d\rho}{\rho} = (\pi + \lambda) \log \frac{1}{2r_0 + 2|P_2|}.\end{aligned}$$

That is,

$$|P_2| \geq r_0 \left(\exp\left(\frac{C_2}{\pi + \lambda} \right) (2r_0)^{\frac{-\lambda}{\pi+\lambda}} - 1 \right).$$

If r_0 is chosen small enough, then it follows that $|P_2| \geq \frac{r_0}{2}$. If at the same time ι and ϵ_0 are small enough, then $P_1 \cap P_2 \neq \emptyset$. In other words, there exists a radius $\rho_1 \in [r_0, \frac{1}{2}]$ such that $|m| \geq \frac{1}{2}$ on $\partial B_{\rho_1}(0)$ with $\deg(m; \partial B_{\rho_1}(0)) = 1$, and moreover,

$$\rho_1 \int_{\partial B_{\rho_1}(0)} e_\epsilon(\boldsymbol{m})\, d\mathcal{H}^1 \leq \pi + \lambda.$$

This inequality implies

$$\int_{\partial B_{\rho_1}(0)} |\nabla m_3^2|\, d\mathcal{H}^1 \leq \frac{(\pi + \lambda)\epsilon}{\rho_1},$$

thus there exists a constant $C_4 = C_4(\lambda)$ such that

$$\sup_{\partial B_{\rho_1}(0)} m_3^2 \leq C_4 \epsilon.$$

Define $\rho_2 = \rho_1 + \epsilon$ and suppose that ϵ_0 is so small that $\rho_2 \leq \frac{3}{4}$. Now we define a map $\tilde{\boldsymbol{m}} : B_{\rho_2}(0) \to S^2$ as follows. For $x \in B_{\rho_1}(0)$, we set $\tilde{\boldsymbol{m}}(x) = \boldsymbol{m}(x)$, and for $x \in B_{\rho_2}(0) \setminus B_{\rho_1}(0)$,

$$\tilde{m}_3 = \frac{|x| - \rho_2}{\rho_1 - \rho_2} m_3 \left(\frac{\rho_1 x}{|x|} \right)$$

and

$$\tilde{m}(x) = \sqrt{1 - \tilde{m}_3^2(x)} \frac{m(\rho_1 x/|x|)}{|m(\rho_1 x/|x|)|}.$$

Then it is readily checked that

$$\int_{B_{\rho_2}(0)\setminus B_{\rho_1}(0)} e_\epsilon(\tilde{\boldsymbol{m}})\, dx \le C_5 \epsilon$$

for a constant $C_5 = C_5(\lambda)$. On $\partial B_{\rho_2}(0)$, we can now write

$$\tilde{\boldsymbol{m}}\left(\rho_2 e^{i\theta}\right) = \left(e^{i(\theta+\psi(\theta))}, 0\right)$$

for a function $\psi \in H^1(0, 2\pi)$ with $\psi(0) = \psi(2\pi)$. Let

$$\psi_0 = \frac{1}{2\pi} \int_0^{2\pi} \psi\, d\theta.$$

We define $\rho_3 = \frac{4}{3}\rho_2$ and we extend $\tilde{\boldsymbol{m}}$ to $B_{\rho_3}(0)$ by a linear interpolation between ψ and ψ_0. More precisely, we set

$$\tilde{\psi}(\rho, \theta) = \frac{\rho - \rho_3}{\rho_2 - \rho_3} \psi(\theta) + \frac{\rho - \rho_2}{\rho_3 - \rho_2} \psi_0$$

and

$$\tilde{\boldsymbol{m}}\left(\rho e^{i\theta}\right) = \left(e^{i(\theta + \tilde{\psi}(\rho,\theta))}, 0\right)$$

for $\rho_2 < \rho < \rho_3$ and $\theta \in [0, 2\pi)$. By the construction of $\tilde{\boldsymbol{m}}$, we have a constant $C_6 = C_6(\lambda)$ such that

$$\begin{aligned}\frac{\rho_2}{2} \int_{\partial B_{\rho_2}(0)} \left|\frac{\partial \tilde{\boldsymbol{m}}}{\partial \theta}\right|^2 d\mathcal{H}^1 = \frac{1}{2} \int_0^{2\pi} (1 + \psi')^2\, d\theta \le \rho_1 \int_{\partial B_{\rho_1}(0)} \frac{e_\epsilon(\boldsymbol{m})}{1 - m_3^2}\, d\mathcal{H}^1 \\ \le (1 + C_6 \epsilon)(\pi + \lambda).\end{aligned}$$

Thus if ϵ_0 is small enough, then we have

$$\int_0^{2\pi} (\psi')^2\, d\theta \le 3\lambda \quad \text{and} \quad \int_0^{2\pi} (\psi - \psi_0)^2\, d\theta \le 6\pi\lambda.$$

Hence we find a universal constant C_7 such that

$$\int_{B_{\rho_3}(0)\setminus B_{\rho_2}(0)} e_\epsilon(\tilde{\boldsymbol{m}})\, dx \le \pi \log \frac{4}{3} + C_7 \lambda.$$

Finally, we extend $\tilde{\boldsymbol{m}}$ to $B_1(0)$ by

$$\tilde{\boldsymbol{m}}(x) = \tilde{\boldsymbol{m}}\left(\frac{\rho_3 x}{|x|}\right), \quad x \in B_1(0)\setminus B_{\rho_3}(0).$$

Then clearly we have

$$\int_{B_1(0)} e_\epsilon(\tilde{\boldsymbol{m}}) \ge I_\epsilon.$$

On the other hand,

$$\int_{B_1(0)} e_\epsilon(\tilde{\boldsymbol{m}})\, dx \le \int_{B_{\rho_1}(0)} e_\epsilon(\boldsymbol{m})\, dx + C_5 \epsilon + \pi \log \frac{1}{\rho_2} + C_7 \lambda.$$

If ϵ_0 is sufficiently small, then we obtain

$$\int_{B_{\rho_1}(0)} e_\epsilon(\boldsymbol{m})\, dx \ge I_\epsilon - \pi \log \frac{1}{\rho_1} - (C_7 + 1)\lambda,$$

using the previous estimates and the definition of ρ_2.

In order to conclude the proof, it now suffices to show that

$$\int_{B_1(0)\setminus B_{\rho_1}(0)} e_\epsilon(\boldsymbol{m})\,dx \geq \pi \log \frac{1}{\rho_1} - \lambda \tag{35}$$

(and replace λ by $\lambda/(C_7+2)$). To this end, note that

$$|[\rho_1, 1] \cap P_1| \geq 1 - \rho_1 - C_3\iota - C_3\epsilon \log \frac{1}{\epsilon}.$$

On the other hand, if $\rho \in P_1$, then

$$\rho \int_{\partial B_\rho(0)} e_\epsilon(\boldsymbol{m})\,d\mathcal{H}^1 \geq \pi - C_8\epsilon \log \frac{1}{\epsilon}$$

for a universal constant C_8 by standard estimates. Therefore,

$$\begin{aligned}\int_{B_1(0)\setminus B_{\rho_1}(0)} e_\epsilon(\boldsymbol{m})\,dx &\geq \pi \int_{[\rho_1,1]\cap P_1} \frac{d\rho}{\rho} - C_8\epsilon \log\frac{1}{\epsilon}\log\frac{1}{\rho_1} \\ &\geq \pi \log \frac{1}{1 - |[\rho_1,1]\cap P_1|} - C_8\epsilon \log\frac{1}{\epsilon}\log\frac{1}{\rho_1}.\end{aligned}$$

Since $\rho_1 \geq r_0$ and r_0 depends only on λ, it follows that (35) is true whenever ϵ_0 and ι are sufficiently small. □

Lemma 5.2. *Let $R_0 > 0$. There exist $r_0 \in (0, R_0]$ and $C > 0$ such that for any $a \in \Omega^d$ with $\rho(a) \geq r_0$ and any $r \in (0, r_0]$, there exist an open set $\tilde{\Omega} \subset \Omega_r(a)$ and a function $\tilde{G} \in W^{1,\infty}(\Omega)$ with the following properties:*

- $|\Omega_r(a)\setminus\tilde{\Omega}| \leq Cr^3$,
- $\nabla^\perp \tilde{G} = j(m_*(\,\cdot\,;a))$ *in* $\tilde{\Omega}$,
- $\tilde{G}$ *is constant on every connected component of* $\Omega\setminus\tilde{\Omega}$, *and*
- $\|\tilde{G}\|_{W^{1,\infty}(\Omega)} \leq C/r$.

Proof. If we use a Hodge decomposition of the form

$$j(m_*(\,\cdot\,;a)) = \nabla F + \nabla^\perp G$$

with $F \in W_0^{1,1}(\Omega)$, then we immediately find $\Delta F = 0$, i.e., $F = 0$. Moreover, we have $\Delta G = 2\pi\delta_a$ in Ω. On $\partial\Omega$, the normal derivative of G is smooth and depends only on $\boldsymbol{g}$. It follows that there exists a function $H \in C^\infty(\overline{\Omega})$ with $\Delta H = 0$ and

$$G(x) = \sum_{\ell=1}^{d} \log|x - a_\ell| - H(x). \tag{36}$$

In order to construct $\tilde{G}$, we now want to truncate G near the points a_ℓ in a suitable way. To this end, let $r \in (0, r_0]$ and define

$$A_\ell = \sup_{B_r(a_\ell)} G, \quad \ell = 1, \ldots, d,$$

and

$$\tilde{B}_\ell = \{x \in B_{r_0}(a_\ell) : \ G(x) \leq A_\ell\}.$$

Moreover, let

$$\tilde{\Omega} = \Omega \setminus \bigcup_{\ell=1}^{d} \tilde{B}_\ell$$

and

$$\tilde{G}(x) = \begin{cases} G(x) & \text{if } x \in \tilde{\Omega}, \\ A_\ell & \text{if } x \in \tilde{B}_\ell,\ \ell = 1, \dots, d. \end{cases}$$

By definition, we have $\tilde{\Omega} \subset \Omega_r(a)$. We claim that whenever r_0 is small enough, the sets $\tilde{B}_\ell$ are closed (and thus they have a positive distance from $\partial B_{\rho_0}(a_\ell)$) and

$$|\tilde{B}_\ell \backslash B_r(a_\ell)| \leq C_1 r^3$$

for $\ell = 1, \dots, d$, where C_1 is a constant that depends only on Ω, $\boldsymbol{g}$, and r_0. Both of these statements follow from the representation (36) of G with relatively easy calculations (more details for similar arguments can be found in a work by Jerrard and Spirn [20, Sect. 8]). Once the claims are verified, it is clear that $\tilde{\Omega}$ and $\tilde{G}$ have the required properties. □

For $\boldsymbol{m} \in H^1(\Omega; S^2)$ we now define

$$\tilde{e}_\epsilon(\boldsymbol{m}) = \frac{1}{2}\left(|\nabla |m||^2 + |\nabla m_3|^2 + \frac{m_3^2}{\epsilon^2} \right).$$

This notation is used in the next theorem and again in Section 6.

Theorem 5.3. *Let $r_0 > 0$ and $C_0 \in \mathbb{R}$. For any $\lambda > 0$ there exist $\epsilon_0 \in (0,1)$ and $\iota > 0$ with the following property. Suppose that $\epsilon \in (0, \epsilon_0]$, $\boldsymbol{m} \in H^1(\Omega; S^2)$ with $\boldsymbol{m}|_{\partial\Omega} = \boldsymbol{g}$, and $a \in \Omega^d$ with $\rho(a) \geq r_0$ satisfy*

$$E_\epsilon(\boldsymbol{m}) \leq \pi d \log \frac{1}{\epsilon} + C_0$$

and

$$\|J(m) - \pi \delta_a\|_{W^{-1,1}(\Omega)} \leq \iota.$$

Then

$$\int_{\Omega_{r_0}(a)} \left(\tilde{e}_\epsilon(\boldsymbol{m}) + \frac{1}{4}\left| \frac{j(m)}{|m|} - j(m_*(\,\cdot\,; a)) \right|^2 \right) dx \leq D_\epsilon(\boldsymbol{m}; a) + \lambda.$$

Proof. We use the notation $\boldsymbol{m}_* = \boldsymbol{m}_*(\,\cdot\,; a)$ throughout this proof. Let $r \in (0, r_0]$ be a fixed number, the value of which will be determined in the course of the proof. Recall that

$$D_\epsilon(\boldsymbol{m}; a) = E_\epsilon(\boldsymbol{m}) - W(a) - d\left(\gamma + \pi \log \frac{1}{\epsilon} \right).$$

There exists a constant C_1 that depends only on r_0 and d, such that

$$W(a) + d\pi \log \frac{1}{r} - \frac{1}{2} \int_{\Omega_r(a)} |\nabla m_*|^2 \, dx \leq C_1 r^2.$$

(An inequality of this form has been proved by Jerrard and Spirn [20, Lemma 12] for Neumann boundary data. The calculations for Dirichlet data are similar.) Using also the fact that the quantity $I_\epsilon - \pi \log \frac{1}{\epsilon}$ is monotone in ϵ, we obtain

$$D_\epsilon(\boldsymbol{m}; a) \geq \int_{\Omega_r(a)} \left(e_\epsilon(\boldsymbol{m}) - \frac{1}{2}|\nabla m_*|^2 \right) dx + \sum_{\ell=1}^{d} \left(\int_{B_r(a_\ell)} e_\epsilon(\boldsymbol{m}) \, dx - I_{\epsilon/r} \right) - C_1 r^2.$$

We now assume that r is so small that $C_1 r^2 \leq \lambda$ and $r \leq 1$.

If we have a function $\phi \in W_0^{1,\infty}(B_r(a_\ell))$ with $\|\nabla\phi\|_{L^\infty(B_r(a_\ell))} \leq 1$, then it also satisfies $\|\phi\|_{W^{1,\infty}(\Omega)} \leq 1$. Hence

$$\|J(m) - \pi\delta_{a_\ell}\|_{\dot{W}^{-1,1}(B_r(a_\ell))} \leq \iota$$

for $\ell = 1, \ldots, d$. As r is fixed, this means that Lemma 5.1 can be applied for the given number λ whenever ι and ϵ_0 are sufficiently small. Thus

$$\int_{B_r(a_\ell)} e_\epsilon(\boldsymbol{m})\,dx - I_{\epsilon/r} \geq -\lambda,$$

and therefore

$$D_\epsilon(\boldsymbol{m}; a) \geq \int_{\Omega_r(a)} \left(e_\epsilon(\boldsymbol{m}) - \frac{1}{2}|\nabla m_*|^2\right)\,dx - (d+1)\lambda.$$

We now want to prove that

$$\int_{\Omega_r(a)} \left(\tilde{e}_\epsilon(\boldsymbol{m}) + \frac{1}{4}\left|\frac{j(m)}{|m|} - j(m_*)\right|^2\right)\,dx \leq \int_{\Omega_r(a)} \left(e_\epsilon(\boldsymbol{m}) - \frac{1}{2}|\nabla m_*|^2\right)\,dx + \lambda, \tag{37}$$

provided that ι, ϵ_0, and r are small enough. Once this inequality is established, the claim of the theorem follows.

In order to prove (37), we first calculate

$$\begin{aligned}\tilde{e}_\epsilon(\boldsymbol{m}) + \frac{1}{4}\left|\frac{j(m)}{|m|} - j(m_*)\right|^2 &= e_\epsilon(\boldsymbol{m}) - \frac{1}{2}|\nabla m_*|^2 - \frac{1}{4}\left|\frac{j(m)}{|m|} - j(m_*)\right|^2 \\ &\quad + (|m|-1)\frac{j(m)}{|m|}\cdot j(m_*) + |j(m_*)|^2 - j(m)\cdot j(m_*).\end{aligned} \tag{38}$$

Clearly

$$\int_{\Omega_r(a)} (|m|-1)\frac{j(m)}{|m|}\cdot j(m_*)\,dx \leq \frac{C_2\epsilon}{r}E_\epsilon(\boldsymbol{m}) \tag{39}$$

for some universal constant C_2. We now use the set $\tilde{\Omega}$ and the function $\tilde{G}$ from Lemma 5.2. We write

$$\begin{aligned}&\int_{\Omega_r(a)} \left(|j(m_*)|^2 - j(m)\cdot j(m_*)\right)\,dx \\ &\quad = \int_{\tilde{\Omega}} j(m_*)\cdot(j(m_*) - j(m))\,dx \\ &\quad\quad + \int_{\Omega_r(a)\setminus\tilde{\Omega}} j(m_*)\cdot\left(j(m_*) - \frac{j(m)}{|m|}\right)\,dx \\ &\quad\quad + \int_{\Omega_r(a)\setminus\tilde{\Omega}} (1-|m|)j(m_*)\cdot\frac{j(m)}{|m|}\,dx.\end{aligned} \tag{40}$$

For the first term on the right-hand side, we have

$$\begin{aligned}\int_{\tilde\Omega} j(m_*)\cdot(j(m_*)-j(m))\,dx &= \int_\Omega \nabla^\perp\tilde G\cdot(j(m_*)-j(m))\,dx\\ &= 2\int_\Omega \tilde G J(m)\,dx - 2\pi\sum_{\ell=1}^d \tilde G(a_\ell)\\ &\le 2\iota\|G\|_{W^{1,\infty}(\Omega)} \le \frac{C_3\iota}{r}.\end{aligned} \tag{41}$$

The second term on the right-hand side of (40) satisfies

$$\begin{aligned}\int_{\Omega_r(a)\setminus\tilde\Omega} j(m_*)\cdot\left(j(m_*)-\frac{j(m)}{|m|}\right)dx &\le \int_{\Omega_r(a)\setminus\tilde\Omega}|j(m_*)|^2\,dx\\ &\quad+\frac14\int_{\Omega_r(a)}\left|\frac{j(m)}{|m|}-j(m_*)\right|^2dx\end{aligned} \tag{42}$$

by Young's inequality. Using the bound on $|\Omega_r(a)\setminus\tilde\Omega|$ from Lemma 5.2, we see that

$$\int_{\Omega_r(a)\setminus\tilde\Omega}|j(m_*)|^2\,dx \le C_4 r \tag{43}$$

for a constant $C_4 = C_4(\Omega,\boldsymbol{g},r_0)$. For the last term in (40) we have again an estimate as in (39). Combining (38)–(43), we finally obtain

$$\begin{aligned}\int_{\Omega_r(a)}\left(\tilde e(\boldsymbol{m})+\frac14\left|\frac{j(m)}{|m|}-j(m_*)\right|^2\right)dx &\le C_5\left(\frac{\iota}{r}+\frac{\epsilon}{r}\log\frac1\epsilon+r\right)\\ &\quad+\int_{\Omega_r(a)}\left(e_\epsilon(\boldsymbol{m})-\frac12|\nabla m_*|^2\right)dx.\end{aligned}$$

If r is chosen sufficiently small, then (37) follows whenever ι and ϵ_0 are small enough. ☐

6. The motion law

In this section we prove Theorem 1.1. To this end, let $\hat a\in C^\infty([0,\infty);\Omega^d)$ be the solution of the initial value problem

$$\hat a(0)=a^0,\quad (1+2i)\pi\hat a_\ell' = -\frac{\partial}{\partial a_\ell}W(\hat a),\quad \ell=1,\dots,d.$$

The most important part of the proof is to show that these curves coincide with the vortex paths found in Theorem 4.1.

Remember that

$$\lim_{\epsilon\searrow0}D_\epsilon(\boldsymbol{m}_\epsilon^0;a^0)=0.$$

It follows at least for a sufficiently small time interval that $\boldsymbol{m}_\epsilon$ is smooth for every sufficiently small $\epsilon>0$. Choose a sequence $\epsilon_k\searrow0$ that satisfies the conclusions of Theorem 4.1, and let a be the corresponding curve in Ω^d. Let $T>0$ be small enough such that there exists a radius $r\in(0,\rho(a^0)/2]$ with $a_\ell(t)\in B_{r/2}(a_\ell^0)$ and $\hat a_\ell\in B_{r/2}(a_\ell^0)$ for $\ell=1,\dots,d$ and for every $t\in[0,T]$.

Choose $\phi, \psi \in C_0^\infty(\Omega)$ such that for every ℓ, both ϕ and ψ are affine with $\nabla\psi = \nabla^\perp\phi$ in $B_r(a_\ell^0)$. We define

$$\xi_k(t) = \int_{\Omega\times\{t\}} \left(\alpha_{\epsilon_k}\psi e_{\epsilon_k}(\boldsymbol{m}_{\epsilon_k}) + \phi\omega(\boldsymbol{m}_{\epsilon_k})\right)\, dx - \pi\sum_{\ell=1}^d \left(\psi(\hat{a}_\ell(t)) + 2\phi(\hat{a}_\ell(t))\right)$$

and

$$\xi(t) = \pi\sum_{\ell=1}^d \left(\psi(a_\ell(t)) + 2\phi(a_\ell(t)) - \psi(\hat{a}_\ell(t)) - 2\phi(\hat{a}_\ell(t))\right).$$

Then we have $\xi_k(t) \to \xi(t)$ for every t.

Lemma 6.1. *There exist a constant C and a sequence $\lambda_k \to 0$ such that for all $t_1, t_2 \in [0,T)$ with $t_1 \le t_2$ and every $k \in \mathbb{N}$,*

$$\begin{aligned}
&\xi_k(t_2) - \xi_k(t_1)\\
&\quad\le C\int_{t_1}^{t_2}\int_{\Omega_r(a^0)} \left(\tilde{e}_\epsilon(\boldsymbol{m}_{\epsilon_k}) + \left|\frac{j(m_{\epsilon_k})}{|m_{\epsilon_k}|} - j(m_*(\,\cdot\,;\hat{a}(t)))\right|^2\right) dx\,dt\\
&\qquad + C\int_{t_1}^{t_2} \|J(m_{\epsilon_k}) - \pi\delta_{\hat{a}}\|_{W^{-1,1}(\Omega)}\, dt + \lambda_k.
\end{aligned}$$

Proof. We use the abbreviation $\boldsymbol{m}_* = \boldsymbol{m}_*(\,\cdot\,;\hat{a})$ and we calculate

$$\begin{aligned}
\pi\sum_{\ell=1}^d \frac{d}{dt}\left(\psi(\hat{a}_\ell(t)) + 2\phi(\hat{a}_\ell(t))\right) &= \pi\sum_{\ell=1}^d \left(2\nabla\phi(\hat{a}_\ell(t)) + \nabla^\perp\phi(\hat{a}_\ell(t))\right)\cdot\hat{a}_\ell'(t)\\
&= -\sum_{\ell=1}^d \nabla^\perp\phi(\hat{a}_\ell(t))\cdot\frac{\partial}{\partial a_\ell}W(\hat{a})\\
&= \int_{\Omega\times\{t\}} \nabla^\perp\nabla\phi\cdot(\nabla m_*\otimes\nabla m_*)\, dx.
\end{aligned}$$

In the last step we have used a well-known representation of the gradient of W [4, Sect. VIII.2]. Moreover, using (26) and (27), we find

$$\begin{aligned}
&\frac{d}{dt}\int_{\Omega\times\{t\}} \left(\alpha_\epsilon\psi e_\epsilon(\boldsymbol{m}_\epsilon) + \phi\omega(\boldsymbol{m}_\epsilon)\right)\, dx\\
&\quad = -\alpha_\epsilon^2\int_{\Omega\times\{t\}} \psi|\boldsymbol{f}_\epsilon(\boldsymbol{m}_\epsilon)|^2\, dx\\
&\qquad - \int_{\Omega\times\{t\}} (\alpha_\epsilon^2\nabla\psi + \nabla^\perp\phi)\cdot\langle\boldsymbol{f}_\epsilon(\boldsymbol{m}_\epsilon), \nabla\boldsymbol{m}_\epsilon\rangle\, dx\\
&\qquad + \alpha_\epsilon\int_{\Omega\times\{t\}} (\nabla\psi - \nabla^\perp\phi)\cdot\langle\boldsymbol{m}_\epsilon\times\boldsymbol{f}_\epsilon(\boldsymbol{m}_\epsilon), \nabla\boldsymbol{m}_\epsilon\rangle\, dx.
\end{aligned}$$

We have

$$\lim_{\epsilon\searrow 0}\left(\alpha_\epsilon^2\int_0^T\int_\Omega (e_\epsilon(\boldsymbol{m}_\epsilon) + |\boldsymbol{f}_\epsilon(\boldsymbol{m}_\epsilon)|^2)\, dx\,dt\right) = 0$$

by (23) and (25), and

$$\lim_{k\to\infty}\left(\alpha_\epsilon\int_0^T\int_{\Omega_r(a^0)} |\nabla\boldsymbol{m}_{\epsilon_k}||\boldsymbol{f}_{\epsilon_k}(\boldsymbol{m}_{\epsilon_k})|\, dx\right) = 0$$

by Theorem 2.1. Furthermore,

$$\int_\Omega \nabla^\perp \phi \cdot \langle \boldsymbol{f}_\epsilon(\boldsymbol{m}), \nabla \boldsymbol{m}\rangle \, dx = -\int_\Omega \nabla^\perp \nabla \phi \cdot (\nabla \boldsymbol{m} \otimes \nabla \boldsymbol{m}) \, dx$$

for any $\boldsymbol{m} \in C^\infty(\Omega; S^2)$. Therefore, it suffices to estimate the integrals

$$\int_{t_1}^{t_2} \int_\Omega \nabla^\perp \nabla \phi \cdot (\nabla \boldsymbol{m}_\epsilon \otimes \nabla \boldsymbol{m}_\epsilon - \nabla m_* \otimes \nabla m_*) \, dx \, dt.$$

For $\boldsymbol{m} = (m, m_3) \in C^\infty(\Omega; S^2)$, we calculate

$$\nabla \boldsymbol{m} \otimes \nabla \boldsymbol{m} = \nabla |m| \otimes \nabla |m| + \nabla m_3 \otimes \nabla m_3 + \frac{j(m)}{|m|} \otimes \frac{j(m)}{|m|}$$

and

$$\begin{aligned} &\frac{j(m)}{|m|} \otimes \frac{j(m)}{|m|} - j(m_*) \otimes j(m_*) \\ &\quad = \left(\frac{j(m)}{|m|} - j(m_*)\right) \otimes \left(\frac{j(m)}{|m|} - j(m_*)\right) \\ &\quad\quad + \left(\frac{j(m)}{|m|} - j(m_*)\right) \otimes j(m_*) + j(m_*) \otimes \left(\frac{j(m)}{|m|} - j(m_*)\right) \\ &\quad = \left(\frac{j(m)}{|m|} - j(m_*)\right) \otimes \left(\frac{j(m)}{|m|} - j(m_*)\right) \\ &\quad\quad + (1 - |m|) \left(\frac{j(m)}{|m|} \otimes j(m_*) + j(m_*) \otimes \frac{j(m)}{|m|}\right) \\ &\quad\quad + (j(m) - j(m_*)) \otimes j(m_*) + j(m_*) \otimes (j(m) - j(m_*)). \end{aligned}$$

It is clear that

$$\left| \int_{\Omega \times \{t\}} (1 - |m_{\epsilon_k}|) \nabla^\perp \nabla \phi \cdot \left(\frac{j(m_{\epsilon_k})}{|m_{\epsilon_k}|} \otimes j(m_*)\right) \, dx \right| \leq C_1 \epsilon_k E_{\epsilon_k}(\boldsymbol{m}_{\epsilon_k})$$

for a constant C_1 that is independent of k and t. In order to estimate

$$\int_{t_1}^{t_2} \int_\Omega \nabla^\perp \nabla \phi \cdot ((j(m_{\epsilon_k}) - j(m_*)) \otimes j(m_*)) \, dx \, dt$$

and

$$\int_{t_1}^{t_2} \int_\Omega \nabla^\perp \nabla \phi \cdot (j(m_*) \otimes (j(m_{\epsilon_k}) - j(m_*))) \, dx \, dt,$$

we first note that in both cases, we can write the integrand in the form

$$\sigma \cdot (j(m_{\epsilon_k}) - j(m_*))$$

for a smooth vector field $\sigma \in C^\infty(\overline{\Omega} \times [0, T]; \mathbb{R}^2)$ that is independent of k. We use a Hodge decomposition

$$-\sigma = \nabla u + \nabla^\perp v,$$

where $u, v \in C^\infty(\overline{\Omega} \times [0,T])$ with $u = 0$ on $\partial\Omega \times [0,T]$. Then

$$\int_{t_1}^{t_2} \int_\Omega \sigma \cdot (j(m_{\epsilon_k}) - j(m_*))\, dx\, dt$$
$$= \int_{t_1}^{t_2} \int_\Omega u \operatorname{div} j(m_{\epsilon_k})\, dx\, dt + 2 \int_{t_1}^{t_2} \left(\int_\Omega v J(m_{\epsilon_k})\, dx - \pi \sum_{\ell=1}^d v(\hat{a}_\ell) \right) dt.$$

Moreover, we compute

$$\operatorname{div} j(m_\epsilon) = (im_\epsilon, f_\epsilon(\boldsymbol{m}_\epsilon)) = \frac{1}{1+\alpha_\epsilon^2} \left(\alpha_\epsilon \left(im_\epsilon, \frac{\partial m_\epsilon}{\partial t} \right) - \frac{\partial m_{\epsilon 3}}{\partial t} \right).$$

Hence

$$\begin{aligned} \int_{t_1}^{t_2} \int_\Omega u \operatorname{div} j(m_{\epsilon_k})\, dx\, dt = {} & \frac{\alpha_\epsilon}{1+\alpha_\epsilon^2} \int_{t_1}^{t_2} \int_\Omega u \left(im_\epsilon, \frac{\partial m_\epsilon}{\partial t} \right) dx\, dt \\ & + \frac{1}{1+\alpha_\epsilon^2} \int_{t_1}^{t_2} \int_\Omega \frac{\partial u}{\partial t} m_{\epsilon 3}\, dx\, dt \\ & - \frac{1}{1+\alpha_\epsilon^2} \int_{\Omega \times \{t_2\}} u m_{\epsilon 3}\, dx \\ & + \frac{1}{1+\alpha_\epsilon^2} \int_{\Omega \times \{t_1\}} u m_{\epsilon 3}\, dx. \end{aligned}$$

Hence there exists a constant C_2, independent of k or t_1, t_2, such that

$$\int_{t_1}^{t_2} \int_\Omega \nabla^\perp \nabla \phi \cdot (j(m_{\epsilon_k}) - j(m_*)) \otimes j(m_*)\, dx\, dt$$
$$\leq C_2 \int_{t_1}^{t_2} \|J(m_{\epsilon_k}) - \pi \delta_{\hat{a}}\|_{W^{-1,1}(\Omega)}\, dt + C_2 \sqrt{\alpha_{\epsilon_k}},$$

and we can draw the same conclusion for $j(m_*) \otimes (j(m_{\epsilon_k}) - j(m_*))$. This finally gives the required estimate. □

Lemma 6.2. *For $t \in [0,T)$, let*

$$\hat{h}_k(t) = \int_{\Omega_r(a^0) \times \{t\}} \left(\tilde{e}_\epsilon(\boldsymbol{m}_{\epsilon_k}) + \frac{1}{8} \left| \frac{j(m_{\epsilon_k})}{|m_{\epsilon_k}|} - j(m_*(\,\cdot\,; \hat{a}(t))) \right|^2 \right) dx$$
$$- D_{\epsilon_k}(\boldsymbol{m}_{\epsilon_k}(\,\cdot\,, t); \hat{a}(t)).$$

Then there exists a constant C such that for almost all $t_1, t_2 \in [0,T)$ with $t_1 \leq t_2$,

$$\limsup_{k \to \infty} \int_{t_1}^{t_2} \hat{h}_k(t)\, dt \leq C \int_{t_1}^{t_2} |\hat{a}(t) - a(t)|\, dt.$$

Proof. In addition to $\hat{h}_k$, we define the functions

$$h_k(t) = \int_{\Omega_r(a^0) \times \{t\}} \left(\tilde{e}_\epsilon(\boldsymbol{m}_{\epsilon_k}) + \frac{1}{4} \left| \frac{j(m_{\epsilon_k})}{|m_{\epsilon_k}|} - j(m_*(\,\cdot\,; a(t))) \right|^2 \right) dx$$
$$- D_{\epsilon_k}(\boldsymbol{m}_{\epsilon_k}(\,\cdot\,, t); a(t)).$$

It is clear that

$$\hat{h}_k(t) \leq h_k(t) + C_1 |\hat{a}(t) - a(t)|$$

for a certain constant C_1 that does not depend on k or t. Thus it suffices to prove the inequality

$$\limsup_{k\to\infty} \int_{t_1}^{t_2} h_k(t)\, dt \le 0.$$

To this end, we first recall that $J(m_{\epsilon_k}(\,\cdot\,,t)) \to \pi\delta_{a(t)}$ in $W^{-1,1}(\Omega)$ for every t. Using Theorem 5.3, we conclude that

$$\limsup_{k\to\infty} h_k(t) \le 0.$$

On the other hand, by Theorem 2.1, we have a constant C_2 that is independent of k or t such that $h_k(t) \le C_2$. Fatou's lemma, applied to $C_2 - h_k$, now gives the required inequality in the limit. □

Next we consider the functions

$$\zeta_k(t) = D_{\epsilon_k}(\boldsymbol{m}_{\epsilon_k}(\,\cdot\,,t);\hat{a}(t))$$

and

$$\chi_k(t) = \|J(m_{\epsilon_k}) - \pi\delta_{\hat{a}}\|_{W^{-1,1}(\Omega)}\,.$$

Let $0 \le t_1 \le t_2 \le T$. Using the identities

$$E_\epsilon(\boldsymbol{m}_\epsilon(\,\cdot\,,t_2)) - E_\epsilon(\boldsymbol{m}_\epsilon(\,\cdot\,,t_1)) = -\frac{\alpha_\epsilon}{1+\alpha_\epsilon^2}\int_{t_1}^{t_2}\int_\Omega \left|\frac{\partial \boldsymbol{m}_\epsilon}{\partial t}\right|^2 dx\, dt$$

and

$$W(\hat{a}(t_1)) - W(\hat{a}(t_2)) = \pi \int_{t_1}^{t_2} |\hat{a}'|^2\, dt,$$

and observing that the functions

$$t \mapsto \frac{\alpha_\epsilon}{1+\alpha_\epsilon^2}\int_{\Omega\times\{t\}} \left|\frac{\partial \boldsymbol{m}_\epsilon}{\partial t}\right|^2 dx$$

are uniformly bounded in $L^1(0,T)$, we obtain a uniform bound for ζ_k in $\mathrm{BV}(0,T)$. Thus we can select a subsequence such that $\zeta_k(t) \to \zeta(t)$ almost everywhere for a function $\zeta : [0,T] \to \mathbb{R}$. Moreover, Theorem 4.1 implies

$$\begin{aligned}
\zeta(t_2) - \zeta(t_1) &\le \pi \int_{t_1}^{t_2} (|\hat{a}'|^2 - |a'|^2)\, dt \\
&= -\pi \int_{t_1}^{t_2} |\hat{a}' - a'|^2\, dt + 2\pi \int_{t_1}^{t_2} \hat{a}' \cdot (\hat{a}' - a')\, dt \\
&\le C_1 \int_{t_1}^{t_2} |\hat{a}' - a'|\, dt
\end{aligned}$$

for almost all t_1, t_2 with $t_1 \le t_2$. That is, we have

$$\zeta' \le C_1 |\hat{a}' - a'|$$

in the sense of distributions.

We define

$$\chi(t) = \sum_{\ell=1}^{d} |\hat{a}_\ell(t) - a_\ell(t)|.$$

Then we know that $\chi_k(t) \to \chi(t)$ almost everywhere. Using similar arguments as in the proof of Proposition 3.1, we also see that χ_k is uniformly bounded in $L^\infty(0,T)$. Hence by Lebesgue's convergence theorem, we have $\chi_k \to \chi$ in $L^1(0,T)$. Using Lemma 6.1 and Lemma 6.2, we therefore obtain

$$\xi(t_2) - \xi(t_1) \leq C_2 \int_{t_1}^{t_2} (\zeta(t) + \chi(t))\, dt$$

for a certain constant C_2. Choosing ϕ and ψ appropriately, this shows that

$$|\hat{a}'(t) - a'(t)| \leq C_3(\zeta(t) + \chi(t))$$

for almost every t. Hence

$$|\hat{a}'(t) - a'(t)| \leq C_4 \int_0^t |\hat{a}'(\tau) - a'(\tau)|\, d\tau,$$

and Gronwall's inequality implies first $\hat{a}' = a'$ and then $\hat{a} = a$ in $[0,T]$. Furthermore, we conclude that $\zeta \leq 0$ in $[0,T]$, and this means in particular that

$$\limsup_{k\to\infty} D_{\epsilon_k}(\boldsymbol{m}_{\epsilon_k}(\,\cdot\,,T); a(T)) \leq 0.$$

Thus we can use the same arguments for T as a new initial time, and we eventually obtain the motion law for all times. Moreover, it follows that $\boldsymbol{m}_{\epsilon_k}$ is smooth for all times.

Finally, we use the fact that the initial value problem that defines $\hat{a}$ has a unique solution. If we choose any sequence $\epsilon_k \searrow 0$, then with these arguments, we find a subsequence which has the properties just described. We conclude that in fact the stronger statement of Theorem 1.1 holds true.

7. Physical background and interpretation

The purpose of this section is to explain how the functional E_ϵ and the corresponding Landau-Lifshitz-Gilbert equations (1) and (2) are related to the theory of micromagnetics. We also mention some approaches to vortex dynamics from theoretical and experimental physics and explain how our main result fits with what can be expected in more realistic models.

We start by describing the model of micromagnetics. More details can be found in the books by Hubert and Schäfer [18] and Bertotti [3]. Suppose that we have a sample of a ferromagnetic material occupying a domain $\Sigma \subset \mathbb{R}^3$. Its magnetization is given by a vector field $\boldsymbol{m} : \Sigma \to \mathbb{R}^3$. At temperatures well below the Curie point, it has constant length. After a renormalization, we can regard $\boldsymbol{m}$ as a map into the unit sphere S^2. To a given magnetization we then assign an energy $\mathcal{E}(\boldsymbol{m})$ which is the sum of the following contributions.

1. The exchange energy is

$$\frac{\epsilon^2}{2} \int_\Sigma |\nabla \boldsymbol{m}|^2\, d\boldsymbol{x},$$

where ϵ is a material constant, called the exchange length. (Typically ϵ is small.) This term is derived from the spin interactions in an underlying atomistic model.

2. The anisotropy energy has the form

$$\int_\Sigma \Phi(\boldsymbol{m})\, d\boldsymbol{x}$$

for a function $\Phi : S^2 \to \mathbb{R}$. As the name suggests, this models effects coming from anisotropies in the crystal lattice.

3. The magnetization generates a stray field $\boldsymbol{h}$ in all of $\mathbb{R}^3$, which gives rise to the third energy contribution. By the static Maxwell equations, the stray field is curl free and can therefore be represented as $\boldsymbol{h} = -\nabla u$ for a potential $u \in H^1(\mathbb{R}^3)$. Moreover, the function u is determined by the equation

$$\Delta u = \operatorname{div} \boldsymbol{m} \quad \text{in } \mathbb{R}^3, \tag{44}$$

which is to be understood in the distribution sense, with $\boldsymbol{m}$ extended by 0 outside of Σ. The magnetostatic energy or stray field energy is then

$$\frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 \, d\boldsymbol{x}.$$

4. If there is an external field $\boldsymbol{H}$, then this is responsible for another energy term, namely

$$-\int_\Sigma \boldsymbol{H} \cdot \boldsymbol{m} \, d\boldsymbol{x}.$$

Thus we obtain the total micromagnetic energy

$$\mathcal{E}(\boldsymbol{m}) = \int_\Sigma \left(\frac{\epsilon^2}{2} |\nabla \boldsymbol{m}|^2 + \Phi(\boldsymbol{m}) - \boldsymbol{H} \cdot \boldsymbol{m} \right) d\boldsymbol{x} + \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 \, d\boldsymbol{x},$$

the last term being subject to (44). Its negative L^2-gradient,

$$\boldsymbol{F}(\boldsymbol{m}) = \epsilon^2 \Delta \boldsymbol{m} - \nabla \Phi(\boldsymbol{m}) - \boldsymbol{H} + \nabla u,$$

is called the effective field.

If we study a sample in the shape of a thin film, such as $\Sigma = \Omega \times (0, \tau)$ for a domain $\Omega \subset \mathbb{R}^2$ and a small number $\tau > 0$, then it is natural to assume that $\boldsymbol{m}$ is independent of the third variable x_3 in order to simplify the functional. The exchange energy then gives rise to

$$\frac{\tau \epsilon^2}{2} \int_\Omega |\nabla \boldsymbol{m}|^2 \, dx,$$

and after a renormalization, we have the first term in the definition of E_ϵ. The second term of E_ϵ, stemming from

$$\tau \int_\Omega \frac{m_3^2}{2} \, dx,$$

has the form of an anisotropy energy that penalizes only the out-of-plane component and is independent of the in-plane directions. However, it does not necessarily arise out of a crystalline anisotropy; in fact it is often more natural to think of it as an approximation of the magnetostatic energy or perhaps a combination of these energies, as the latter can typically not be neglected for very thin films, while the anisotropy contribution in so-called soft films is very small.

It can be seen from (44) that the magnetostatic energy effectively penalizes the distributional divergence of $\boldsymbol{m}$, a part of which comes from the normal part of $\boldsymbol{m}$ on the boundary of Σ, called surface charge. In the case of a thin film, a large part of the boundary is $\Omega \times \{0, \tau\}$, and here the third component m_3 is penalized. Moreover, if τ becomes very small, then the surface contribution will eventually dominate the rest of the magnetostatic energy. In fact, $\frac{\tau}{2}\int_\Omega m_3^2\,dx$ is the leading order contribution, followed by a nonlocal term $\frac{\tau^2}{4}\|\operatorname{div} m\|^2_{\dot{H}^{-1/2}}$ that enforces tangential magnetization at the lateral boundary. More precisely, by virtue of a critical Sobolev embedding, the specific energy of the normal component $m\cdot\nu$ scales with an additional logarithmic factor as $\tau\to 0$, see [9, 32] for details.

Reducing the magnetostatic energy via boundary effects eventually leads to topological singularities and the formation of domain patterns. Vortices, also known as Bloch lines in this context, generally appear within domain wall networks. But depending on the specific regime $\tau,\epsilon\to 0$, the scenario can be quite diverse. In [32], the boundary penalty is so weak that the global minimizer is very different and develops boundary vortices. For an extensive discussion of scaling regimes and the energy of certain features of a thin-film ferromagnet, we refer the reader to DeSimone et al. [10]; results for specific regimes can be found in [9, 22, 23, 24, 31, 32, 33]. In very small magnetic systems such as magnetic nanodots, however, the formation of internal vortices is energetically preferable to the formation of domain walls or boundary vortices [42]. This is the situation we intend to describe. We replace the nonlocal magnetostatic energy by the local integral above and we impose Dirichlet boundary conditions for $\boldsymbol{m}$. Our setting allows for interior vortices but no boundary interaction, which fits the physical expectations in such nanoregimes but still lacks rigorous mathematical support. Note that in this model the nonlocal magnetostatic energy has been reduced to local contributions of the boundaries $\Omega\times\{0,\tau\}$ and a constraint coming from $\partial\Omega\times(0,\tau)$; any contribution of the nonsingular part of the divergence $\operatorname{div} m$ has been neglected. A somewhat more realistic model might be to use a boundary penalty for $m\cdot\nu$ instead of the Dirichlet condition, compare [31, 32].

After explaining the origin of our time-independent model, we turn to the evolution equations. A model for the dynamics of the magnetization is given by the Landau-Lifshitz-Gilbert equation. There are several ways to represent it; perhaps the most common form in the physics literature is the Gilbert form [12],

$$\frac{\partial \boldsymbol{m}}{\partial t} + \boldsymbol{m}\times \boldsymbol{F}(\boldsymbol{m}) = \alpha\,\boldsymbol{m}\times\frac{\partial \boldsymbol{m}}{\partial t}. \tag{45}$$

This includes a phenomenological damping term, namely the right-hand side of the equation. The dimensionless damping constant $\alpha>0$ is typically small. If we want to solve (45) for initial data $\boldsymbol{m}^0:\Sigma\to S^2$, then we note that for a solution of (45), the derivative $\frac{\partial \boldsymbol{m}}{\partial t}$ is always tangential to the sphere. Hence we expect that $\boldsymbol{m}$ continues to take values on S^2.

Taking the vector product with $\boldsymbol{m}$ on both sides of (45), we obtain

$$\alpha\frac{\partial \boldsymbol{m}}{\partial t} + \boldsymbol{m}\times\frac{\partial \boldsymbol{m}}{\partial t} + \boldsymbol{m}\times(\boldsymbol{m}\times\boldsymbol{F}(\boldsymbol{m})) = 0. \tag{46}$$

Combining (45) and (46) yields the equation

$$(1+\alpha^2)\frac{\partial \boldsymbol{m}}{\partial t} = -\alpha\,\boldsymbol{m}\times(\boldsymbol{m}\times\boldsymbol{F}(\boldsymbol{m})) - \boldsymbol{m}\times\boldsymbol{F}(\boldsymbol{m}), \tag{47}$$

which is the original form introduced by Landau and Lifshitz in [26]. In fact, all three equations (45)–(47) are equivalent, at least for classical solutions. If we define

$$\boldsymbol{f}(\boldsymbol{m}) = -\boldsymbol{m} \times (\boldsymbol{m} \times \boldsymbol{F}(\boldsymbol{m})),$$

which is the orthogonal projection of $\boldsymbol{F}(\boldsymbol{m})$ onto the tangent space of S^2, then we can write (47) in the form

$$(1+\alpha^2)\frac{\partial \boldsymbol{m}}{\partial t} = \alpha\, \boldsymbol{f}(\boldsymbol{m}) - \boldsymbol{m} \times \boldsymbol{f}(\boldsymbol{m}). \tag{48}$$

Observing that

$$\Delta \boldsymbol{m} + |\nabla \boldsymbol{m}|^2 \boldsymbol{m}$$

is the tangential part of $\Delta \boldsymbol{m}$ (which can be verified by differentiating the equation $\langle \boldsymbol{m}, \nabla \boldsymbol{m}\rangle = 0$), we see that equation (2) is (48) for the functional E_ϵ instead of $\mathcal{E}$.

Returning to a more physical point of view, recall that our energy E_ϵ is a model for small, thin magnetic elements whose energetically most favored state is a vortex. These are very interesting from a technological point of view, as the vortex has both a chirality (clockwise or counterclockwise) and a polarity (up or down), making it possible to store two bits of data in a single small element. For a short survey of recent advances in theoretical and experimental physics relating to vortices in small elements, we refer to [2].

The idea to describe the magnetization dynamics by an effective equation for a simpler system, e.g. by an ODE for the vortex or domain wall coordinates (treating them as a quasiparticle) essentially goes back to Thiele [44] and was used for vortex systems by Huber [17]. However, only very little rigorous theory exists in this context, while mathematical results mainly address the motion of domain walls, see e.g. [5, 35]. Our equation of motion for the vortices, (4), corresponds precisely to Huber's results in the case of no boundary conditions. Our choice of a logarithmic α_ϵ is also in perfect agreement with [17].

In the case of a disc-shaped domain, $\Omega = B_1(0)$, the renormalized energy can be calculated explicitly. For a single vortex at $a \in B_1(0)$ and Dirichlet boundary conditions $\boldsymbol{m} = \frac{(x^\perp,0)}{|x|}$, it is given by $W(a) = \pi \log \frac{1}{1-|a|^2}$. It follows that (4) describes a spiralling motion into the center. Such spiral motions have been found numerically for the full problem [2]; the frequency of the gyrotropic part can also be measured experimentally [34]. In the case of a disc, the central vortex has one interesting property: only if the vortex is central ($a = 0$), the associated canonical harmonic map m_* is divergence free. If we make a better approximation of $\mathcal{E}$ where the divergence is also penalized by $\|\operatorname{div} m\|^2_{\dot{H}^{-1/2}}$, see [9], we expect this to correspond to an additional energy contribution that is quadratic in a, at least for small $|a|$. Given that $W(a) \approx \pi |a|^2$ for small $|a|$, adding another quadratic term does not change the local form of the energy landscape; this implies that in the special case of a disc, our results should yield the correct qualitative behavior even if our approximation of the nonlocal magnetostatic energy by a local term is not completely justified.

Finally, we mention that very recently, there has been considerable interest in fast switching of vortex core polarities using an applied field pulse. The fastest known switching path involves the nucleation and annihilation of a pair of vortices, see [14]. These processes involve bubbling events that our present analytical techniques are not yet sophisticated enough to describe.

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