PROOF OF THE CARATHÉODORY CONJECTURE BY MEAN CURVATURE FLOW IN THE SPACE OF ORIENTED AFFINE LINES

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1. Introduction

A conjecture attributed to Constantin Carathéodory[1] states:

The Carathéodory Conjecture: *Every closed convex C^3-smooth surface in Euclidean 3-space has at least two umbilic points.*

In this paper we prove the following:

Main Theorem. *Let p be an isolated umbilic point on a C^3-smooth surface S in Euclidean 3-space. Then the winding number of the principal foliation about p is less than or equal to 1.*

Aside from establishing the Carathéodory conjecture, this result has interest in its own right, as it is well-known to be equivalent to a conjecture of Loewner on solutions of the wave equation. For more on these conjectures and their history, see Berger [2] and references therein.

Recall that an umbilic point is a point where the second fundamental form (a symmetric 2-tensor) of the surface has a double eigenvalue. Away from such points, the eigenvectors of the second fundamental form define a pair of orthogonal foliations of the surface - the principal foliations. The winding number of the principal foliations about an isolated umbilic point is called the index of the umbilic point - which, due to the fact that the principal foliations may not be orientable, is an element of $\frac{1}{2}\mathbb{Z}$.

Our proof of the Main Theorem depends upon a reformulation of the problem in terms of isolated complex points on Lagrangian surfaces in the space of oriented geodesics of Euclidean 3-space, which we identify with TS^2. Here complex and Lagrangian refer to the neutral Kähler structure on TS^2 introduced by the authors in [7].

We relate the index of an isolated complex point on an otherwise totally real surface in any almost-complex 4-manifold to the space of holomorphic discs whose boundary is contained in the surface. Indeed, if there exists such a holomorphic disc, and it persists for all small deformations of the boundary surface, the index of the associated elliptic boundary value problem is greater than zero. We show through the Keller-Maslov index of the boundary that, in our setting, this is exactly the condition required for the Carathéodory conjecture to hold. We thus seek to prove the existence of stable holomorphic discs attached along the boundary to the Lagrangian surface in TS^2.

To establish the existence of such holomorphic discs we flow to them by the mean curvature flow associated to the neutral Kähler metric on TS^2. In particular, we prove long-time existence of the mean curvature flow for n-dimensional spacelike submanifolds in $n+m$-dimensional manifolds with metrics of signature (n, m) under certain conditions. In our case these conditions are met, and so we establish long-time existence for the mean curvature flow in TS^2 for positive discs with boundary lying in a totally real surface.

Moreover, we prove that for small enough initial disc, the flow is asymptotically holomorphic in time. By altering the complex structure we construct a sequence of holomorphic discs which, by virtue of our estimates, remain in a compact set. Crucially, we establish an a priori area bound for positive discs with boundary

[1]Hans Hamburger, Berliner Mathematische Gesellschaft, Berlin, 26th March 1924

contained on the deformation of a Lagrangian surface. Thus, the compactness of the space of J-holomorphic curves with boundary leads to a convergent sequence of holomorphic discs. While the limit disc may be bubbled - bubbling off other holomorphic discs - continuity up to the boundary ensures that the Keller-Maslov index is retained in the limit. In fact, this is all we require for our proof.

Throughout, we prove existence of the flow by a priori estimates. This allows us to conclude that the solution persists under perturbation of the boundary surface. Thus we have demonstrated the existence of stable holomorphic discs and the Main Theorem follows.

The next section contains the reformulation and proof of the Main Theorem. In the following sections we supply the details of the proof.

2. Strategy and Proof

2.1. **Reformulation of the Conjecture in $\mathbf{TS^2}$.** The space of oriented geodesics in Euclidean 3-space $\mathbb{E}^3$, namely the oriented affine lines in the underlying $\mathbb{R}^3$, may be identified with the total space of the tangent bundle to the 2-sphere.

This 4-manifold is endowed with a natural Kähler structure $(\mathbb{J}, \Omega, \mathbb{G})$ which is invariant under the action induced on TS^2 by the Euclidean group acting on $\mathbb{E}^3$. Throughout this paper, we denote the Kähler surface $(TS^2, \mathbb{J}, \Omega, \mathbb{G})$ simply by TS^2. We now briefly summarize the properties of this structure we require - further details can be found in [7] and references therein.

Given an oriented C^k-smooth surface S in $\mathbb{E}^3$ with $k \geq 1$, the set of oriented lines normal to S gives rise to a C^{k-1}-smooth surface Σ in TS^2. Such a surface Σ is Lagrangian: $\Omega|_\Sigma = 0$. Indeed, the well-known converse holds by Frobenius integrability (see for example [6]):

Proposition 1. *A surface Σ in TS^2 is Lagrangian iff there exists a surface S in $\mathbb{E}^3$ which is orthogonal to the oriented lines of Σ.*

Given one such orthogonal surface S in $\mathbb{E}^3$, we have a one parameter family of parallel surfaces which are also orthogonal to the oriented lines of Σ. Moreover, a point on S is umbilic iff the corresponding points on the parallel surfaces are also umbilic. Indeed, this property allows us to reformulate the Conjecture entirely in TS^2, as we will see below.

Let S be a C^3-smooth oriented surface in $\mathbb{E}^3$, and Σ the corresponding surface in TS^2 formed by the normal lines to S. The canonical projection $\pi : TS^2 \to S^2$ restricted to Σ is just the Gauss map of the surface, and so we have:

Proposition 2. *The surface S is non-flat (has non-zero Gauss curvature) iff Σ is the graph of a section of the bundle $\pi : TS^2 \to S^2$. In particular, the surface Σ in TS^2 formed by the oriented normal lines of a convex surface S is the graph of a section.*

A point p on a surface Σ in an almost complex 4-manifold $(\mathbb{M}, J)$ is said to be *complex* if $J : T_p\mathbb{M} \to T_p\mathbb{M}$ leaves $T_p\Sigma$ invariant. In our situation:

Proposition 3. *A point γ on a Lagrangian surface Σ in TS^2 is complex iff the point on the orthogonal surface S in $\mathbb{E}^3$ with oriented normal line γ is umbilic.*

Moreover, the index of an isolated umbilic point on S is precisely the index of the associated complex point on the Lagrangian surface Σ as defined in [6].

Here, an umbilic point on $S \subset \mathbb{E}^3$ is a point where the second fundamental form has a double eigenvalue and the $\frac{1}{2}\mathbb{Z}$-valued index of an isolated umbilic point is the winding number of the eigen-directions about the point.

Thus the Main Theorem is equivalent to:

Main Theorem. *An isolated complex point γ on the graph of a Lagrangian section of $TS^2 \to S^2$ has index less than or equal to 1.*

Note that, given a surface $S \subset \mathbb{E}^3$ with isolated umbilic point, by a Möbius transformation of $\mathbb{E}^3$, S can be locally mapped to a convex surface with isolated umbilic point of the same index. Thus, while we prove the Main Theorem in the convex case, we can in fact drop the assumption of convexity of the surface in $\mathbb{E}^3$.

2.2. **The manifold of J-holomorphic discs with boundary.** We now discuss holomorphic discs with boundary lying on a totally real surface in an almost complex surface $(\mathbb{M}, J)$. Throughout, we denote by $C^{k,\alpha}$ the usual Hölder spaces in the appropriate function spaces.

Definition 1. For an almost complex surface $(\mathbb{M}, J)$ define the set of Hölder boundary conditions by

$$\mathcal{B} = \{\Sigma \subset \mathbb{M} \mid \Sigma \text{ is a totally real embedded surface of smoothness } C^{k,\alpha}\ \},$$

and the set of parameterized discs with boundary condition by

$$\mathcal{F} = \{(f, \Sigma) \mid f : D \to \mathbb{M},\ f \in C^{k,\alpha}(D) \cap C^0(\overline{D}) \text{ and } f(\partial D) \subset \Sigma \in \mathcal{B}\}.$$

The spaces $\mathcal{F}$ and $\mathcal{B}$ are Banach manifolds with respect to the $C^{k,\alpha}$ norm and the projection $\pi : \mathcal{F} \to \mathcal{B} : \pi(f, \Sigma) = \Sigma$ is a Banach bundle. For $(f, \Sigma) \in \mathcal{F}$ define $\bar{\partial}_J f = \frac{1}{2}(df \circ j - J \circ df)$ where j is the standard complex structure on D. Then, $\bar{\partial}_J f \in \Omega^{01}(f^*T\mathbb{M})$ and we define

$$\Omega^{01}(\mathcal{F}) = \bigcup_{(f,\Sigma)\in\mathcal{F}} \Omega^{01}(f^*T\mathbb{M}).$$

This is a Banach vector bundle over $\mathcal{F}$ and the operator $\bar{\partial}_J$ is a section of this bundle.

Definition 2. The set of holomorphic discs with boundary condition is defined by

$$\mathcal{M}_J = \{(f, \Sigma) \in \mathcal{F} \mid \bar{\partial}_J f = 0\ \}.$$

Consider the linearization of $\bar{\partial}_J$ at $(f, \Sigma) \in \mathcal{F}$ with respect to any connection on $\Omega^{01}(\mathcal{F})$:

$$\nabla_{(f,\Sigma)}\bar{\partial}_J : \Gamma(f^*TM, f^*T\Sigma) \to \Omega^{01}(f^*TM),$$

where $\Gamma(f^*T\mathbb{M}, f^*T\Sigma)$ denotes sections of $f^*T\mathbb{M} \to D$ with boundary values in the sub-bundle $f^*T\Sigma \to \partial D$. The key points about this operator are:

Proposition 4. [10] *$\nabla_{(f,\Sigma)}\bar{\partial}_J$ is Fredholm. If $\nabla_{(f,\Sigma)}\bar{\partial}_J$ is surjective then $\mathcal{M}_J \subset \mathcal{F}$ is a Banach manifold with $T_{(f,\Sigma)}\mathcal{M}_J = T_\Sigma\mathcal{B} \oplus ker\nabla_{(f,\Sigma)}\overline{\partial}_J$.*

There exists $\mathcal{B}_0 \subset \mathcal{B}$ dense and open such that for all $(f, \Sigma_0) \in \pi^{-1}(\mathcal{B}_0) \cap \mathcal{M}_J$ we have: $\nabla_{(f,\Sigma_0)}\bar{\partial}_J$ is surjective.

Definition 3. [10] For $(f,\Sigma)\in\mathcal{F}$, define the *Keller-Maslov index* $\mu(f,\Sigma)\in\mathbb{Z}$ to be the relative Chern class

$$\mu(f,\Sigma)=c_1(f^*T\mathbb{M},f^*T\Sigma)=c_1\left(f^*T^{10}\mathbb{M}\bigcup_{f^*T\Sigma}\overline{f^*T^{10}\mathbb{M}}\right),$$

where we double the base D to $D\cup\bar{D}$ and identify the fibres by conjugation about $T\Sigma\subset T^{10}\mathbb{M}$ over ∂D.

Proposition 5. *The index of the Fredholm operator $\nabla_{(f,\Sigma)}\bar{\partial}_J$ is ind$(\nabla_{(f,\Sigma)}\bar{\partial})=\mu(f,\Sigma)-1$. In addition, if $\nabla_{(f,\Sigma)}\bar{\partial}_J$ is surjective, then the projection $\pi:\mathcal{M}_J\to\mathcal{B}$ is Fredholm with index ind$(\nabla_{(f,\Sigma)}\pi)=\mu(f,\Sigma)-1$.*

Proof. The first claim follows from the index formula for the corresponding Riemann-Hilbert boundary value problem, see [10].

The second claim is seen as follows. By Proposition 4, $\mathcal{M}_J$ is a Banach manifold and since $\nabla_{(f,\Sigma)}\pi:T_{(f,\Sigma)}\mathcal{M}_J\equiv T_\Sigma\mathcal{B}\oplus ker\nabla_{(f,\Sigma)}\overline{\partial}_J\to T_\Sigma\mathcal{B}$ we have $ker(\nabla_{(f,\Sigma)}\pi)=ker\nabla_{(f,\Sigma)}\overline{\partial}_J$ from which the claim follows. □

Theorem 1 (Sard-Smale)**.** *If ind$(\nabla_{(f,\Sigma)}\pi)\leq 0$ for $(f,\Sigma)\in\mathcal{U}\subset\mathcal{M}_J$, then $\pi(\mathcal{U})\subset\mathcal{B}$ has measure zero.*

2.3. **Proof of the Main Theorem.** Given a Lagrangian surface $\Sigma\subset TS^2$ with isolated complex point $\gamma\in\Sigma$, we seek to construct a holomorphic disc with boundary lying in Σ by mean curvature flow. In particular, we consider the following parabolic boundary value problem.

P.B.V.P.

Consider a family of immersed positive discs in TS^2 given by $f_s:D\to TS^2$: $f_s(\xi,\bar{\xi})=(\xi,\eta=F(s,\xi,\bar{\xi}))$ such that

$$\frac{df_s}{ds}^{\perp}=H,$$

with initial condition

$$F(0,\xi,\bar{\xi})=F_0(\xi,\bar{\xi}),\qquad \textit{that is}\qquad D_0=\Sigma_0,$$

and boundary conditions

(i) $f_s(\partial D)\subset\tilde{\Sigma}$,

(ii) $-\epsilon|\lambda|+|\sigma|=|\tilde{\sigma}|$ *on* $f_s(\partial D)$,

where H is the mean curvature vector associated with the immersion f_s in $(TS^2,\mathbb{G})$, Σ_0 and $\tilde{\Sigma}$ are some given positive discs, $\epsilon>0$ and λ, σ and $\tilde{\sigma}$ are the twist and shear of $f_s(D)$ and $\tilde{\Sigma}$ (cf. Definition 9).

We claim that for small enough initial discs the above flow converges to a holomorphic disc with boundary on $\tilde{\Sigma}$. We prove this as follows.

In Theorem 2 we establish long-time existence of the mean curvature flow of a compact n-dimensional spacelike submanifold Σ of a manifold $\mathbb{M}$ endowed with a metric of signature (n,m), subject to the conditions:

(A) the submanifold Σ remains within a compact set of $\mathbb{M}$
(B) timelike curvature condition (3.1) holds

This result is extended in Theorem 3 from compact submanifolds to submanifolds with boundary constrained to lie in an n-dimensional submanifold, and an additional Neumann condition.

In order to apply this result to mean curvature flow of positive discs in the space of oriented affine lines TS^2, a number of technical issues must be addressed, namely:

(1) show that conditions (A) and (B) hold for spacelike graphs in TS^2
(2) determine boundary and initial surfaces
(3) ensure parabolicity of the boundary conditions

To resolve issue (1) we show that (A) holds in Proposition 25. While the timelike curvature condition does not hold in full generality for TS^2, we prove in Proposition 30 that it does hold along a solution of **P.B.V.P**.

Proposition 18 and Proposition 36 with $C_0 = 0$ demonstrate that we cannot use the Lagrangian surface Σ generated by normals to the convex surface in $\mathbb{E}^3$ as either initial or boundary surface in our flow, as it is neither positive definite nor allows positive discs to be attached to it.

Rather, for the boundary condition, we add a linear holomorphic twist about the complex point (equation (4.9)), which produces a disc $\tilde{\Sigma}$ that is positive for small enough radius and allows other positive discs to be attached to it (Proposition 32). This perturbed surface has an isolated complex point of the same index as the original Lagrangian surface and so we aim to find stable holomorphic discs attached to it. The existence of a compatible initial disc is proven in Proposition 35 and so we have addressed issue (2).

The C^3 smoothness requirement on $S \subset \mathbb{E}^3$ ensures that the mean curvature vectors of Σ, $\tilde{\Sigma}$ adn Σ_0 are C^0, the minimum requirement for **P.B.V.P.** to admit a solution.

To resolve issue (3), we formulate a general set of higher codimensional boundary value problems and prove in Proposition 15 that they are parabolic. Then, in Proposition 33, we show that the boundary conditions we implement in the TS^2 flow are of this type.

We have therefore proven long-time existence in the setting in which we require it. Moreover, in Theorem 4 we prove that, for small enough initial disc, the flowing disc is asymptotically holomorphic. Thus we can deform the complex structure so that, for late enough time, the flowing disc is holomorphic. From this we extract a sequence of J-holomorphic discs for which, if there exists an a priori area bound, compactness results (Theorem 5) imply that we can extract a holomorphic disc in the limit. In Proposition 36 such an area bound is shown to exist precisely when the boundary surface is Lagrangian with a linear holomorphic twist. The limit disc $f_\infty(D)$, which may be bubbled, is holomorphic with respect to our original $\mathbb{J}$ and continuity up to the boundary implies that the Keller-Maslov index of the holomorphic discs is retained in the limit. These results are established in Theorem 6. Interpolation arguments show that the flow converges to this disc.

To complete the proof of the Main Theorem we prove in Proposition 34 that the boundary of the flowing disc does not cross the complex point. Thus we can attach a holomorphic disc $f(D)$ to $\tilde{\Sigma}$ about the complex point γ. Moreover, for any small perturbation of the boundary surface $\tilde{\Sigma}$ we can carry out the mean curvature flow and arrive at a holomorphic disc attached to the perturbation. By virtue of Proposition 5 and Theorem 1 this implies that $\text{ind}(\nabla_{(f,\Sigma)}\pi) = \mu(f,\Sigma) - 1 > 0$.

Let I be the index of the isolated complex point and μ be the Keller-Maslov index of $(f_\infty, \tilde{\Sigma})$. Then, considering the Euler number of the sphere consisting of $f_\infty(D)$ and the $\tilde{\Sigma}$ joined along the boundary of D we have

$$I + \tfrac{1}{2}\mu = 2.$$

Thus,

$$I = 2 - \tfrac{1}{2}\mu < \tfrac{3}{2},$$

which establishes the Main Theorem.

3. Mean Curvature Flow of Spacelike Submanifolds

In this section we establish a long-time existence result for mean curvature flow of spacelike surfaces in indefinite manifolds. Throughout we utilize the summation convention on repeated indices, except for the quantity ψ_α, defined below. In some instances we include summation signs for clarity. Note that raising and lowering normal indices (Greek indices) changes the sign of the component.

3.1. **Setting.** Let $\mathbb{M}$ be an $n+m$-dimensional manifold endowed with a metric $\mathbb{G}$ of signature (n,m). We assume throughout that there exists a *multi-time function* $t : \mathbb{M} \to \mathbb{R}^m$ of maximal rank with components t_α for $\alpha = 1...m$ such that

$$\mathbb{G}(\overline{\nabla} t_\alpha, \overline{\nabla} t_\alpha) < 0 \qquad\qquad \forall \alpha = 1...m,$$

where all geometric quantities associated with $\mathbb{G}$ will be denoted with a bar.

Definition 4. The manifold $(\mathbb{M},\mathbb{G})$ is said to satisfy the *timelike curvature condition* if, for any spacelike n-plane P at a point in $\mathbb{M}$, the Riemann curvature tensor satisfies

$$\mathbb{G}(\overline{R}(X,\tau_i)X,\tau_i) \;\geq k\; \mathbb{G}(X,X), \tag{3.1}$$

for some positive constant k, where $\{\tau_i\}_{i=1}^n$ form an orthonormal basis for P and X is any timelike vector orthogonal to P.

Note 1. This generalises the timelike convergence condition of the codimension one case employed in [4]:

$$\overline{R}ic(X,X) \geq 0.$$

3.2. **Background frame.** We fix an orthonormal frame on $(\mathbb{M},\mathbb{G})$:

$$\{e_i, T_\alpha\}_{i,\alpha=1}^{n,m} \qquad \text{s.t.} \qquad \mathbb{G}(e_i,e_j) = \delta_{ij} \qquad \mathbb{G}(T_\alpha,T_\beta) = -\delta_{\alpha\beta} \qquad \mathbb{G}(e_i,T_\alpha) = 0,$$

with

$$T_\alpha = -\psi_\alpha \mathbb{G}(\overline{\nabla} t_\alpha, \cdot) \qquad\qquad \psi_\alpha^{-2} = -\mathbb{G}(\overline{\nabla} t_\alpha, \overline{\nabla} t_\alpha).$$

Definition 5. Given a contravariant tensor B on $\mathbb{M}$ we define its norm by

$$\|B\|^2 = \sum_{i_1,\dots i_l=1}^{n} [B(e_{i_1}, e_{i_2}, ..., e_{i_l})]^2 + \sum_{\beta_1,\dots\beta_l=1}^{m} [B(T_{\beta_1}, T_{\beta_2}, ..., T_{\beta_l})]^2.$$

Similarly, for a covariant tensor B we dualise it with the metric $\mathbb{G}$ and define its norm as above.

Higher derivative norms are also defined:

$$\|B\|_k^2 = \sum_{j=0}^{k} \|\overline{\nabla}^j B\|^2.$$

For a mixed tensor, we occasionally use the induced metric on the spacelike components to define a norm on the timelike components. That is, if $B_{\alpha\beta ijk}$ is a tensor of the indicated type, then we define

$$|B_{\alpha\beta}|^2 = \sum_{i=1}^{n} [B_{\alpha\beta}(e_i, e_i, e_i)]^2.$$

3.3. **Immersed spacelike submanifolds.** Let $f : \Sigma \to \mathbb{M}$ be a spacelike immersion of an n-dimensional manifold Σ, and let g be the metric induced on Σ by $\mathbb{G}$.

Definition 6. A second orthonormal basis for $(\mathbb{M}, \mathbb{G})$ along Σ is *adapted* to the submanifold if:

$$\{\tau_i, \nu_\alpha\}_{i,\alpha=1}^{n,m} \qquad \text{s.t.} \qquad \mathbb{G}(\tau_i, \tau_j) = \delta_{ij} \qquad \mathbb{G}(\nu_\alpha, \nu_\beta) = -\delta_{\alpha\beta} \qquad \mathbb{G}(\tau_i, \nu_\alpha) = 0,$$

where $\{\tau_i\}_{i=1}^n$ form an orthonormal basis for (Σ, g), and $\{\nu_\alpha\}_{\alpha=1}^m$ span the normal space.

The *second fundamental form* of the immersion is

$$A_{ij\alpha} = \mathbb{G}(\overline{\nabla}_{\tau_i}\nu_\alpha, \tau_j) = -\mathbb{G}(\overline{\nabla}_{\tau_i}\tau_j, \nu_\alpha),$$

while the *mean curvature vector* is

$$H_\alpha = g^{ij} A_{ij\alpha}.$$

We have the following two equations for the splitting of the connection

$$\overline{\nabla}_{\tau_i}\tau_j = \nabla_{\tau_i}\tau_j - A_{ij}^{\alpha}\nu_\alpha \tag{3.2}$$

$$\overline{\nabla}_{\tau_i}\nu_\alpha = A_{i\alpha}^{j}\tau_j + C_{i\alpha}^{\beta}\nu_\beta, \tag{3.3}$$

where $C_{i\alpha}^{\beta}$ are the components of the normal connection

$$\nabla_{\tau_i}^{\perp}\nu_\alpha = C_{i\alpha}^{\beta}\nu_\beta.$$

We also have the Gauss and Codazzi-Mainardi equations

$$\bar{R}_{ijkl} = R_{ijkl} + A_{kj}^{\alpha}A_{il\alpha} - A_{lj}^{\alpha}A_{ik\alpha}, \tag{3.4}$$

$$-\bar{R}_{kij}^{\alpha} = \nabla_i A_{jk}^{\alpha} - \nabla_j A_{ik}^{\alpha} + A_{jk}^{\beta}C_{i\beta}^{\alpha} - A_{ik}^{\beta}C_{j\beta}^{\alpha}. \tag{3.5}$$

3.4. **The group $O(n,m)$.** Introduce the notation

$$X_{ij} = \mathbb{G}(\tau_i, e_j) \quad W_{i\beta} = \mathbb{G}(\tau_i, T_\beta) \quad U_{\alpha j} = -\mathbb{G}(\nu_\alpha, e_j) \quad V_{\alpha\beta} = -\mathbb{G}(\nu_\alpha, T_\beta).$$

Thus

$$e_i = X_{ij}\tau_j + U_{\alpha i}\nu_\alpha \quad T_\beta = W_{i\beta}\tau_i + V_{\alpha\beta}\nu_\alpha,$$

and the $(n+m)\times(n+m)$ dimensional matrix

$$M = \begin{pmatrix} X & W \\ -U & -V \end{pmatrix},$$

is an element of the orthogonal group $O(n,m)$.

Proposition 6. *The $O(n,m)$ condition on M is*

$$X^TX = I_n + U^TU \qquad V^TV = I_m + W^TW \qquad U^TV = X^TW. \tag{3.6}$$

Proof. This follows from the condition

$$M^T \begin{pmatrix} I_n & 0 \\ 0 & -I_m \end{pmatrix} M = \begin{pmatrix} I_n & 0 \\ 0 & -I_m \end{pmatrix}.$$

□

The vectors $\{\tau_i\}_1^n$ span the tangent space of Σ, while $\{\nu_\alpha\}_1^m$ span the normal bundle. We are free to rotate these frames within these two spaces, and this corresponds to left action of $O(n)$ and $O(m)$ within $O(n,m)$.

Similarly, we consider rotations of $\{e_i\}_1^n$ that preserve the n-dimensional vector space that they span, along with rotations of $\{T_\beta\}_1^m$ that preserves the m-dimensional space they span. These correspond to right actions of $O(n)$ and $O(m)$ within $O(n,m)$. Note that the positive definite norm in Definition 5 is preserved by these rotations.

Proposition 7. *By rotations of the frames $\{e_i, T_\alpha\}$ and $\{\tau_j, \nu_\beta\}$ which preserve the tangent and normal bundles of Σ, as well as the tensor norm of Definition 5, we can simplify the matrix $M \in O(n,m)$ for $n \geq m$ to*

$$M = \begin{pmatrix} I_{n-m} & 0 & 0 \\ 0 & D_1 & \pm D_4A^T \\ 0 & D_3A & D_2 \end{pmatrix},$$

where $A \in O(m)$ is a transposition matrix, D_1, D_2, D_3 and D_4 are diagonal matrices satisfying

$$D_1^2 = I_m + D_3^2 \qquad D_2^2 = I_m + D_4^2 \qquad |D_1|^2 = |D_2|^2,$$

and $\pm$ of a diagonal matrix means a free choice of sign on the entries of the matrix.

Proof. Consider first the matrix $X_{ij} = <\tau_i, e_j>$. The matrix X^TX is symmetric and non-negative definite and so it has a well-defined square root, namely a symmetric $n \times n$ matrix which we denote by $\sqrt{X^TX}$. By the first equation of (3.6), X is invertible since $\det(X) \geq 1$ and so we can define the $n \times n$ matrix $A = \sqrt{X^TX}X^{-1}$. Then

$$A^TI_nA = (X^{-1})^T\sqrt{X^TX}\sqrt{X^TX}X^{-1} = (X^{-1})^TX^TXX^{-1} = I_n,$$

so that $A \in O(n)$. Define a new frame by $\{A_{ij}\tau_j, \nu_\alpha\}$ and then

$$\tilde{X}_{ij} = A_{ik} < \tau_k, e_j >= \sqrt{X^TX}X^{-1}X = \sqrt{X^TX},$$

which is symmetric. Now we can act on both the left and right of $\tilde{X}$ by $O(n)$ to diagonalise it.

A similar argument yields a diagonalisation of $V_{\alpha\beta}$.

After diagonalisation of X, the first of equations (3.6) implies that the matrix U^TU is diagonal. Thus the n m-dimensional vectors $\{U_{\alpha i}\nu_\alpha\}_{i=1}^n$ are mutually orthogonal and, since $n \geq m$, we conclude that $n-m$ of these vectors must be zero.

After a reordering of the basis elements, the matrix M then decomposes into

$$M = \begin{pmatrix} I_{n-m} & 0 & W_2 \\ 0 & X_1 & W_1 \\ 0 & U_1 & V \end{pmatrix}.$$

The last of equations (3.6) now implies that $W_2 = 0$ and we reduce the problem to the square case:

$$X_1^T X_1 = I_m + U_1^T U_1 \qquad V^T V = I_m + W_1^T W_1 \qquad U_1^T V = X_1^T W_1.$$

In fact, to indicate that X^1 and V are diagonal, let us write $X_1 = D_1$ and $V = D_2$. Thus

$$D_1^2 = I_m + U_1^T U_1, \tag{3.7}$$

$$D_2^2 = I_m + W_1^T W_1, \tag{3.8}$$

$$U_1^T D_2 = D_1 W_1. \tag{3.9}$$

Equations (3.7) and (3.8) imply that there exists diagonal matrices D_3 and D_4 (with entries defined up to a sign) such that

$$U_1 = D_3 A \qquad W_1 = D_4 B \qquad \text{for } A, B \in O(m).$$

Thus equations (3.7), (3.8) and (3.9) now read

$$D_1^2 = I_m + D_3^2, \tag{3.10}$$

$$D_2^2 = I_m + D_4^2, \tag{3.11}$$

$$A^T D_2 D_3 = D_1 D_4 B. \tag{3.12}$$

Taking the transpose of this last equation and multiplying back on the right we find that

$$A^T D_2^2 D_3^2 A = D_1^2 D_4^2. \tag{3.13}$$

However, if $A \in O(m)$ that sends a diagonal matrix to a diagonal matrix, then A must be a transposition. Similarly

$$B^T D_1^2 D_4^2 B = D_2^2 D_3^2,$$

and so $A = \pm B^T$.

Denote the diagonal elements of D_1, D_2, D_3 and D_4 by λ_i, μ_i, a_i and b_i, respectively, where $i = n - m + 1, ...n$. Then equations (3.10), (3.11) and (3.13) read

$$\lambda_i^2 = 1 + a_i^2 \qquad \mu_i^2 = 1 + b_i^2 \qquad \mu_i^2 a_i^2 = \lambda_{p(i)}^2 b_{p(i)}^2,$$

where p is the permutation of $(n - m + 1, ...n)$ determined by the transposition A. Combining these three equations we get

$$a_i^2 + a_i^2 b_i^2 = a_{p(i)}^2 + a_{p(i)}^2 b_{p(i)}^2,$$

which when summed yields

$$\sum_i a_i^2 = \sum_i b_i^2 \qquad \text{and} \qquad \sum_i \lambda_i^2 = \sum_i \mu_i^2.$$

Thus $|D_1|^2 = |D_2|^2$ as claimed.

□

Definition 7. The function v is defined to be

$$v^2=\sum_{\alpha,\beta}V^{\alpha\beta}V_{\alpha\beta}.$$

This is a generalization of the *tilt function* in the codimension one case [1].

We now use the normal form to construct estimates of the norms of the adapted frames:

Proposition 8. *For an adapted frame* $\{\tau_i,\nu_\alpha\}$ *we have*

$$\|\tau_i\|^2\leq[n^2+m(m-1)^2]v^2 \qquad \|\nu_\alpha\|^2\leq m^3v^2,$$

for all $i=1,2,...n$ *and* $\alpha=1,2,...m$.

Proof. First consider an adapted frame $\{\overset{\circ}{\tau}_i,\overset{\circ}{\nu}_\alpha\}$ for which the matrix M has the form given in Proposition 7. For a general adapted frame $\{\tau_i,\nu_\alpha\}$

$$\tau_i=A_i^j\,\overset{\circ}{\tau}_j \qquad \nu_\alpha=B_\alpha^\beta\,\overset{\circ}{\nu}_\beta,$$

where $A\in O(n)$ and $B\in O(m)$. Then

$$\begin{aligned}
\|\tau_i\|^2&=\sum_j(\mathbb{G}(\tau_i,e_j))^2+\sum_\alpha(\mathbb{G}(\tau_i,T_\alpha))^2\\
&=\sum_j\left[\sum_k A_i^k\mathbb{G}(\overset{\circ}{\tau}_k,e_j)\right]^2+\sum_\alpha\left[\sum_k A_i^k\mathbb{G}(\overset{\circ}{\tau}_i,T_\alpha)\right]^2\\
&\leq\sum_j\left[\sum_k |A_i^k|\,|\mathbb{G}(\overset{\circ}{\tau}_k,e_j)|\right]^2+\sum_\alpha\left[\sum_k |A_i^k|\,|\mathbb{G}(\overset{\circ}{\tau}_i,T_\alpha)|\right]^2\\
&\leq\sum_j\left[\sum_k\,|\mathbb{G}(\overset{\circ}{\tau}_k,e_j)|\right]^2+\sum_\alpha\left[\sum_k\,|\mathbb{G}(\overset{\circ}{\tau}_i,T_\alpha)|\right]^2\\
&\leq\sum_j\left[\sum_k\,|X_{kj}|\right]^2+\sum_\alpha\left[\sum_k\,|W_{k\alpha}|\right]^2\\
&\leq\left[\sum_k\,|X_{kk}|\right]^2+\sum_\alpha\left[\sum_{k=n-m+1}^n\,|W_{k\alpha}|\right]^2\\
&\leq n^2v^2+m(m-1)^2v^2\\
&\leq[n^2+m(m-1)^2]v^2.
\end{aligned}$$

Similarly for ν_α.

□

3.5. **The height functions.** Let $u_\alpha:\Sigma\rightarrow\mathbb{R}$ be the *height function* $u_\alpha=t_\alpha\circ f$. Then

Proposition 9. *For all* $\alpha=1...m$

$$\nabla u_\alpha=\overline{\nabla}t_\alpha+\psi_\alpha^{-1}\sum_\beta V_{\beta\alpha}\nu_\beta,$$

$$\nabla u_\alpha \cdot \nabla u_\beta = \psi_\alpha^{-1}\psi_\beta^{-1}\left(\sum_\gamma V_{\gamma\alpha}V_{\gamma\beta} - \delta_{\alpha\beta}\right).$$

Proof. From the definition of u_α and T_α we have

$$\nabla u_\alpha = \overline{\nabla} t_\alpha + \psi_\alpha^{-1}\sum_\beta V_{\beta\alpha}\nu_\beta = \psi_\alpha^{-1}(\sum_\beta V_{\beta\alpha}\nu_\beta - T_\alpha),$$

and so

$$\begin{aligned}\nabla u_\alpha \cdot \nabla u_\beta =& \psi_\alpha^{-1}\psi_\beta^{-1}\ \mathbb{G}\left(\sum_\gamma V_{\gamma\alpha}\nu_\gamma - T_\alpha, \sum_\delta V_{\delta\beta}\nu_\delta - T_\beta\right)\\ &= \psi_\alpha^{-1}\psi_\beta^{-1}\left(\sum_\gamma V_{\gamma\alpha}V_{\gamma\beta} - \delta_{\alpha\beta}\right).\end{aligned}$$

as claimed. □

Proposition 10.

$$\triangle u_\gamma = -\psi_\gamma^{-1}V_{\alpha\gamma}H^\alpha + g^{ij}\overline{\nabla}_i\overline{\nabla}_j t_\gamma.$$

$$\begin{aligned}\triangle V_{\alpha\beta} =& V_{\gamma\beta}(A_{ij\gamma}A^{ij}_\alpha - < \overline{R}(\tau_i,\nu_\gamma)\tau_i,\nu_\alpha >) - \tilde{\nabla}_{T_\beta}H_\alpha - A^{ij}_\alpha T_\beta(g_{ij})\\ &+ \tfrac{1}{2}(\overline{\nabla}\mathcal{L}_{T_\beta}\mathbb{G})(\nu_\alpha,\tau_i,\tau_i) - (\overline{\nabla}\mathcal{L}_{T_\beta}\mathbb{G})(\tau_i,\nu_\alpha,\tau_i) - (\overline{\nabla}T_\beta)(H,\nu_\alpha)\\ &- 2C_{i\alpha}{}^{\gamma} < \nu_\gamma, \overline{\nabla}_{T_\beta}\tau_i > + (\nabla_i C_{i\alpha}{}^{\gamma} + C_{i\alpha}{}^{\delta}C_{i\delta}{}^{\gamma})V_{\gamma\beta},\end{aligned}$$

where $\triangle$ is the Laplacian of the induced metric $\triangle = g^{ij}\nabla_i\nabla_j$.

Proof. The first statement follows from a straightforward generalization of the codimension one case [4].

For the second statement we follow Bartnik [1] and fix a point $p \in \Sigma$ and choose an orthonormal frame $\{\tau_i\}$ on Σ such that $(\nabla_i\tau_j)(p) = 0$. Extend this frame in a neighbourhood of Σ by $\mathcal{L}_{T_\beta}\tau_i = 0$ for fixed β. Then

$$\begin{aligned}-\triangle V_{\alpha\beta} &= \triangle < \nu_\alpha, T_\beta >\\ &= \tau_i\tau_i < \nu_\alpha, T_\beta >\\ &= \tau_i(< \overline{\nabla}_{\tau_i}\nu_\alpha, T_\beta > + < \nu_\alpha, \overline{\nabla}_{\tau_i}T_\beta >)\\ &= \tau_i(A^j_{i\alpha} < \tau_j, T_\beta > + C^\gamma_{i\alpha} < \nu_\gamma, T_\beta > + < \nu_\alpha, \overline{\nabla}_{\tau_i}T_\beta >)\\ &=< \overline{R}(\tau_i, T_\beta)\tau_i, \nu_\alpha > + < \nu_\alpha, \overline{\nabla}_{T_\beta}\overline{\nabla}_{\tau_i}\tau_i > + < \overline{\nabla}_{\tau_i}\nu_\alpha, \overline{\nabla}_{T_\beta}\tau_i >)\\ &\quad + (\overline{\nabla}_{\tau_i}H_\alpha + < \overline{R}(\tau_i,\tau_j)\nu_\alpha, \tau_i > - A^\gamma_{ij}C^\alpha_{i\gamma} + H^\gamma C^\alpha_{j\gamma}) < \tau_j, T_\beta >\\ &\quad + C^\gamma_{i\alpha}(< \overline{\nabla}_{\tau_i}\nu_\gamma, T_\beta > + < \nu_\gamma, \overline{\nabla}_{\tau_i}T_\beta >) + < \nu_\gamma, T_\beta > \overline{\nabla}_{\tau_i}C^\gamma_{i\alpha}\\ &\quad + A^j_{i\alpha}(< \overline{\nabla}_{\tau_i}\tau_j, T_\beta > + < \tau_j, \overline{\nabla}_{\tau_i}T_\beta >)\\ &=< \overline{R}(\tau_i,\nu_\gamma)\tau_i, \nu_\alpha >< \nu_\gamma, T_\beta > + < \nu_\alpha, \overline{\nabla}_{T_\beta}\overline{\nabla}_{\tau_i}\tau_i > + 2A^j_{i\alpha} < \tau_j, \overline{\nabla}_{T_\beta}\tau_i >\\ &\quad + 2C^\gamma_{i\alpha} < \nu_\gamma, \overline{\nabla}_{\tau_i}T_\beta > + < \tau_i, T_\beta > \overline{\nabla}_i H_\alpha + H^\gamma C_{i\gamma\alpha} < \tau_i, T_\beta >\\ &\quad + A^{ij}_\alpha A^\gamma_{ij} < \nu_\gamma, T_\beta > + C^\gamma_{i\alpha}C^\delta_{i\gamma} < \nu_\delta, T_\beta > + < \nu_\gamma, T_\beta > \overline{\nabla}_{\tau_i}C^\gamma_{i\alpha}\\ &= -V_{\gamma\beta}(A_{ij\gamma}A^{ij}_\alpha + < \overline{\mathrm{R}}(\tau_i,\nu_\gamma)\tau_i, \nu_\alpha >) + \tilde{\nabla}_{T_\beta}H_\alpha\\ &\quad < \nu_\alpha, \overline{\nabla}_{T_\beta}\overline{\nabla}_i\tau_i > + A^{ij}_\alpha T_\beta < \tau_i, \tau_j >\\ &\quad + 2C_{i\alpha}{}^{\gamma} < \nu_\gamma, \overline{\nabla}_{T_\beta}\tau_i > + (\nabla_i C_{i\alpha}{}^{\gamma} - C_{i\alpha}{}^{\delta}C_{i\delta}{}^{\gamma})V_{\gamma\beta}. \qquad (3.14)\end{aligned}$$

To complete the proof we now use the following:

Lemma 1.

$$T_\beta < \tau_i, \overline{\nabla}_i \nu_\alpha > = - < \overline{\nabla}_i \tau_i, \overline{\nabla}_{T_\beta} \nu_\alpha > + \tfrac{1}{2}(\overline{\nabla}\mathcal{L}_{T_\beta}\mathbb{G})(\nu_\alpha, \tau_i, \tau_i) \\ - (\overline{\nabla}\mathcal{L}_{T_\beta}\mathbb{G})(\tau_i, \nu_\alpha, \tau_i) - < \overline{\nabla}_H T_\beta, \nu_\alpha > .$$

Proof. The proof of this follows the codimension one case (Proposition 2.1 of [1]). □

To complete the proof of the proposition we note that

$$\begin{aligned} < \nu_\alpha, \overline{\nabla}_{T_\beta} \overline{\nabla}_i \tau_i > &= T_\beta < \nu_\alpha, \overline{\nabla}_i \tau_i > - < \overline{\nabla}_{T_\beta} \nu_\alpha, \overline{\nabla}_i \tau_i > \\ &= -T_\beta < \overline{\nabla}_i \nu_\alpha, \tau_i > - < \overline{\nabla}_{T_\beta} \nu_\alpha, \overline{\nabla}_i \tau_i > \\ &= -\tfrac{1}{2}(\overline{\nabla}\mathcal{L}_{T_\beta}\mathbb{G})(\nu_\alpha, \tau_i, \tau_i) + (\overline{\nabla}\mathcal{L}_{T_\beta}\mathbb{G})(\tau_i, \nu_\alpha, \tau_i) \\ &\quad + < \overline{\nabla}_H T_\beta, \nu_\alpha >, \end{aligned}$$

where in the last equailty we have used Lemma 1. Substituting this in equation (3.14) then yields the result.

□

3.6. **Mean curvature flow.** Let $f_s : \Sigma \to \mathbb{M}$ be a family of n-dimensional space-like immersed surface in an $n+m$-dimensional manifold $\mathbb{M}$ with a metric $\mathbb{G}$ of signature (n, m). In addition, we assume that $n \geq m$, the case $n < m$ follows by similar arguments.

Then f_s moves by mean curvature flow if it satisfies the equation

$$\frac{df}{ds}^{\perp} = \left(g^{jk} \overline{\nabla}_{\frac{\partial f}{\partial x^j}} \frac{\partial f}{\partial x^k} \right)^{\perp} = H, \tag{3.15}$$

where $\overline{\nabla}$ is the Levi-Civita connection associated with the ambient metric on $\mathbb{M}$, H is the mean curvature vector $H = H^\alpha \nu_\alpha$ and $^\perp$ is the projection perpendicular to the tangent space of Σ.

The flow of the functions u_γ and v are given by

Proposition 11.

$$\left(\frac{d}{ds} - \triangle \right) u_\gamma = -g^{ij} \overline{\nabla}_i \overline{\nabla}_j t_\gamma, \tag{3.16}$$

$$v \left(\frac{d}{ds} - \triangle \right) v \leq - V^{\alpha\beta} V_{\gamma\beta}(A_{ij\gamma} A^{ij}_\alpha - < \overline{R}(\tau_i, \nu_\gamma)\tau_i, \nu_\alpha >) + A^{ij}_\alpha \mathcal{L}_{T_\beta} g_{ij} V^{\alpha\beta} \\ - \tfrac{1}{2}(\overline{\nabla}\mathcal{L}_{T_\beta}\mathbb{G})(\nu_\alpha, \tau_i, \tau_i) V^{\alpha\beta} + (\overline{\nabla}\mathcal{L}_{T_\beta}\mathbb{G})(\tau_i, \nu_\alpha, \tau_i) V^{\alpha\beta} \\ + 2 C_{i\alpha}{}^{\gamma} < \nu_\gamma, \overline{\nabla}_{T_\beta} \tau_i > V^{\alpha\beta} - C_{i\alpha}{}^{\delta} C_{i\delta}{}^{\gamma} V_{\gamma\beta} V^{\alpha\beta}.$$

Proof. Generalizing Proposition 3.1 of [4], note the time derivatives are

$$\frac{du_\gamma}{ds} = -\psi_\gamma^{-1} V_{\alpha\gamma} H^\alpha,$$

$$\frac{dV_{\alpha\beta}}{ds} = -\overline{\nabla}_{T_\beta} H_\alpha - H^\gamma < \overline{\nabla}_{\nu_\gamma} T_\beta, \nu_\alpha > .$$

The flow of u_γ then follows immediately from Proposition 10.

To find the flow of v note that

$$v\left(\frac{d}{ds}-\triangle\right)v=V^{\alpha\beta}\left(\frac{d}{ds}-\triangle\right)V_{\alpha\beta}+\frac{1}{v^2}\left[(v_\alpha\nabla V_\alpha)\cdot(v_\beta\nabla V_\beta)-V_\alpha^2|\nabla V_\beta|^2\right],$$

where we sum over α and β and diagonalised $V_{\alpha\beta}=\mathrm{diag}(V_1,...V_m)$. By the Cauchy-Schwarz inequality we have

$$v\left(\frac{d}{ds}-\triangle\right)v\leq V^{\alpha\beta}\left(\frac{d}{ds}-\triangle\right)V_{\alpha\beta}.$$

Now contracting the second equation of Proposition 10 with $V^{\alpha\beta}$ yields the result. □

Proposition 12. *Assume that $\mathbb{M}$ satisfies the timelike curvature condition. Let Σ_s be a smooth solution of (3.15) on the interval $0\leq s<s_0$ such that Σ_s is contained in a smooth compact subset of $\mathbb{M}$ for all $0\leq s<s_0$. Then the function v satisfies the a priori estimate*

$$v(p,s)\leq(m+\sup_{\Sigma_0}v)\sup_{(q,s)\in\Sigma\times[0,s_0]}\exp[K(u(q,s)-u(p,s))],$$

for some positive constant $K(n,m,\|t\|_3,|\psi|,\|\overline{R}\|,|H|,k)$, where $u=\sum_\alpha u_\alpha$.

Proof. Let K>0 be a constant to be determined later and set

$$C_K=(1+\sup_{\Sigma_0}v)\sup_{\Sigma\times[0,s_0]}\exp(Ku).$$

Consider the function $f=v\exp(Ku)$. Suppose, for the sake of contradiction, that the function f reaches C_K for the first time at $(p_1,s_1)\in M\times(0,s_0]$. Then at this point $v\geq m+1$ and

$$\left(\frac{d}{ds}-\triangle\right)f\dot{\geq}0\qquad\qquad\nabla f\dot{=}0.$$

Here and throughout a dot over an inequality or equality will refer to evaluation at the point (p_1,s_1). Working out these two equations we have

$$\left(\frac{d}{ds}-\triangle\right)v+Kv\left(\frac{d}{ds}-\triangle\right)u-2K\nabla u\cdot\nabla v-K^2v|\nabla u|^2\dot{\geq}0, \tag{3.17}$$

$$\nabla v+Kv\nabla u\dot{=}0. \tag{3.18}$$

Substituting the second of these in the first we obtain

$$Kv\left(\frac{d}{ds}-\triangle\right)u\dot{\geq}-\left(\frac{d}{ds}-\triangle\right)v-K^2v|\nabla u|^2. \tag{3.19}$$

Now, from Proposition 11 and the estimates in Proposition 8

$$\left(\frac{d}{ds}-\triangle\right)u=-g^{ij}\overline{\nabla}_i\overline{\nabla}_jt\leq\|\overline{\nabla}_i\overline{\nabla}_jt\|.\|\tau_i\|.\|\tau_j\|\leq C_1v^2, \tag{3.20}$$

where $C_1=C_1(n,m,\|t\|_2)$.

At p_1 we can set $C_{i\alpha}{}^{\beta}=0$ and then, Proposition 11 and the timelike curvature condition imply that

$$v\left(\frac{d}{ds}-\triangle\right)v\ \dot{\leq}-V_\alpha^2|A_\alpha|^2+C_2(\|T\|_1)|A_\alpha|V_\alpha+C_3(n,m,\|T\|_2)v^4$$
$$\leq-(1-\epsilon)V_\alpha^2|A_\alpha|^2+C_4(\epsilon,n,m,\|T\|_2)v^4, \tag{3.21}$$

for any $\epsilon>0$. Here we have utilised the gauge choice $V_{\alpha\beta}=V_\alpha\delta_{\alpha\beta}$ on some of the terms and summation is over α.

Now, from the Schwartz and arithmetic-geometric mean inequalities

$$V_\alpha^2|A_\alpha|^2\geq\left(1+\frac{1}{n}\right)\lambda_\alpha^2V_\alpha^2-H_\alpha^2V_\alpha^2, \tag{3.22}$$

where λ_α is the eigenvalue of $A_{ij\alpha}$ with the maximum absolute value.

On the other hand we compute

$$\nabla_iV_{\alpha\beta}=-A^j_{i\alpha}<\tau_j,T_\beta>-<\nu_\alpha,\overline{\nabla}_iT_\beta>,$$

and so

$$v\nabla_iv=V^{\alpha\beta}\nabla_iV_{\alpha\beta}=-A^j_{i\alpha}W_{j\beta}V^{\alpha\beta}-<\nu_\alpha,\overline{\nabla}_iT_\beta>V^{\alpha\beta}.$$

Thus

$$|\nabla v|\leq|\lambda_\alpha|V_\alpha+C_5(n,m,\|T\|_1)v^2,$$

and so

$$|\nabla v|^2\leq(1+\epsilon)V_\alpha^2\lambda_\alpha^2+C_6(\epsilon,n,m,\|T\|_1)v^4.$$

Rearranging this last inequality

$$V_\alpha^2\lambda_\alpha^2\geq\frac{1}{1+\epsilon}|\nabla v|^2-C_6v^4. \tag{3.23}$$

Combining inequalities (3.22) and (3.23) we get

$$V_\alpha^2|A_\alpha|^2\geq\left(1+\frac{1}{n}\right)\left[\frac{1}{1+\epsilon}|\nabla v|^2-C_5v^4\right]-|H_\alpha|^2V_\alpha^2,$$

which, when substituted in inequality (3.21) yields

$$v\left(\frac{d}{ds}-\triangle\right)v\ \dot{\leq}-\left(1+\frac{1}{n}\right)\frac{1-\epsilon}{1+\epsilon}|\nabla v|^2+C_7(n,|H|)v^2+C_6v^4,$$

and, by virtue of equation (3.18),

$$|\nabla v|^2\doteq K^2v^2|\nabla u|^2,$$

yielding

$$\left(\frac{d}{ds}-\triangle\right)v\ \dot{\leq}-\left(1+\frac{1}{n}\right)\frac{1-\epsilon}{1+\epsilon}K^2v|\nabla u|^2+C_7v+C_6v^3. \tag{3.24}$$

Substituting inequalities (3.20) and (3.24) in (3.19) we get

$$KC_1v^2\ \dot{\geq}\left[\left(1+\frac{1}{n}\right)\frac{1-\epsilon}{(1+\epsilon)}-1\right]K^2|\nabla u|^2+C_7+C_6v^2,$$

for any $\epsilon>0$.

Now for $0<\epsilon<1/(1+2n)$

$$\left(1+\frac{1}{n}\right)\frac{1-\epsilon}{1+\epsilon}-1>0,$$

and so using Proposition 9

$$|\nabla u|^2=\sum_{\alpha,\beta}\nabla u_\alpha\cdot\nabla u_\beta=\sum_\alpha\psi_\alpha^{-2}(V_\alpha^2-1)\geq \min_\alpha\psi_\alpha^{-2}(v^2-m),$$

we have

$$KC_1v^2\dot{\geq}C_8(\epsilon,n,|\psi|)K^2(v^2-m)+C_7+C_6v^2,$$

which can be rearranged to

$$v^2\dot{\leq}\frac{mC_8K^2-C_7}{C_7K^2-C_1K+C_6},$$

which for large K violates $v\geq m+1$. □

For tensors H_α and $A_{ij\alpha}$ we define a positive norm by

$$|H|_+^2=-H_\alpha H^\alpha \qquad |A|_+^2=-A_{ij\alpha}A^{ij\alpha},$$

and similarly for their gradients.

Proposition 13. [3] *Under the mean curvature flow*

$$\left(\frac{d}{ds}-\triangle\right)|H|_+^2=-2|\tilde{\nabla}H|_+^2-2|A|_+^2|H|_+^2-2H^\alpha H^\beta\bar{R}_{i\alpha i\beta},$$

$$\left(\frac{d}{ds}-\triangle\right)|A|_+^2=-2|\tilde{\nabla}A|_+^2-2|A|_+^4+A*A*\overline{R}+A*\overline{\nabla}\,\overline{R},$$

where $\tilde{\nabla}$ *is the covariant derivative in both the tangent and normal bundles and* $*$ *represents linear combinations of contractions of the tensors involved.*

Proposition 14. *Under the mean curvature flow*

$$|H|_+^2\leq C_1(1+s^{-1}),$$

$$|A|_+^2\leq C_2(1+s^{-1}),$$

where $C_1=C_1(n,k)$ *and* $C_2=C_1(n,\|\overline{R}\|_1)$.

Proof. From the previous proposition and the timelike curvature condition (3.1) we conclude that

$$\left(\frac{d}{ds}-\triangle\right)|H|_+^2\leq-2n^{-1}|H|_+^4+2k|H|_+^2,$$

while

$$\left(\frac{d}{ds}-\triangle\right)|A|_+^2\leq-2|A|_+^4+C_3|A|_+^2+C_4|A|_+\leq-|A|_+^4+C_5,$$

The result then follows by a suitable modification of Lemma 4.5 of Ecker and Huisken [4]. □

Theorem 2. *Let* Σ_0 *be a smooth compact* n*-dimensional spacelike submanifold of an* $n+m$ *dimensional manifold* $\mathbb{M}$ *satisfying the timelike curvature condition. Then there exists a unique family* $f_s(\Sigma)$ *of smooth compact* n*-dimensional spacelike submanifolds satisfying the initial value problem (3.15) on an interval* $0\leq s<s_0$*. Moreover, if* $f_s(\Sigma)$ *remains in a smooth compact region of* $\mathbb{M}$ *as* $s\to s_0$*, the solution can be extended beyond* s_0*.*

Proof. Having bounded the second fundamental form in Proposition 14, bounds on the higher derivatives and hence long-time existence, follow from standard parabolic bootstrapping arguments as in [4]. □

3.7. **Boundary conditions.** While our considerations so far have been on flowing compact submanifolds, the results can be extended to the case where the flowing submanifold has boundary, so long as we implement suitable boundary conditions. For dimensional reasons, and because mean curvature flow is a second order PDE, we must have m boundary conditions in total.

Consider the case of $m-1$ Dirichlet conditions and one Neumann condition. That is, let $\tilde{\Sigma}$ be a fixed n-dimensional submanifold in an $n+m$-dimensional manifold $\mathbb{M}$ and consider the Dirichlet requirement that a given immersed n-dimensional spacelike submanifold $f:\Sigma\rightarrow\mathbb{M}$ has boundary in $\tilde{\Sigma}$.

Definition 8. An *independent* Neumann boundary condition is a Neumann condition

$$N(f)=0 \qquad\qquad \text{along } f(\partial\Sigma),$$

where N is some linear combination of the slopes of Σ which is linearly independent from the derived Dirichlet condition .

Consider the following flow.

Let $f_s:\Sigma\rightarrow\mathbb{M}$ be a family of immersed n-dimensional spacelike submanifolds of $\mathbb{M}$ such that

$$\frac{df}{ds}^{\perp}=H,$$

with initial condition

$$f_0(\Sigma)=\Sigma_0,$$

and boundary conditions

(i) $\partial f_s(\Sigma)\subset\tilde{\Sigma}$,
(ii) $N(f_s)=0$,

where H is the mean curvature vector associated with $f_s(\Sigma)$, Σ_0 and $\tilde{\Sigma}$ are some given n-dimensional submanifolds, and boundary condition (ii) consists of an independent Neumann condition.

Proposition 15. *The above boundary value problem is parabolic.*

Proof. To establish this we verify that the well-known Lopatinski-Shapiro conditions are satisfied. Let $(x^1,...,x^{n-1},\nu)$ be coordinates that flatten out the boundary, so that the boundary is given by $\nu=0$. The Lopatinski-Shapiro condition requires that there exists a unique solution of the ODE associated with the above problem, namely:

$$\frac{d^2f^\alpha}{d^2t}-\sum_j(\zeta^j)^2f^\alpha=0,$$

$$\lim_{t\rightarrow\infty}f^\alpha=0,$$

$$\text{(i)}\qquad A_{\alpha k}\frac{df^\alpha}{dt}(0)+B_{\alpha kj}\zeta^j f^\alpha(0)=g_k \qquad \text{(ii)}\qquad C_\alpha\frac{df^\alpha}{dt}(0)+D_{\alpha j}\zeta^j f^\alpha(0)=h,$$

where the initial conditions (i) and (ii) are derived from the linearisation of the Dirichlet condition ($j,k=1,...,n-1$), along with the Neumann condition. The first equation can be integrated to

$$f^\alpha=E_1^\alpha\exp(-|\zeta|t)+E_2^\alpha\exp(|\zeta|t).$$

The second condition implies that $E_2^\alpha=0$ and it remains to satisfy the initial conditions

$$(i)\qquad -A_{\alpha k}|\zeta|E_1^\alpha+B_{\alpha kj}\zeta^jE_1^\alpha=g_k \qquad (ii)\qquad -C_\alpha|\zeta|E_1^\alpha+D_{\alpha j}\zeta^jE_1^\alpha=h.$$

Since the Neumann condition is assumed linearly independent from the linearized Dirichlet conditions, there exists a unique solution to this ODE.

□

Thus we can extend Theorem 2 to the case where the flowing submanifold has the above boundary constraints:

Theorem 3. *Let Σ_0, $\tilde{\Sigma}$ be smooth n-dimensionalsubmanifolds of an $n+m$ dimensional manifold $\mathbb{M}$ satisfying the timelike curvature condition. Assume that Σ_0 is spacelike, $\partial(\Sigma_0)$ lies on $\tilde{\Sigma}$ and an independent Neumann condition holds along the boundary. Then there exists a unique family $f_s(\Sigma)$ of smooth compact n-dimensional spacelike submanifolds satisfying the above parabolic boundary value problem on an interval $0\leq s<s_0$. Moreover, if $f_s(\Sigma)$ remains in a smooth compact region of $\mathbb{M}$ as $s\to s_0$, the solution can be extended beyond s_0.*

4. Mean Curvature Flow in TS^2

In this section we establish the main estimates we require to prove convergence of mean curvature flow in TS^2 as stated in section 2.3. Throughout we use the term positive surface to mean spacelike surface: the induced metric is positive definite.

4.1. **The neutral Kähler metric.**

In order to compute geometric quantities we introduce local coordinates on TS^2. These are readily supplied by lifting the standard complex coordinate ξ (obtained by stereographic projection from the south pole on S^2) to complex coordinates (ξ,η) on TS^2. In particular, we identify $(\xi,\eta)\in\mathbb{C}^2$ with the vector

$$\eta\frac{\partial}{\partial\xi}+\bar\eta\frac{\partial}{\partial\bar\xi}\in T_\xi S^2.$$

These coordinates are holomorphic with respect to the complex structure $\mathbb{J}$:

$$\mathbb{J}\left(\frac{\partial}{\partial\xi}\right)=i\frac{\partial}{\partial\xi}\qquad\qquad \mathbb{J}\left(\frac{\partial}{\partial\eta}\right)=i\frac{\partial}{\partial\eta},$$

and the symplectic 2-form and neutral metric have the following local expressions:

$$\Omega=4(1+\xi\bar\xi)^{-2}\mathbb{R}\mathrm{e}\left(d\eta\wedge d\bar\xi-\frac{2\bar\xi\eta}{1+\xi\bar\xi}d\xi\wedge d\bar\xi\right),$$

$$\mathbb{G}=4(1+\xi\bar\xi)^{-2}\mathbb{I}\mathrm{m}\left(d\bar\eta d\xi+\frac{2\bar\xi\eta}{1+\xi\bar\xi}d\xi d\bar\xi\right). \qquad (4.1)$$

We are interested in graphs of local sections of the bundle $\pi : TS^2 \to S^2$. Such local sections are given by $\xi \mapsto (\xi, \eta = F(\xi, \bar{\xi}))$, for some function $F : \mathbb{C} \to \mathbb{C}$.

Definition 9. We introduce the complex slopes of F:

$$\sigma = -\partial \bar{F} \qquad\qquad \rho = \theta + i\lambda = (1+\xi\bar{\xi})^2 \partial (F(1+\xi\bar{\xi})^{-2}).$$

Here, and throughout, ∂ represents differentiation with respect to ξ. The functions λ and σ are commonly referred to as the *twist* and *shear* of the underlying family Σ of oriented lines in $\mathbb{E}^3$.

We note the following two identities, which follow from these definitions:

$$-(1+\xi\bar{\xi})^2 \partial \left[\frac{\bar{\sigma}}{(1+\xi\bar{\xi})^2} \right] = \bar{\partial}\rho + \frac{2F}{(1+\xi\bar{\xi})^2}, \tag{4.2}$$

$$\mathbb{I}\mathrm{m}\, \partial \left\{ (1+\xi\bar{\xi})^2 \partial \left[\frac{\bar{\sigma}}{(1+\xi\bar{\xi})^2} \right] \right\} = \partial\bar{\partial}\lambda + \frac{2\lambda}{(1+\xi\bar{\xi})^2}. \tag{4.3}$$

The geometric significance of λ and σ are as follows:

Proposition 16. [6] *A surface Σ given by a local section $\eta = F(\xi, \bar{\xi})$ is Lagrangian iff $\lambda = 0$ and is holomorphic iff $\sigma = 0$.*

For the induced metric we have:

Proposition 17. *The metric induced on the graph of a section by the Kähler metric is given in coordinates $(\xi, \bar{\xi})$ by;*

$$g = \frac{2}{(1+\xi\bar{\xi})^2} \begin{bmatrix} i\sigma & -\lambda \\ -\lambda & -i\bar{\sigma} \end{bmatrix},$$

with inverse

$$g^{-1} = \frac{(1+\xi\bar{\xi})^2}{2(\lambda^2 - \sigma\bar{\sigma})} \begin{bmatrix} i\bar{\sigma} & -\lambda \\ -\lambda & -i\sigma \end{bmatrix}.$$

Proof. This follows from pulling back the neutral metric (4.1) along a local section $\eta = F(\xi, \bar{\xi})$. □

Proposition 18. *The induced metric on a Lagrangian surface is Lorentz, except at complex points, where it is degenerate.*

Proof. By the previous Proposition the determinant of the induced metric is $2(1+\xi\bar{\xi})^{-2}(\lambda^2 - \sigma\bar{\sigma})$ and the result follows. □

Note 2. In Section 2.1 we have seen that umbilic points on surfaces in $\mathbb{E}^3$ give rise to complex points on Lagrangian surfaces in TS^2 and now we see that these correspond to degeneracies in the induced Lorentz metric. Moreover, the principal foliation corresponds exactly to the null directions of the Lorentz metric. Thus the Carathéodory conjecture bounds the winding number of the light cone of certain Lorentz surfaces with isolated degenerate points, and the hyperbolic nature (and hence difficulty) of the problem becomes evident.

Note 3. Roughly speaking, our strategy is to maximise the area of an immersed surface by reducing the shear and increasing the twist. In fact, the previous Proposition was generalized [9]:

Let $(\mathbb{M}, \mathbb{J}, \Omega, \mathbb{G})$ *be any neutral Kähler surface and let* $p \in \mathbb{M}$ *and* $v_1, v_2 \in T_p\mathbb{M}$ *span a plane. Then*

$$\Omega(v_1, v_2)^2 - \varsigma^2(v_1, v_2) = det\ \mathbb{G}(v_i, v_j),$$

where $\varsigma^2(v_1, v_2) \geq 0$ *with equality iff* $span_{\mathbb{R}}\{v_1, v_2\}$ *is a complex line in* $T_p\mathbb{M}$.

In order to continue, we introduce geometric tools which will prove useful later.

4.2. **Orthonormal frames.** Let $\Sigma \to TS^2$ be an immersed surface and assume that the induced metric is not degenerate on Σ, so that for $\gamma \in \Sigma$ we have the orthogonal splitting $T_\gamma TS^2 = T_\gamma\Sigma \oplus N_\gamma\Sigma$.

Definition 10. Given a surface Σ in TS^2, a frame $\{e_{(a)}\}_{a=1}^4$ is *adapted* if $\{e_{(1)}, e_{(2)}\}$ is a basis for the tangent space TΣ and $\{e_{(3)}, e_{(4)}\}$ is a basis for the normal bundle NΣ, and

$$\mathbb{G}(e_{(a)}, e_{(b)}) = \text{diag}(1, \epsilon, -\epsilon, -1),$$

where $\epsilon = 1$ for $\mathbb{G}|_\Sigma$ positive definite and $\epsilon = -1$ for $\mathbb{G}|_\Sigma$ Lorentz. For mean curvature flow of positive discs we have $\epsilon = 1$.

Proposition 19. *If* Σ *is a non-degenerate surface given by the graph* $\xi \to (\xi, \eta = F(\xi, \bar{\xi}))$*, then the following vector fields form an adapted basis:*

$$e_{(1)} = 2\mathbb{R}e\left[\alpha_1\left(\frac{\partial}{\partial\xi} + \partial F\frac{\partial}{\partial\eta} + \partial\bar{F}\frac{\partial}{\partial\bar{\eta}}\right)\right],$$

$$e_{(2)} = 2\mathbb{R}e\left[\alpha_2\left(\frac{\partial}{\partial\xi} + \partial F\frac{\partial}{\partial\eta} + \partial\bar{F}\frac{\partial}{\partial\bar{\eta}}\right)\right],$$

$$e_{(3)} = 2\mathbb{R}e\left[\alpha_2\left(\frac{\partial}{\partial\xi} + (\bar{\partial}\bar{F} - 2(F\partial u - \bar{F}\bar{\partial}u))\frac{\partial}{\partial\eta} - \partial\bar{F}\frac{\partial}{\partial\bar{\eta}}\right)\right],$$

$$e_{(4)} = 2\mathbb{R}e\left[\alpha_1\left(\frac{\partial}{\partial\xi} + (\bar{\partial}\bar{F} - 2(F\partial u - \bar{F}\bar{\partial}u))\frac{\partial}{\partial\eta} - \partial\bar{F}\frac{\partial}{\partial\bar{\eta}}\right)\right],$$

for

$$\alpha_1 = \frac{e^{-u-\frac{1}{2}\phi i+\frac{1}{4}\pi i}}{\sqrt{2}[-\lambda - |\sigma|]^{\frac{1}{2}}} \qquad \alpha_2 = \frac{e^{-u-\frac{1}{2}\phi i-\frac{1}{4}\pi i}}{\sqrt{2}[\epsilon(-\lambda + |\sigma|)]^{\frac{1}{2}}},$$

where $\bar{\partial}F = -|\sigma|e^{-i\phi}$ *and we have introduced* $e^{2u} = 4(1 + \xi\bar{\xi})^{-2}$*. Note that when* $|\sigma| = 0$*, then* ϕ *is just a gauge freedom for the frame.*

Using the same notation as above:

Proposition 20. *The dual basis of 1-forms is:*

$$\theta^{(1)} = \mathbb{I}m\left[(\alpha_1\partial\bar{F} + \bar{\alpha}_1(\bar{\partial}\bar{F} - 2(F\partial u - \bar{F}\bar{\partial}u)))d\xi - \bar{\alpha}_1 d\eta\right]e^{2u},$$

$$\theta^{(2)} = \epsilon\ \mathbb{I}m\left[(\alpha_2\partial\bar{F} + \bar{\alpha}_2(\bar{\partial}\bar{F} - 2(F\partial u - \bar{F}\bar{\partial}u)))d\xi - \bar{\alpha}_2 d\eta\right]e^{2u},$$

$$\theta^{(3)} = \epsilon\ \mathbb{I}m\left[(\alpha_2\partial\bar{F} - \bar{\alpha}_2\partial F)d\xi + \bar{\alpha}_2 d\eta\right]e^{2u},$$

$$\theta^{(4)} = \mathbb{I}m\left[(\alpha_1\partial\bar{F} - \bar{\alpha}_1\partial F)d\xi + \bar{\alpha}_1 d\eta\right]e^{2u}.$$

4.3. **The second fundamental form.** Now consider the Levi-Civita connection $\overline{\nabla}$ associated with $\mathbb{G}$ and for $X, Y \in T\Sigma$ we have the orthogonal splitting

$$\overline{\nabla}_X Y = \nabla_X Y + H(X, Y),$$

where $H : T\Sigma \times T\Sigma \to N\Sigma$ is the second fundamental form of the immersed surface Σ.

Proposition 21. *The second fundamental form is:*

$$H(e_{(a)}, e_{(b)}) = 2\mathbb{R}e\left[\beta_{ab}\left(\frac{\partial}{\partial\xi} + (\bar\partial\bar F - 2(F\partial u - \bar F\bar\partial u))\frac{\partial}{\partial\eta} - \partial\bar F\frac{\partial}{\partial\bar\eta}\right)\right],$$

where

$$\beta_{11} = \big(i\lambda\partial|\sigma| - \sigma\bar\partial|\sigma| + i\lambda\partial\lambda - \sigma\bar\partial\lambda + |\sigma|(|\sigma|+\lambda)(\partial\phi - ie^{i\phi}\bar\partial\phi + 2i\partial u - 2e^{i\phi}\bar\partial u)\big)$$

$$\Big/\big(2e^{2u+i\phi}(|\sigma|+\lambda)^2(-|\sigma|+\lambda)\big),$$

$$\beta_{22} = \big(-i\lambda\partial|\sigma| + \sigma\bar\partial|\sigma| + i\lambda\partial\lambda - \sigma\bar\partial\lambda + |\sigma|(|\sigma|-\lambda)(\partial\phi + ie^{i\phi}\bar\partial\phi + 2i\partial u + 2e^{i\phi}\bar\partial u)\big)$$

$$\Big/\big(2\epsilon e^{2u+i\phi}(|\sigma|-\lambda)^2(-|\sigma|-\lambda)\big),$$

$$\beta_{12} = \big(-|\sigma|\partial|\sigma| + i\lambda e^{i\phi}\bar\partial|\sigma| + \lambda\partial\lambda - i\sigma\bar\partial\lambda\big)$$

$$\Big/\left(2e^{2u+i\phi}(|\sigma|^2-\lambda^2)\sqrt{\epsilon(\lambda^2-|\sigma|^2)}\right).$$

Proof. Consider the parallel and perpendicular projection operators ${}^{\parallel}P : TTS^2 \to T\Sigma$ and ${}^{\perp}P : TTS^2 \to N\Sigma$. These are given in terms of an adapted frame by

$${}^{\parallel}P^k_j = \delta^k_j - e^k_{(3)}\theta^{(3)}_j - e^k_{(4)}\theta^{(4)}_j \qquad {}^{\perp}P^k_j = \delta^k_j - e^k_{(1)}\theta^{(1)}_j - e^k_{(2)}\theta^{(2)}_j.$$

The parallel projection operator has the following coordinate description:

$$\begin{aligned}
{}^{\parallel}P^{\xi}_{\bar\eta} &= -\tfrac{1}{2\Delta}\bar\sigma & {}^{\parallel}P^{\xi}_{\bar\xi} &= -\tfrac{1}{2\Delta}(\bar\partial\bar F + \lambda i)\bar\sigma,\\
{}^{\parallel}P^{\xi}_{\eta} &= -\tfrac{1}{2\Delta}\lambda i & {}^{\parallel}P^{\xi}_{\xi} &= \tfrac{1}{2\Delta}[(\partial F - 2\lambda i)\lambda i - |\sigma|^2],\\
{}^{\parallel}P^{\eta}_{\bar\eta} &= \tfrac{1}{2\Delta}\bar\sigma(\partial F - \lambda i) & {}^{\parallel}P^{\eta}_{\bar\xi} &= \tfrac{1}{2\Delta}[-\bar\sigma[\partial F\bar\partial\bar F - |\sigma|^2 - \lambda i(\bar\partial\bar F - \partial F)] + 2\lambda^2],\\
{}^{\parallel}P^{\eta}_{\eta} &= -\tfrac{1}{2\Delta}[\lambda i\partial F + |\sigma|^2] & {}^{\parallel}P^{\eta}_{\xi} &= \tfrac{1}{2\Delta}\lambda i[(\partial F - 2\lambda i)\partial F - |\sigma|^2].
\end{aligned}$$

where $\Delta = \lambda^2 - |\sigma|^2$, while the perpendicular projection operator is

$${}^{\perp}P^{\xi}_{\xi} = {}^{\parallel}P^{\eta}_{\eta} \qquad {}^{\perp}P^{\xi}_{\bar\xi} = -{}^{\parallel}P^{\xi}_{\bar\xi} \qquad {}^{\perp}P^{\xi}_{\eta} = -{}^{\parallel}P^{\xi}_{\eta} \qquad {}^{\perp}P^{\xi}_{\bar\eta} = -{}^{\parallel}P^{\xi}_{\bar\eta},$$

$${}^{\perp}P^{\eta}_{\xi} = -{}^{\parallel}P^{\eta}_{\xi} \qquad {}^{\perp}P^{\eta}_{\bar\xi} = -{}^{\parallel}P^{\eta}_{\bar\xi} \qquad {}^{\perp}P^{\eta}_{\eta} = {}^{\parallel}P^{\xi}_{\xi} \qquad {}^{\perp}P^{\eta}_{\bar\eta} = -{}^{\parallel}P^{\eta}_{\bar\eta}.$$

In terms of an adapted frame the second fundamental form

$$H_{(ab)}{}^{j} = {}^{\perp}P^j_k\, e^l_{(a)}\nabla_l\, e^k_{(b)}.$$

The result follows by computation. □

Proposition 22. *The mean curvature vector of the surface Σ is:*

$$H = 2\mathbb{R}e\left[\gamma\left(\frac{\partial}{\partial\xi}+(\bar{\partial}\bar{F}-2(F\partial u-\bar{F}\bar{\partial}u))\frac{\partial}{\partial\eta}-\partial\bar{F}\frac{\partial}{\partial\bar{\eta}}\right)\right],$$

where

$$\gamma = \big(-\lambda(-i\lambda\partial|\sigma|+\sigma\bar{\partial}|\sigma|)-|\sigma|(i\lambda\partial\lambda-\sigma\bar{\partial}\lambda)-|\sigma|(|\sigma|^2-\lambda^2)(\partial\phi+2i\partial u)\big) \Big/\big(e^{2u+i\phi}(|\sigma|^2-\lambda^2)^2\big).$$

Proof. The mean curvature vector of the surface Σ is the trace of the second fundamental form, which is

$$H^j = H_{(11)}{}^j+\epsilon H_{(22)}{}^j.$$

The result follows from computing this with the aid of the previous Proposition. □

Note 4. We can also write the mean curvature vector component (see [8] for a variational derivation of this formula)

$$H^\xi = \epsilon\frac{2e^{-2u}}{\sqrt{|\lambda^2-|\sigma|^2|}}\left[ie^{-2u}\partial\left(\frac{\bar{\sigma}e^{2u}}{\sqrt{|\lambda^2-|\sigma|^2|}}\right)-\bar{\partial}\left(\frac{\lambda}{\sqrt{|\lambda^2-|\sigma|^2|}}\right)\right]. \tag{4.4}$$

Corollary 1. *A holomorphic graph has vanishing mean curvature.*

Proof. This follows from inserting $\sigma=0$ in equation (4.4). □

4.4. **Mean curvature flow.** Consider **P.B.V.P.** of section 2.3 for discs in TS^2 attached to a real surface $\tilde{\Sigma}$.

Proposition 23. *For a graph in TS^2, the mean curvature flow is*

$$\begin{aligned}\dot{F} =& g^{jk}\partial_j\partial_k F+\frac{i\bar{\sigma}}{\Delta}\big((\sigma\xi-\bar{\rho}\bar{\xi})(1+\xi\bar{\xi})+\bar{F}-\bar{\xi}^2F\big)\\ =& \frac{(1+\xi\bar{\xi})^2}{2(\lambda^2-\sigma\bar{\sigma})}\left(-2\bar{\sigma}\partial\lambda-i\bar{\sigma}\bar{\partial}\sigma+2\lambda\partial\bar{\sigma}+i\sigma\bar{\partial}\bar{\sigma}+\frac{4i\bar{\sigma}(\sigma\xi+\lambda i\bar{\xi})}{1+\xi\bar{\xi}}\right).\end{aligned} \tag{4.5}$$

Proof. Consider a surface $f:\Sigma\times[0,s_0)\to\mathbb{M}$ such that $f_s(\xi,\bar{\xi})=(\xi,\bar{\xi},F_s(\xi,\bar{\xi}),\bar{F}_s(\xi,\bar{\xi}))$. Then

$$\frac{\partial f}{\partial s}=\frac{\partial F}{\partial s}\frac{\partial}{\partial\eta}+\frac{\partial\bar{F}}{\partial s}\frac{\partial}{\partial\bar{\eta}}.$$

Projecting normal to Σ

$$\begin{aligned}\frac{\partial f}{\partial s}^{\perp} =& ({}^{\perp}P^{\xi}_{\eta}\dot{F}+{}^{\perp}P^{\xi}_{\bar{\eta}}\dot{\bar{F}})\frac{\partial}{\partial\xi}+({}^{\perp}P^{\eta}_{\eta}\dot{F}+{}^{\perp}P^{\eta}_{\bar{\eta}}\dot{\bar{F}})\frac{\partial}{\partial\eta}\\ &+({}^{\perp}P^{\bar{\xi}}_{\bar{\eta}}\dot{\bar{F}}+{}^{\perp}P^{\bar{\xi}}_{\eta}\dot{F})\frac{\partial}{\partial\bar{\xi}}+({}^{\perp}P^{\bar{\eta}}_{\bar{\eta}}\dot{\bar{F}}+{}^{\perp}P^{\bar{\eta}}_{\eta}\dot{F})\frac{\partial}{\partial\bar{\eta}},\end{aligned}$$

and so the mean curvature flow is

$${}^{\perp}P^{\xi}_{\eta}\dot{F}+{}^{\perp}P^{\xi}_{\bar{\eta}}\dot{\bar{F}}=H^\xi,$$

or from the expressions of the projection operators given in the proof of Proposition 21

$$\frac{\lambda i}{2\Delta}\dot{F}-\frac{\bar{\sigma}}{2\Delta}\dot{\bar{F}}=H^\xi.$$

Combining this with its complex conjugate we get

$$\dot{F} = -2\lambda i H^{\xi} + 2\bar{\sigma} H^{\bar{\xi}}. \tag{4.6}$$

Using the expression (4.4) for the mean curvature we get that

$$H^{\xi} = \frac{(1+\xi\bar{\xi})^2}{4\Delta^2}\Big[2\left(i\partial\bar{\sigma} - \bar{\partial}\lambda - \frac{2i\bar{\xi}\bar{\sigma}}{1+\xi\bar{\xi}}\right)\Delta \\ - 2i\lambda\bar{\sigma}\partial\lambda + i\sigma\bar{\sigma}\partial\bar{\sigma} + \bar{\sigma}^2\partial\sigma + 2\lambda^2\bar{\partial}\lambda - \lambda\sigma\bar{\partial}\bar{\sigma} - \lambda\bar{\sigma}\bar{\partial}\sigma\Big],$$

and the second equality stated in the Proposition follows from inserting this in equation (4.6).

To see that the first equality in the Proposition holds, we compute

$$\begin{aligned} g^{jk}\partial_j\partial_k F &= \frac{(1+\xi\bar{\xi})^2}{2\Delta}\left(i\bar{\sigma}\partial^2 F - 2\lambda\partial\bar{\partial}F - i\sigma\bar{\partial}^2 F\right) \\ &= \frac{(1+\xi\bar{\xi})^2}{2\Delta}\left[i\bar{\sigma}\partial\left(\theta + i\lambda + \frac{2\bar{\xi}F}{1+\xi\bar{\xi}}\right) + 2\lambda\partial\bar{\sigma} + i\sigma\bar{\partial}\bar{\sigma}\right] \\ &= \frac{(1+\xi\bar{\xi})^2}{2\Delta}\Big[-2\bar{\sigma}\partial\lambda - i\bar{\sigma}\bar{\partial}\sigma + i\sigma\bar{\partial}\bar{\sigma} + 2\lambda\partial\bar{\sigma} \\ &\qquad + i\bar{\sigma}\left(\frac{2(\sigma\xi + \rho\bar{\xi})}{1+\xi\bar{\xi}} - \frac{2(\bar{F} - \bar{\xi}^2 F)}{(1+\xi\bar{\xi})^2}\right)\Big], \end{aligned}$$

where we have used identity (4.3) in the more convenient form

$$\partial\theta = i\partial\lambda - (1+\xi\bar{\xi})^2\partial\left(\frac{\bar{\sigma}}{(1+\xi\bar{\xi})^2}\right) - \frac{2F}{(1+\xi\bar{\xi})^2}.$$

Thus

$$g^{jk}\partial_j\partial_k F + \frac{i\bar{\sigma}}{\Delta}\left((\sigma\xi - \bar{\rho}\bar{\xi})(1+\xi\bar{\xi}) + \bar{F} - \bar{\xi}^2 F\right) \\ = \frac{(1+\xi\bar{\xi})^2}{2(\lambda^2 - \sigma\bar{\sigma})}\left(-2\bar{\sigma}\partial\lambda - i\bar{\sigma}\bar{\partial}\sigma + i\sigma\bar{\partial}\bar{\sigma} + 2\lambda\partial\bar{\sigma} + \frac{4i\bar{\sigma}(\sigma\xi + \lambda i\bar{\xi})}{1+\xi\bar{\xi}}\right).$$

as claimed. □

Proposition 24. *Under the mean curvature flow the shear evolves by:*

$$\frac{\partial\sigma}{\partial t} = g^{jk}\partial_j\partial_k\sigma + \frac{H_1(1+\xi\bar{\xi})^2 + 2H_2(1+\xi\bar{\xi}) + 2H_3}{2\Delta^2},$$

where

$$\begin{aligned} H_1 = &-4\lambda\sigma\partial\lambda\bar{\partial}\lambda - 2i\lambda\bar{\sigma}\partial\lambda\partial\sigma + 2(\lambda^2 + y)\partial\lambda\bar{\partial}\sigma + 2i\lambda\sigma\partial\lambda\partial\bar{\sigma} + 2\lambda^2\bar{\partial}\lambda\partial\sigma \\ &+ 2\sigma^2\bar{\partial}\lambda\partial\bar{\sigma} + i\bar{\sigma}^2(\partial\sigma)^2 - 2\lambda\bar{\sigma}\partial\sigma\bar{\partial}\sigma - 2\lambda\sigma\bar{\partial}\sigma\partial\bar{\sigma} - i\sigma^2(\partial\bar{\sigma})^2, \end{aligned}$$

$$\begin{aligned} H_2 = &-2\sigma\partial\lambda\left(2i\lambda\bar{\sigma}\bar{\xi} + (\lambda^2 + \sigma\bar{\sigma})\xi\right) + 2\sigma\bar{\partial}\lambda\left(\lambda^2 - \sigma\bar{\sigma}\right)\bar{\xi} + \partial\sigma\left(i\bar{\sigma}\bar{\xi}(3\lambda^2 - \sigma\bar{\sigma}) + 2\lambda^3\xi\right) \\ &+ \bar{\partial}\sigma\left(i\sigma\xi - 2\lambda\bar{\xi}\right)(\lambda^2 - \sigma\bar{\sigma}) + 2\partial\bar{\sigma}\left(i\sigma^2\bar{\sigma}\bar{\xi} + \lambda\sigma^2\xi\right), \end{aligned}$$

and

$$H_3 = -\sigma\left(i\sigma\xi^2 - 3i\bar{\sigma}\bar{\xi}^2 - 4\lambda\xi\bar{\xi}\right)(\lambda^2 - \sigma\bar{\sigma}).$$

In addition,

$$\frac{\partial\rho}{\partial t} = g^{jk}\partial_j\partial_k\rho + \frac{H_4(1+\xi\bar{\xi})^3 + H_5(1+\xi\bar{\xi})^2 + H_6(1+\xi\bar{\xi}) + H_7}{2(1+\xi\bar{\xi})\Delta^2},$$

where

$$H_4 = 4\lambda\bar{\sigma}\left(\partial\lambda\right)^2 - 2\bar{\sigma}^2\partial\lambda\partial\sigma + 2i\lambda\bar{\sigma}\partial\lambda\bar{\partial}\sigma - 4\lambda^2\partial\lambda\partial\bar{\sigma} - 2\sigma\bar{\sigma}\partial\lambda\partial\bar{\sigma} - 2i\lambda\sigma\partial\lambda\bar{\partial}\bar{\sigma}$$
$$- i\bar{\sigma}^2\partial\sigma\bar{\partial}\sigma + 2\lambda\bar{\sigma}\partial\sigma\partial\bar{\sigma} + i\lambda^2\partial\sigma\bar{\partial}\bar{\sigma} - i\lambda^2\bar{\partial}\sigma\partial\bar{\sigma} + 2\lambda\sigma\left(\partial\bar{\sigma}\right)^2 + i\sigma^2\partial\bar{\sigma}\bar{\partial}\bar{\sigma},$$

$$H_5 = -4\partial\lambda\left(2i\lambda\sigma\bar{\sigma}\xi - \lambda^2\bar{\sigma}\bar{\xi} - \sigma\bar{\sigma}^2\bar{\xi}\right) + 2\partial\sigma\left(i\lambda^2\bar{\sigma}\xi + i\sigma\bar{\sigma}^2\xi - 2\lambda\bar{\sigma}^2\bar{\xi}\right)$$
$$+ 2\left(2\partial\bar{\sigma}i\lambda^2\sigma\xi - 2\partial\bar{\sigma}\lambda\sigma\bar{\sigma}\bar{\xi} + \bar{\partial}\bar{\sigma}i\lambda^2\sigma\bar{\xi} - \bar{\partial}\bar{\sigma}i\sigma^2\bar{\sigma}\bar{\xi}\right),$$

$$H_6 = -4(i(\lambda^2 - \sigma\bar{\sigma}) + \lambda\theta)(\lambda^2 - \sigma\bar{\sigma}),$$

and

$$H_7 = 4i(F\sigma\xi - \bar{F}\bar{\sigma}\bar{\xi})(\lambda^2 - \sigma\bar{\sigma}).$$

Proof. The proofs of these statements follow from differentiation of the flow equation (4.5). We illustrate this for the flow of σ, leaving the flow of ρ to the reader.

We start by splitting the expression into convenient terms:

$$-\dot{\bar{\sigma}} = \bar{\partial}\dot{F} = E_1 + E_2 + E_3 + E_4,$$

where E_1 is second order in the derivatives of λ and σ, E_2 and E_3 are the quadratic and linear first order terms, and E_4 is the zeroth order terms. We now compute each of these terms in turn.

So, differentiating equation (4.5) we have

$$E_1 = \frac{(1+\xi\bar{\xi})^2}{2\Delta}\left(-2\bar{\sigma}\bar{\partial}\partial\lambda - i\bar{\sigma}\bar{\partial}\bar{\partial}\sigma + 2\lambda\bar{\partial}\partial\bar{\sigma} + i\sigma\bar{\partial}\bar{\partial}\bar{\sigma}\right).$$

At this point we exploit the 3-jet identity (4.3) which we write in the more favorable form

$$\partial\bar{\partial}\lambda = \mathbb{Im}\left[\bar{\partial}\bar{\partial}\sigma - \frac{2\xi}{1+\xi\bar{\xi}}\bar{\partial}\sigma + \frac{2\xi^2}{(1+\xi\bar{\xi})^2}\sigma\right] - \frac{2\lambda}{(1+\xi\bar{\xi})^2}.$$

Inserting this in the expression for E_1 yields

$$E_1 = \frac{(1+\xi\bar{\xi})^2}{2\Delta}\left(-i\bar{\sigma}\partial\partial\bar{\sigma} + 2\lambda\partial\bar{\partial}\bar{\sigma} + i\sigma\bar{\partial}\bar{\partial}\bar{\sigma} - \frac{2i\bar{\sigma}(\xi\bar{\partial}\sigma - \bar{\xi}\partial\bar{\sigma})}{1+\xi\bar{\xi}} + \frac{2i\bar{\sigma}(\xi^2\sigma - \bar{\xi}^2\bar{\sigma} - 2i\lambda)}{(1+\xi\bar{\xi})^2}\right).$$

The first three terms of this, noting the expression for g^{-1} in Proposition 17, are easily seen to be the rough Laplacian of $-\bar{\sigma}$:

$$E_1 = -g^{jk}\partial_j\partial_k\bar{\sigma} + \frac{i\bar{\sigma}}{\Delta}\left((\bar{\xi}\partial\bar{\sigma} - \xi\bar{\partial}\sigma)(1+\xi\bar{\xi}) + \xi^2\sigma - \bar{\xi}^2\bar{\sigma} - 2i\lambda\right).$$

We note that in the final sets of expressions, the lower order terms introduced into E_1 by the 3-jet identity will have to be added to E_3 and E_4.

Moving to the quadratic first order term we compute from differentiating equation (4.5) that

$$
\begin{aligned}
E_2 =& \frac{(1+\xi\bar\xi)^2}{2\Delta}\Big(-2\bar\partial\bar\sigma\partial\lambda - i\bar\partial\bar\sigma\bar\partial\sigma + 2\bar\partial\lambda\partial\bar\sigma + i\bar\partial\sigma\bar\partial\bar\sigma \\
&\qquad - \frac{1}{\Delta}\bar\partial\Delta(-2\bar\sigma\partial\lambda - i\bar\sigma\bar\partial\sigma + 2\lambda\partial\bar\sigma + i\sigma\bar\partial\bar\sigma)\Big) \\
=& \frac{(1+\xi\bar\xi)^2}{2\Delta^2}\Big(4\lambda\bar\sigma\bar\partial\lambda\partial\lambda + 2i\lambda\bar\sigma\bar\partial\lambda\bar\partial\sigma - 2(\lambda^2+\sigma\bar\sigma)\bar\partial\lambda\partial\bar\sigma - 2i\lambda\sigma\bar\partial\lambda\bar\partial\bar\sigma \\
&\qquad - 2\lambda^2\bar\partial\bar\sigma\partial\lambda + 2\lambda\sigma\bar\partial\bar\sigma\partial\bar\sigma + i\sigma^2(\bar\partial\bar\sigma)^2 - i\bar\sigma^2(\bar\partial\sigma)^2 \\
&\qquad - 2\bar\sigma^2\bar\partial\sigma\partial\lambda + 2\lambda\bar\sigma\bar\partial\sigma\partial\bar\sigma\Big) \\
=& -\frac{(1+\xi\bar\xi)^2}{2\Delta^2}\bar H_1.
\end{aligned}
$$

This establishes the quadratic first order term, once we recall that $\bar\partial\dot F = -\dot{\bar\sigma}$.

Moving to the linear first order term

$$
\begin{aligned}
E_3 =& \frac{(1+\xi\bar\xi)\xi}{\Delta}\left(-2\bar\sigma\partial\lambda - i\bar\sigma\bar\partial\sigma + 2\lambda\partial\bar\sigma + i\sigma\bar\partial\bar\sigma\right) \\
&+ \frac{1+\xi\bar\xi}{\Delta}\left(2i(\sigma\xi + i\lambda\bar\xi)\bar\partial\bar\sigma + 2i\bar\sigma(\xi\bar\partial\sigma + i\bar\xi\bar\partial\lambda)\right) \\
&- \frac{1+\xi\bar\xi}{\Delta^2}\left[2i\bar\sigma(\sigma\xi + i\lambda\bar\xi)(2\lambda\bar\partial - \sigma\bar\partial\bar\sigma - \bar\sigma\bar\partial\sigma)\right] \\
=& \frac{1+\xi\bar\xi}{\Delta^2}\Big\{ -2\xi\bar\sigma(\lambda^2-\sigma\bar\sigma)\partial\lambda + [2\bar\sigma^2(i\sigma\xi - \lambda\bar\xi) + i\xi\bar\sigma(\lambda^2-\sigma\bar\sigma)]\bar\partial\sigma \\
&+ 2\xi\lambda(\lambda^2-\sigma\bar\sigma)\partial\bar\sigma + [i\xi\sigma(3\lambda^2-\sigma\bar\sigma) - 2\bar\xi\lambda^3]\bar\partial\bar\sigma \\
&+ 2[\bar\xi\bar\sigma(\lambda^2+\sigma\bar\sigma) - 2i\xi\lambda\sigma\bar\sigma]\bar\partial\lambda\Big\}.
\end{aligned}
$$

Adding the linear term from E_1 we compute that

$$
\begin{aligned}
-\bar H_3 =& \frac{\Delta^2}{1+\xi\bar\xi}E_3 + i\bar\xi\bar\sigma(\lambda^2-\sigma\bar\sigma)\partial\bar\sigma - i\xi\bar\sigma(\lambda^2-\sigma\bar\sigma)\bar\partial\sigma \\
=& \frac{1+\xi\bar\xi}{\Delta^2}\Big\{ -2\xi\bar\sigma(\lambda^2-\sigma\bar\sigma)\partial\lambda + 2\bar\sigma^2(i\sigma\xi - \lambda\bar\xi)\bar\partial\sigma \\
&+ (\lambda^2-\sigma\bar\sigma)(2\xi\lambda + i\bar\xi\bar\sigma)\partial\bar\sigma + [i\xi\sigma(3\lambda^2-\sigma\bar\sigma) - 2\bar\xi\lambda^3]\bar\partial\bar\sigma \\
&+ 2[\bar\xi\bar\sigma(\lambda^2+\sigma\bar\sigma) - 2i\xi\lambda\sigma\bar\sigma]\bar\partial\lambda\Big\},
\end{aligned}
$$

as claimed in the Proposition.

Finally, we work out the zero order term by looking again at the derivative of equation (4.5) :

$$
E_4 = \frac{2i\bar\sigma}{\Delta}\Big(\sigma\xi^2 + i\lambda(1+2\xi\bar\xi)\Big),
$$

and taking into account the zero order term of E_1, we have

$$
-\bar H_4 = \Delta^2\ E_4 + i\bar\sigma(\xi^2\sigma - \bar\xi^2\bar\sigma - 2i\lambda)\Delta = \bar\sigma(3i\xi^2\sigma - i\bar\xi^2\bar\sigma - 4\xi\bar\xi\lambda)\Delta,
$$

as claimed. □

For later use, introduce the flat metric

$$< d\xi, d\bar{\xi} >= 1 \qquad\qquad < d\xi, d\xi >=< d\bar{\xi}, d\bar{\xi} >= 0,$$

on Σ via its coordinate ξ. If we denote the flat norm and inner product by $|.|$ and $< \cdot, \cdot >$, and the norm and inner product of g by $\|.\|$ and $<< \cdot, \cdot >>$, we have the following estimates which will prove useful:

Lemma 2.

$$\frac{(1+\xi\bar{\xi})^2(-\lambda-|\sigma|)}{\lambda^2-|\sigma|^2}|X|^2 \leq \|X\|^2 \leq \frac{(1+\xi\bar{\xi})^2(-\lambda+|\sigma|)}{\lambda^2-|\sigma|^2}|X|^2.$$

We now prove that the flow remains in a compact set.

Proposition 25. *For the mean curvature flow in TS^2 there exists positive constants C_1 and C_2 depending only on the size of the initial disc and F_0 such that*

$$F\bar{F} + C_1\xi\bar{\xi} < C_2,$$

for all time.

Proof. A straightforward computation shows that

$$\left(\frac{d}{ds} - \triangle_0\right)\xi\bar{\xi} = \frac{\lambda(1+\xi\bar{\xi})^2}{\lambda^2-\sigma\bar{\sigma}},$$

and

$$\left(\frac{d}{ds} - \triangle_0\right)F\bar{F} = -2g^{jk}\partial_j F\partial_k\bar{F} + \frac{[\bar{F}(i\sigma\bar{\sigma}\xi-(i\bar{\sigma}\theta+\lambda\bar{\sigma})\bar{\xi})-F(i\sigma\bar{\sigma}\bar{\xi}-(i\sigma\theta-\lambda\sigma)\xi)](1+\xi\bar{\xi})}{\lambda^2-\sigma\bar{\sigma}} - \frac{F^2 i\sigma - F\bar{F}i\sigma\xi^2 + F\bar{F}i\bar{\sigma}\bar{\xi}^2 - \bar{F}^2 i\bar{\sigma}}{\lambda^2-\sigma\bar{\sigma}},$$

where we introduce $\triangle_0$ for the rough Laplacian:

$$\triangle_0 = g^{jk}\partial_j\partial_k.$$

Let

$$I_1 = \frac{[\bar{F}(i\sigma\bar{\sigma}\xi-(i\bar{\sigma}\theta+\lambda\bar{\sigma})\bar{\xi})-F(i\sigma\bar{\sigma}\bar{\xi}-(i\sigma\theta-\lambda\sigma)\xi)](1+\xi\bar{\xi})}{\lambda^2-\sigma\bar{\sigma}}.$$

Define the function $\mathcal{D}:\mathbb{C}^2\rightarrow\mathbb{R}$ by $\mathcal{D}(\eta,\xi)=\eta\bar{\eta}+C_1\xi\bar{\xi}$ for some positive constant C_1 to be determined later. We consider the pullback of this function to the flowing disc. At a turning point p of $\mathcal{D}$ we have

$$\partial\mathcal{D} = F\partial\bar{F} + \bar{F}\partial F + C_1\bar{\xi} = F\bar{\partial}\bar{F} + \bar{F}\bar{\partial}F + C_1\xi = 0,$$

which can be written

$$\bar{F}\theta = F\sigma - i\bar{F}\lambda - C_1\bar{\xi} - \frac{2\bar{\xi}F\bar{F}}{1+\xi\bar{\xi}},$$

$$F\theta = \bar{F}\bar{\sigma} + iF\lambda - C_1\xi - \frac{2\xi F\bar{F}}{1+\xi\bar{\xi}}.$$

Substituting these in the definition of I_1 we find that, at a turning point,

$$I_1 = \frac{2[\bar{F}\bar{\sigma}(i\sigma\xi - \lambda\bar{\xi}) - F\sigma(i\bar{\sigma}\bar{\xi} + \lambda\xi)](1+\xi\bar{\xi})}{\lambda^2 - \sigma\bar{\sigma}} + \frac{i(C_1(1+\xi\bar{\xi}) + 2F\bar{F})(\bar{\sigma}\bar{\xi}^2 - \sigma\xi^2)}{\lambda^2 - \sigma\bar{\sigma}}.$$

Moreover, at p

$$\bar{F}dF = \bar{F}\partial F d\xi + \bar{F}\bar{\partial}F\, d\bar{\xi} = (-F\partial\bar{F} - C_1\bar{\xi})d\xi + \bar{F}\bar{\partial}F\, d\bar{\xi} = (F\sigma - C_1\bar{\xi})d\xi - \bar{F}\bar{\sigma}d\bar{\xi},$$

and so

$$<< \bar{F}dF, \bar{\xi}d\xi - \xi d\bar{\xi} >> = -\frac{[\bar{F}\bar{\sigma}(i\sigma\xi - \lambda\bar{\xi}) - F\sigma(i\bar{\sigma}\bar{\xi} + \lambda\xi)](1+\xi\bar{\xi})^2}{2(\lambda^2 - \sigma\bar{\sigma})} - \frac{C_1(1+\xi\bar{\xi})^2(i\bar{\sigma}\bar{\xi}^2 + \lambda\xi\bar{\xi})}{2(\lambda^2 - \sigma\bar{\sigma})}.$$

We conclude that, at p,

$$I_1 = -\frac{2}{1+\xi\bar{\xi}} << \bar{F}dF, \bar{\xi}d\xi - \xi d\bar{\xi} >> + \frac{2}{1+\xi\bar{\xi}} << Fd\bar{F}, \bar{\xi}d\xi - \xi d\bar{\xi} >> + \frac{2C_1(1+\xi\bar{\xi})(i\bar{\sigma}\bar{\xi}^2 + \lambda\xi\bar{\xi} - i\sigma\xi^2)}{\lambda^2 - \sigma\bar{\sigma}} + \frac{2iF\bar{F}(\bar{\sigma}\bar{\xi}^2 - \sigma\xi^2)}{\lambda^2 - \sigma\bar{\sigma}}.$$

Now combining these, we find that at a turning point

$$\begin{aligned}\left(\frac{d}{ds} - \triangle_0\right) F\bar{F} &= -2 << dF, d\bar{F} >> -2 << \bar{F}dF, \frac{\bar{\xi}d\xi - \xi d\bar{\xi}}{1+\xi\bar{\xi}} >> \\ &\quad + 2 << Fd\bar{F}, \frac{\bar{\xi}d\xi - \xi d\bar{\xi}}{1+\xi\bar{\xi}} >> + \frac{2C_1(1+\xi\bar{\xi})(i\bar{\sigma}\bar{\xi}^2 + \lambda\xi\bar{\xi} - i\sigma\xi^2)}{\lambda^2 - \sigma\bar{\sigma}} \\ &\quad - \frac{F^2 i\sigma + F\bar{F}i\sigma\xi^2 - F\bar{F}i\bar{\sigma}\bar{\xi}^2 - \bar{F}^2 i\bar{\sigma}}{\lambda^2 - \sigma\bar{\sigma}} \\ &= -2 << dF - F\frac{\bar{\xi}d\xi - \xi d\bar{\xi}}{1+\xi\bar{\xi}}, d\bar{F} + \bar{F}\frac{\bar{\xi}d\xi - \xi d\bar{\xi}}{1+\xi\bar{\xi}} >> \\ &\quad - \frac{2F\bar{F}}{(1+\xi\bar{\xi})^2} << \bar{\xi}d\xi - \xi d\bar{\xi}, \bar{\xi}d\xi - \xi d\bar{\xi} >> \\ &\quad + \frac{2C_1(1+\xi\bar{\xi})(i\bar{\sigma}\bar{\xi}^2 + \lambda\xi\bar{\xi} - i\sigma\xi^2)}{\lambda^2 - \sigma\bar{\sigma}} \\ &\quad - \frac{F^2 i\sigma + F\bar{F}i\sigma\xi^2 - F\bar{F}i\bar{\sigma}\bar{\xi}^2 - \bar{F}^2 i\bar{\sigma}}{\lambda^2 - \sigma\bar{\sigma}}.\end{aligned}$$

However

$$<< \bar{\xi}d\xi - \xi d\bar{\xi}, \bar{\xi}d\xi - \xi d\bar{\xi} >>= \frac{(1+\xi\bar{\xi})^2}{2(\lambda^2 - \sigma\bar{\sigma})}(i\bar{\sigma}\bar{\xi}^2 + 2\lambda\xi\bar{\xi} - i\sigma\xi^2),$$

and so

$$\begin{aligned}\left(\frac{d}{ds} - \triangle_0\right) F\bar{F} &= -2 << dF - F\frac{\bar{\xi}d\xi - \xi d\bar{\xi}}{1+\xi\bar{\xi}}, d\bar{F} + \bar{F}\frac{\bar{\xi}d\xi - \xi d\bar{\xi}}{1+\xi\bar{\xi}} >> \\ &\quad + \frac{2C_1(1+\xi\bar{\xi})(i\bar{\sigma}\bar{\xi}^2 + \lambda\xi\bar{\xi} - i\sigma\xi^2)}{\lambda^2 - \sigma\bar{\sigma}} - \frac{F^2 i\sigma - \bar{F}^2 i\bar{\sigma}}{\lambda^2 - \sigma\bar{\sigma}} \\ &\quad - \frac{2\lambda\xi\bar{\xi}F\bar{F}}{\lambda^2 - \sigma\bar{\sigma}}.\end{aligned}$$

From the negativity of the first term on the right hand side of the last equation we have that

$$\left(\frac{d}{ds}-\triangle_0\right)(F\bar{F}+C_1\xi\bar{\xi})\leq\frac{2C_1(1+\xi\bar{\xi})(i\bar{\sigma}\bar{\xi}^2+\lambda\xi\bar{\xi}-i\sigma\xi^2)}{\lambda^2-\sigma\bar{\sigma}}-\frac{F^2i\sigma-\bar{F}^2i\bar{\sigma}}{\lambda^2-\sigma\bar{\sigma}}$$
$$-\frac{2\lambda\xi\bar{\xi}F\bar{F}}{\lambda^2-\sigma\bar{\sigma}}+\frac{C_1\lambda(1+\xi\bar{\xi})^2}{\lambda^2-\sigma\bar{\sigma}},$$

and we now estimate each of these terms.

First, comparing with the flat metric

$$\begin{aligned}\frac{2C_1(1+\xi\bar{\xi})(i\bar{\sigma}\bar{\xi}^2+\lambda\xi\bar{\xi}-i\sigma\xi^2)}{\lambda^2-\sigma\bar{\sigma}}&=\frac{4C_1}{1+\xi\bar{\xi}}\|\bar{\xi}d\xi+\xi d\bar{\xi}\|^2+\frac{6\lambda C_1(1+\xi\bar{\xi})\xi\bar{\xi}}{\lambda^2-\sigma\bar{\sigma}}\\&\leq\frac{4C_1(1+\xi\bar{\xi})(-\lambda-|\sigma|)}{\lambda^2-\sigma\bar{\sigma}}|\bar{\xi}d\xi+\xi d\bar{\xi}|^2\\&=\frac{8C_1\xi\bar{\xi}(1+\xi\bar{\xi})(-\lambda-|\sigma|)}{\lambda^2-\sigma\bar{\sigma}}\\&\leq\frac{8C_1\xi\bar{\xi}(1+\xi\bar{\xi})(-\lambda+|\sigma|)}{\lambda^2-\sigma\bar{\sigma}},\end{aligned}$$

and

$$\begin{aligned}-\frac{F^2i\sigma-\bar{F}^2i\bar{\sigma}}{\lambda^2-\sigma\bar{\sigma}}-\frac{2\lambda\xi\bar{\xi}F\bar{F}}{\lambda^2-\sigma\bar{\sigma}}&=\frac{2}{(1+\xi\bar{\xi})^2}\|\bar{F}d\xi+F\,d\bar{\xi}\|^2+\frac{2\lambda(1-\xi\bar{\xi})F\bar{F}}{\lambda^2-\sigma\bar{\sigma}}\\&\leq\frac{4F\bar{F}(-\lambda+|\sigma|)(1+\xi\bar{\xi})^2}{\lambda^2-\sigma\bar{\sigma}}+\frac{2\lambda(1-\xi\bar{\xi})F\bar{F}}{\lambda^2-\sigma\bar{\sigma}}\\&\leq\frac{8F\bar{F}(-\lambda+|\sigma|)(1+\xi\bar{\xi})^2}{\lambda^2-\sigma\bar{\sigma}}.\end{aligned}$$

Finally, we conclude that

$$\left(\frac{d}{ds}-\triangle_0\right)(F\bar{F}+C_1\xi\bar{\xi})\leq\frac{8(F\bar{F}+C_1\xi\bar{\xi})(-\lambda+|\sigma|)(1+\xi\bar{\xi})^2}{\lambda^2-\sigma\bar{\sigma}}+\frac{C_1\lambda(1+\xi\bar{\xi})^2}{\lambda^2-\sigma\bar{\sigma}},$$

which is negative for

$$C_1>\frac{16|F_{\max}|^2}{1-16R^2_{\max}},$$

where $|F_{\max}|^2$ is the maximum modulus of F and $R_{\max}$ the maximum radius of the initial disc (assumed to be less than 1/16).

Thus, at a maximum value the function $\mathcal{D}$ is decreasing and the bound follows. □

4.5. **The multi-time function.** We now construct the multi-time function on the compact set in which the flow stays (*cf.* section 3.1).

Proposition 26. *Define the open ball*

$$B=\{(\xi,\eta)\in\mathbb{C}^2\mid C_1\xi\bar{\xi}+\eta\bar{\eta}<C_2\quad\textit{and}\quad\xi\bar{\xi}<1\ \},$$

for positive constants C_1 and C_2. Then the function $t:B\to\mathbb{R}^2=\mathbb{C}$ defined by

$$t=\eta+iC_3\xi,$$

for any positive constant C_3 satisfying $C_3^2>C_2$ is a multi-time function on B.

Proof. Let $\xi = Re^{i\vartheta}$, $t_1 = \mathbb{R}\mathrm{e}(t)$ and $t_2 = \mathbb{I}\mathrm{m}(t)$. For any $\alpha = A_1 + iA_2 \in \mathbb{C}$, consider the surface $t^{-1}(\alpha)$, or equivalently, $\eta = \alpha - iC_3\xi$. On this surface

$$\mathbb{G}(\overline{\nabla}t_1, \overline{\nabla}t_1) = \mathbb{G}(\overline{\nabla}t_2, \overline{\nabla}t_2) = \tfrac{1}{2}(1+R^2)(C_3R^2 - 2R(A_2\cos\vartheta - A_1\sin\vartheta) - C_3). \tag{4.7}$$

On $t^{-1}(\alpha) \cap B$ we have

$$C_1R^2 + (\alpha - iC_3\xi)(\bar{\alpha} + iC_3\bar{\xi}) = (C_1 + C_3^2)R^2 - 2RC_3(A_2\cos\vartheta - A_1\sin\vartheta) + A_1^2 + A_2^2 < C_2.$$

Thus

$$\begin{aligned} 0 >& \frac{C_1 + C_3^2}{C_3}R^2 - 2R(A_2\cos\vartheta - A_1\sin\vartheta) + \frac{A_1^2 + A_2^2 - C_2}{C_3} \\ &> C_3R^2 - 2R(A_2\cos\vartheta - A_1\sin\vartheta) - \frac{C_2}{C_3}, \end{aligned}$$

and so if $C_3^2 > C_2$ we conclude that on $t^{-1}(\alpha) \cap B$

$$C_3R^2 - 2R(A_2\cos\vartheta - A_1\sin\vartheta) - C_3 < 0.$$

From equation (4.7) we have then that for t restricted to B

$$\mathbb{G}(\overline{\nabla}t_1, \overline{\nabla}t_1) = \mathbb{G}(\overline{\nabla}t_2, \overline{\nabla}t_2) < 0.$$

Thus t is a multi-time function on B.

□

4.6. **Asymptotic holomorphicity.** Consider again the mean curvature flow on TS^2.

Proposition 27. *For the flow there exists a positive constant C_1 such that*

$$\frac{|\sigma|^2}{\lambda^2} < C_1 = \sup_{\Sigma_0} \frac{|\sigma|^2}{\lambda^2}.$$

Proof. We compute the flow of this quantity to be

$$\begin{aligned} \left(\frac{d}{ds} - \triangle_0\right)\frac{|\sigma|^2}{\lambda^2} = & -2\frac{\lambda^2 - 3|\sigma|^2}{\lambda^2 - |\sigma|^2}\left\|d\left(\tfrac{|\sigma|}{\lambda}\right)\right\|^2 + \frac{4|\sigma|}{\lambda(1+\xi\bar{\xi})} << d\left(\tfrac{|\sigma|}{\lambda}\right), d(1+\xi\bar{\xi}) >> \\ & - \frac{2|\sigma|^2}{\lambda^2}\left\|d\phi - 2\frac{jd(1+\xi\bar{\xi})}{1+\xi\bar{\xi}}\right\|^2 + \frac{2|\sigma|(1+\xi\bar{\xi})}{\lambda^2 - |\sigma|^2} < d\left(\tfrac{|\sigma|}{\lambda}\right), d(1+\xi\bar{\xi}) > \\ & + \frac{4|\sigma|^2}{\lambda^3}. \end{aligned}$$

At a maximum of this quantity $d(|\sigma|/\lambda) = 0$ and thus

$$\left(\frac{d}{ds} - \triangle_0\right)\frac{|\sigma|^2}{\lambda^2} = -\frac{2|\sigma|^2}{\lambda^2}\left\|d\phi - 2\frac{jd(1+\xi\bar{\xi})}{1+\xi\bar{\xi}}\right\|^2 + \frac{4|\sigma|^2}{\lambda^3} \leq 0.$$

Thus it is decreasing at a maximum and the result follows. □

Proposition 28. *The flow satisfies*

$$\left(\frac{d}{ds}-\triangle_0\right)\frac{|\sigma|^2}{\lambda^2-|\sigma|^2}=-2\frac{(\lambda^2+|\sigma|^2)}{(\lambda^2-|\sigma|^2)^3}\Big\|\lambda d|\sigma|-|\sigma|d\lambda\Big\|^2-2\frac{\lambda^2|\sigma|^2}{(\lambda^2-|\sigma|^2)^2}\Big\|d\phi\Big\|^2$$
$$+\frac{4\lambda|\sigma|}{(\lambda^2-|\sigma|^2)^3}\;\mathbb{R}e\,[H_8], \tag{4.8}$$

where

$$H_8=(1+\xi\bar{\xi})\left[i\lambda\bar{\sigma}\bar{\xi}\partial|\sigma|-i|\sigma|\bar{\sigma}\bar{\xi}\partial\lambda+2\lambda|\sigma|(i\lambda\xi-\bar{\sigma}\bar{\xi})\partial\phi\right]$$
$$-2i\lambda\sigma|\sigma|\xi^2-2|\sigma|^3+2(1+2\xi\bar{\xi})\lambda^2|\sigma|,$$

and ϕ is the argument of σ. Here the norm $\|.\|$ is taken with respect to the induced metric given in Proposition 17.

Proof. This follows from the fact that

$$\left(\frac{d}{ds}-\triangle_0\right)\frac{|\sigma|^2}{\lambda^2-|\sigma|^2}=\frac{\lambda}{(\lambda^2-|\sigma|^2)^2}\left[\lambda\sigma\left(\frac{d}{ds}-\triangle_0\right)\bar{\sigma}+\lambda\bar{\sigma}\left(\frac{d}{ds}-\triangle_0\right)\sigma-2|\sigma|^2\left(\frac{d}{ds}-\triangle_0\right)\lambda\right]$$
$$-\frac{2|\sigma|^2(3\lambda^2+|\sigma|^2)}{(\lambda-|\sigma|^2)^3}\|d\lambda\|^2-\frac{2\lambda^2(\lambda^2+|\sigma|^2)}{(\lambda-|\sigma|^2)^3}<<d\sigma,d\bar{\sigma}>>$$
$$+\frac{4\lambda(\lambda^2+|\sigma|^2)}{(\lambda-|\sigma|^2)^3}<<\sigma d\bar{\sigma}+\bar{\sigma}d\sigma,d\lambda>>-\frac{2\lambda^2\sigma^2}{(\lambda-|\sigma|^2)^2}\|d\bar{\sigma}\|^2$$
$$-\frac{2\lambda^2\bar{\sigma}^2}{(\lambda-|\sigma|^2)^2}\|d\sigma\|^2,$$

and the flow equations given in Proposition 24. □

We now prove:

Proposition 29. *By choosing a small enough initial disc, under the mean curvature flow we have the following estimate:*

$$\left(\frac{d}{ds}-\triangle_0\right)\frac{|\sigma|^2}{\lambda^2-|\sigma|^2}\leq\frac{4\lambda}{(\lambda^2+|\sigma|^2)}\frac{|\sigma|^4}{(\lambda^2-|\sigma|^2)^2},$$

Proof. Our starting point is equation (4.8), which we rewrite in the form

$$\left(\frac{d}{ds}-\triangle_0\right)\frac{|\sigma|^2}{\lambda^2-|\sigma|^2}=I_1+I_2+I_3+I_4,$$

where

$$I_1=-2\frac{(\lambda^2+|\sigma|^2)}{(\lambda^2-|\sigma|^2)^3}\Big\|\lambda d|\sigma|-|\sigma|d\lambda-\frac{\lambda^2-|\sigma|^2}{\lambda^2+|\sigma|^2}\frac{\lambda|\sigma|}{1+\xi\bar{\xi}}d(1+\xi\bar{\xi})\Big\|^2,$$
$$I_2=2\frac{(1+\xi\bar{\xi})\lambda^2|\sigma|}{(\lambda^2-|\sigma|^2)^3}<d(1+\xi\bar{\xi}),\lambda d|\sigma|-|\sigma|d\lambda>,$$
$$I_3=-2\frac{\lambda^2|\sigma|^2}{(\lambda^2-|\sigma|^2)^3}\Big\|d\phi-2(1+\xi\bar{\xi})^{-1}j[d(1+\xi\bar{\xi})]\Big\|^2,$$
$$I_4=\frac{|\sigma|^2}{2(\lambda^2-|\sigma|^2)^2(\lambda^2+|\sigma|^2)}\Big\{-i|\sigma|\lambda^2(\xi^2e^{i\phi}-\bar{\xi}^2e^{-i\phi})+4\lambda\{[2-\xi\bar{\xi}]\lambda^2+2|\sigma|^2\}\Big\}.$$

Here $\sigma=|\sigma|e^{i\phi}$ and we have introduced the flat complex structure $j(d\xi)=id\xi$.

First we estimate I_1 using the flat metric and Lemma 2:

$$I_1 \leq -2\frac{(1+\xi\bar{\xi})^2(\lambda^2+|\sigma|^2)}{(\lambda^2-|\sigma|^2)^3(-\lambda+|\sigma|)}\left|\lambda d|\sigma| - |\sigma| d\lambda - \frac{\lambda^2-|\sigma|^2}{\lambda^2+|\sigma|^2}\frac{\lambda|\sigma|}{1+\xi\bar{\xi}}d(1+\xi\bar{\xi})\right|^2,$$

and so after completing the squares

$$I_1 + I_2 \leq -2\tfrac{(1+\xi\bar{\xi})^2(\lambda^2+|\sigma|^2)}{(\lambda^2-|\sigma|^2)^3(-\lambda+|\sigma|)}\left\{\left|\lambda d|\sigma| - |\sigma| d\lambda + \tfrac{(-\lambda+|\sigma|)(-\lambda+2|\sigma|)\lambda|\sigma|}{2(\lambda^2+|\sigma|^2)(1+\xi\bar{\xi})}d(1+\xi\bar{\xi})\right|^2 \right.$$
$$\left. - \tfrac{\lambda(-\lambda+|\sigma|)^2(-3\lambda-4|\sigma|)}{4(\lambda^2+|\sigma|^2)^2}\tfrac{\lambda^2|\sigma|^2}{(1+\xi\bar{\xi})^2}\left|d(1+\xi\bar{\xi})\right|^2\right\}.$$

Clearly I_3 is negative, so we discard it. To estimate I_4 we use

$$-2\xi\bar{\xi} \leq i(\xi^2 e^{i\phi} - \bar{\xi}^2 e^{-i\phi}) \leq 2\xi\bar{\xi}.$$

Thus

$$I_4 \leq \tfrac{|\sigma|^2}{2(\lambda^2-|\sigma|^2)^2(\lambda^2+|\sigma|^2)}\left\{2|\sigma|\lambda^2\xi\bar{\xi} + 2\lambda\{[4-2\xi\bar{\xi}]\lambda^2 + 4|\sigma|^2\}\right\}.$$

Combining the estimates

$$\left(\frac{d}{ds} - \triangle_0\right)\frac{|\sigma|^2}{\lambda^2-|\sigma|^2} \leq \frac{4\lambda|\sigma|^2}{(\lambda^2-|\sigma|^2)^2} + \frac{\lambda^2|\sigma|^2(\lambda^3+2\lambda^2|\sigma|-2\lambda|\sigma|^2-|\sigma|^3)\xi\bar{\xi}}{(\lambda^2-|\sigma|^2)^3(\lambda^2+|\sigma|^2)}.$$

By choosing a small enough disc we can achieve $-\lambda \geq 3|\sigma|$ throughout the flow (*cf.* Proposition 27), so that

$$\left(\frac{d}{ds} - \triangle_0\right)\frac{|\sigma|^2}{\lambda^2-|\sigma|^2} \leq \frac{4\lambda|\sigma|^2}{(\lambda^2-|\sigma|^2)^2} + \frac{\lambda^5|\sigma|^2\xi\bar{\xi}}{9(\lambda^2-|\sigma|^2)^3(\lambda^2+|\sigma|^2)}$$
$$\leq \frac{4\lambda}{\lambda^2+|\sigma|^2}\frac{|\sigma|^4}{(\lambda^2-|\sigma|^2)^2}.$$

This completes the proof. □

Proposition 30. *The timelike curvature condition holds during the flow.*

Proof. Given a P, chose an adapted orthonormal frame $(e_{(i)} = \tau_i, e_{(3)}, e_{(4)})$ and for a timelike vector $X = X^{(3)}e_{(3)} + X^{(4)}e_{(4)}$ compute

$$<\overline{R}(X,\tau_i)X,\tau_i> = \frac{2|\sigma|}{\lambda^2-|\sigma|^2}(-(X^{(3)})^2 + (X^{(3)})^2) \geq \frac{2|\sigma|}{\lambda^2-|\sigma|^2}|X|^2.$$

From the previous Proposition we have that

$$\left(\frac{d}{ds} - \triangle_0\right)\frac{|\sigma|^2}{\lambda^2-|\sigma|^2} \leq 0,$$

and so we have the a priori bound

$$\frac{|\sigma|^2}{\lambda^2-|\sigma|^2} \leq C_1.$$

In addition, since the area form increases point wise under the flow

$$\lambda^2 - |\sigma|^2 \geq C_2,$$

and therefore

$$\left(\frac{|\sigma|}{\lambda^2-|\sigma|^2}\right)^2 = \frac{|\sigma|^2}{\lambda^2-|\sigma|^2}\frac{1}{\lambda^2-|\sigma|^2} \leq \frac{C_1}{C_2}.$$

Thus

$$\mathbb{G}(\overline{R}(X,\tau_i)X,\tau_i)\ \geq k|X|^2,$$

as claimed. ∎

Proposition 31. *The following estimates hold*

$$|\lambda|<C_{10}\qquad |\sigma|<C_{11}\qquad |\theta|<C_{12},$$

for positive constants C_{10},C_{11},C_{12}*, depending on* $n,m,\|t\|_3,|\psi|,\|\overline{R}\|,f_0$.

Proof. This follows from the main gradient estimate for v, established in Proposition 12, which we can now apply since we know that the timelike curvature condition holds. For completeness, we prove this by obtaining the explicit expression for v with respect to the background frame $\{e_{(1)},e_{(2)},T_{(1)},T_{(2)}\}$ associated to the multi-time function t defined in Proposition 26.

We find that (*cf.* Section 3.2):

$$\psi^{-2}\equiv\psi_1^{-2}=\psi_2^{-2}=\tfrac{1}{2}(1+R^2)(C_3+2R(A_2\cos\theta-A_1\sin\theta)-C_3R^2),$$

and the background orthonormal frame is

$$e_{(1)}=\tfrac{1}{2}(1+R^2)^2\psi\ \mathbb{R}\mathrm{e}\left[i\frac{\partial}{\partial\xi}+C_3\frac{\partial}{\partial\eta}\right],$$

$$e_{(2)}=\tfrac{1}{2}(1+R^2)^2\psi\ \mathbb{R}\mathrm{e}\left[\frac{\partial}{\partial\xi}-iC_3\frac{\partial}{\partial\eta}\right],$$

$$T_{(1)}=\tfrac{1}{2}\psi\ \mathbb{R}\mathrm{e}\left[-i(1+R^2)^2\frac{\partial}{\partial\xi}+(4\psi^{-2}-C_3(1+R^2)^2)\frac{\partial}{\partial\eta}\right],$$

$$T_{(2)}=\tfrac{1}{2}\psi\ \mathbb{R}\mathrm{e}\left[(1+R^2)^2\frac{\partial}{\partial\xi}+i(4\psi^{-2}-C_3(1+R^2)^2)\frac{\partial}{\partial\eta}\right].$$

Now, consider a frame $\{\tau_1,\tau_2,\nu_1,\nu_2\}$ adapted to the flowing surface (*cf.* Definition 6). As the surface is a graph, we use the expressions for the frame supplied by Proposition 19 with $\tau_1=e_{(1)}$, $\tau_2=e_{(2)}$, $\nu_1=e_{(3)}$, $\nu_2=e_{(4)}$. The transformation matrix between the timelike vectors of the background and adapted frame is (*cf.* section 3.3):

$$V_{\alpha\beta}=-\mathbb{G}(\nu_\alpha,T_\beta),$$

and

$$v^2=V_{33}^2+V_{34}^2+V_{43}^2+V_{44}^2.$$

Thus an a priori bound on v implies an a priori bound on each of the components of the matrix $V_{\alpha\beta}$.

Suppose for the sake of contradiction that at some point and time λ goes to infinity. By a translation we can assume that this occurs at $(\xi,\eta)=(0,0)$. At the origin we have the following expressions for the transition matrix:

$$V_{33}+iV_{34}=\frac{(1+i)(|\sigma|+\lambda-C_3)-(1-i)\theta}{[2C_3(-|\sigma|-\lambda)]^{\frac{1}{2}}}e^{-i\phi/2},$$

$$V_{44}+iV_{43}=\frac{(1-i)(-|\sigma|+\lambda-C_3)-(1+i)\theta}{[2C_3(|\sigma|-\lambda)]^{\frac{1}{2}}}e^{i\phi/2}.$$

Thus, if λ goes to infinity, then so does v, which contradicts Proposition 12. Similarly for the other slopes $|\sigma|$ and θ. ∎

Theorem 4. *By choosing a small enough initial disc, under the mean curvature flow the disc becomes asymptotically holomorphic:*

$$\lim_{s\to\infty} |\sigma| = 0.$$

Proof. By the previous propositions, our general results on mean curvature flow can be applied, in particular we can apply Proposition 31.

Then by Propositions 29 and 31 we have

$$\left(\frac{d}{ds} - \triangle_0\right)\frac{|\sigma|^2}{\lambda^2 - |\sigma|^2} \leq \frac{4\lambda}{(\lambda^2 + |\sigma|^2)}\frac{|\sigma|^4}{(\lambda^2 - |\sigma|^2)^2} \leq -C\frac{|\sigma|^4}{(\lambda^2 - |\sigma|^2)^2},$$

and we can again apply Lemma 4.5 of [4] to get the result. □

4.7. **Boundary and initial conditions.** In our case we would like the boundary disc to be the Lagrangian disc Σ with the isolated complex point, but, as the metric will be Lorentz or degenerate on such a disc (see Proposition 18), it will not be spacelike and cannot be used as a boundary condition. Instead we perturb the disc around the complex point to make it positive, and attach the flow disc to this surface.

More specifically, suppose Σ is given by $\eta = F(\xi, \bar{\xi})$ and the point γ lies at the origin $\xi = 0$. Define the perturbed surface $\tilde{\Sigma}$ by "adding a linear holomorphic twist":

$$\eta = \tilde{F} = F - iC_0\xi, \tag{4.9}$$

where C_0 is a real positive constant.

Proposition 32. *The induced metric on the surface $\tilde{\Sigma}$ is positive definite in a neighbourhood of the origin. The origin is an isolated complex point on $\tilde{\Sigma}$ with the same index as that of the complex point on Σ.*

Proof. The fact that $\tilde{\Sigma}$ has an isolated complex point of the same index at the origin follows from the fact that the perturbation is holomorphic, and hence $\tilde{\sigma} = \sigma$.

In addition, computing the twist

$$\tilde{\lambda} = \mathbb{I}\mathrm{m}\ (1 + \xi\bar{\xi})^2\partial\left(\frac{\tilde{F}}{(1+\xi\bar{\xi})^2}\right) = \mathbb{I}\mathrm{m}\ (1 + \xi\bar{\xi})^2\partial\left(\frac{F - iC_0\xi}{(1+\xi\bar{\xi})^2}\right) = -C_0\frac{1-\xi\bar{\xi}}{1+\xi\bar{\xi}},$$

where we have used the fact that Σ is Lagrangian. Thus, at the origin $\tilde{\lambda}^2(0) - |\tilde{\sigma}(0)|^2 = C_0^2 > 0$ and so in a small enough neighbourhood, the metric is positive definite. □

Note 5. In order for the induced metric to be positive (rather than negative) definite we have arranged that $\tilde{\lambda} < 0$.

We now prove that:

Proposition 33. *The boundary condition $-\epsilon\lambda + |\sigma| = |\tilde{\sigma}|$ in* **P.B.V.P.** *is an independent Neumann condition.*

Proof. By Proposition 15, this follows if the Neumann condition is linearly independent of the derived Dirichlet condition. To see that this is the case, consider the condition that two graphs $f, \tilde{f} : D \to TS^2$ intersect on a curve. If the graph functions are F and $\tilde{F}$, then the tangent spaces are spanned by

$$T_\gamma\Sigma = \mathrm{Span}_{\alpha\in\mathbb{C}}\ \mathbb{R}\mathrm{e}\ \alpha\left(\frac{\partial}{\partial\xi} + \partial F\frac{\partial}{\partial\eta} + \partial\bar{F}\frac{\partial}{\partial\bar{\eta}}\right),$$

$$T_\gamma\tilde{\Sigma} = \mathrm{Span}_{\tilde{\alpha}\in\mathbb{C}}\ \mathbb{R}\mathrm{e}\ \tilde{\alpha}\left(\frac{\partial}{\partial\xi} + \partial\tilde{F}\frac{\partial}{\partial\eta} + \partial\bar{\tilde{F}}\frac{\partial}{\partial\bar{\eta}}\right).$$

If these tangent spaces intersect in a one dimensional set, then there exists α and $\tilde{\alpha}$ such that

$$\mathbb{R}\mathrm{e}\ \alpha\left(\frac{\partial}{\partial\xi} + \partial F\frac{\partial}{\partial\eta} + \partial\bar{F}\frac{\partial}{\partial\bar{\eta}}\right) = \mathbb{R}\mathrm{e}\ \tilde{\alpha}\left(\frac{\partial}{\partial\xi} + \partial\tilde{F}\frac{\partial}{\partial\eta} + \partial\bar{\tilde{F}}\frac{\partial}{\partial\bar{\eta}}\right).$$

Comparing components we find that $\alpha = \tilde{\alpha}$ and

$$\alpha\partial F + \bar{\alpha}\bar{\partial}F = \alpha\partial\tilde{F} + \bar{\alpha}\bar{\partial}\tilde{F},$$

or

$$\alpha\partial(F - \tilde{F}) + \bar{\alpha}\bar{\partial}(F - \tilde{F}) = 0.$$

Thus, a necessary and sufficient condition for the tangent planes to intersect on a line is that $|\partial(F - \tilde{F})| = |\bar{\partial}(F - \tilde{F})|$, or, in our notation, $|\rho - \tilde{\rho}| = |\sigma - \tilde{\sigma}|$. This is clearly linearly independent of the Neumann condition $-\epsilon\lambda + |\sigma| = |\tilde{\sigma}|$.

□

Proposition 34. *The boundary of the flowing disc never crosses the complex point on the boundary surface* $\tilde{\Sigma}$.

Proof. This follows from the boundary condition $-\epsilon\lambda + |\sigma| = |\tilde{\sigma}|$, since if at some $s = s_1$ the boundary of the flowing disc reaches $\xi = 0$, then $-\epsilon\lambda(s_1, 0) + |\sigma(s_1, 0)| = |\tilde{\sigma}(0)| = 0$. Thus $\lambda(s_1, 0) = 0$ and the disc is no longer positive definite which contradicts long-time existence for the flow established earlier. □

We turn now to initial conditions.

Proposition 35. *For ϵ small enough, there exists an initial positive disc such that $f_0(\partial D) \subset \tilde{\Sigma}$ is a simple closed curve containing the complex point on $\tilde{\Sigma}$ and which is compatible with the given boundary condition $-\epsilon\lambda + |\sigma| = |\tilde{\sigma}|$. Moreover, the initial disc can be chosen arbitrarily C^1-close to the boundary disc $\tilde{\Sigma}$, and $\pi(f_0(\partial D))$ is contained in the a disc of radius less than* $1/4$.

Proof. Given a surface $\tilde{\Sigma}$ with boundary $\partial\tilde{\Sigma}$, consider the set $U_{\epsilon_1}(\tilde{\Sigma})$ of C^1 perturbations Σ of $\tilde{\Sigma}$ that fix $\partial\tilde{\Sigma}$ and

$$\|\Sigma - \tilde{\Sigma}\|_{C_1} < \epsilon_1.$$

For $p \in \partial\tilde{\Sigma}$, define the set of graphical planes on the boundary

$$\mathcal{P}_p = \{\ P \ \text{ a linear plane in } \ T_p\tilde{\Sigma} \mid \pi|_P \ \text{ has rank 2 }\},$$

and

$$\mathcal{P} = \bigcup_{p\in\partial\tilde{\Sigma}} \mathcal{P}_p.$$

This has a bundle structure $\mathcal{P} \to \partial\tilde{\Sigma}$ and $T_{\partial\tilde{\Sigma}}\tilde{\Sigma}$ is a section of this bundle which we denote by $\tilde{s}$.

Now, there exists $\delta > 0$ such that if s is a section of $\mathcal{P}$ with

$$\|s - \tilde{s}\|_{C_1} < \delta,$$

then there exists $\Sigma \in U_\delta(\tilde{\Sigma})$ with $s = T_{\partial\Sigma}\Sigma$.

Our boundary condition $-\epsilon\lambda + |\sigma| = |\tilde{\sigma}|$ is satisfied at $\epsilon = 0$ if we take the boundary and initial disc to be the same. Thus, by the preceding result, there exists an $\epsilon > 0$ such that we can perturb $\tilde{\Sigma}$ to a nearby disc satisfying the boundary condition. Moreover, since positivity is an open condition, we can choose ϵ small enough so that the perturbed disc is also positive.

□

5. Convergence of the Evolution to a $\mathbb{J}$-Holomorphic Disc

5.1. Compactness of the manifold of J-holomorphic discs with boundary.

Let $(\mathbb{M}, \mathbb{G})$ be a 4-manifold with neutral metric and $\pi_2(\mathbb{M}) = 0$. Fix a $C^{k,\alpha}$ surface $\tilde{\Sigma} \to \mathbb{M}$ and let G be *any* fixed Riemannian metric on $\mathbb{M}$.

Define

$$\mathcal{J} = \{J \mid J \text{ is a } C^{k,\alpha} \text{ almost complex structure on } \mathbb{M} \text{ s.t. } \tilde{\Sigma} \text{ is totally real }\},$$

and

$$\mathcal{M}_{\tilde{\Sigma}} = \{(f, J) \mid (f, \tilde{\Sigma}) \in \mathcal{F},\ J \in \mathcal{J},\ \text{and } \bar{\partial}_J f = 0\},$$

where $\mathcal{F}$ is defined as in section 2.2.

Definition 11. A sequence of immersed discs $f_n(D) \subset \mathbb{M}$ is said to be *uniformly positive* if $\exists C_2 > 0$ such that

$$\mathbb{G}(X_n, X_n) > C_2\, G(X_n, X_n) \qquad \forall X_n \in Tf_n(D).$$

Note that this property is independent of the Riemannian metric G (with different C_2).

Definition 12. $C \subset D$ is called a *regular* set of curves in D if C consists of $c_k : [0, 1] \to D$, a finite number of disjoint embedded smooth curves with boundary in ∂D.

We now state a compactness theorem for the space of J-holomorphic discs with boundary in a totally real surface.

Theorem 5. *Let $(f_n, J_n) \in \mathcal{M}_{\tilde{\Sigma}}$ be a sequence such that*

(1) *$f_n(D)$ lies in a fixed compact set of $\mathbb{M}$ for all n,*
(2) *$J_n \to \mathbb{J}$ uniformly in $C^{k,\alpha}$,*
(3) *$f_n(D)$ is uniformly positive,*
(4) *there exists a constant C_1 such that, for all n,*

$$Area_{\mathbb{G}}(f_n(D)) < C_1.$$

Then there exists a subsequence n', again denoted by n, and

(i) *diffeomorphisms $\phi_n \in C^\infty(D, D)$,*
(ii) *complex structures j_n on D,*

(iii) *a regular set of curves* $C \subset D$,
(iv) *a complex structure* j_∞ *on* $D - C$,
(v) *a limit map* $f_\infty \in C^0((D, \partial D), (\mathbb{M}, \tilde{\Sigma})) \cap C^{k+1,\alpha}(D - C, \mathbb{M})$,

such that

(a) $\phi_n(j_n) \to j_\infty$ *uniformly in* C^∞ *on compact subsets of* $D - C$,
(b) $f_\infty|_{D-C} = j_\infty$*-*$\mathbb{J}$*−holomorphic,*
(c) *the restriction of* f_∞ *to any component of* C *is constant,*
(d) $f_n \circ \phi_n \to f_\infty$ *uniformly in* C^0 *on the closure of* D *and uniformly in* $C^{k+1,\alpha'}$ *for each* $\alpha' \in (0, \alpha)$ *on compact subsets of the closure of* $D - C$,
(e) *The homotopy classes of* $f_n(\partial D)$ *and* $f_\infty(\partial D)$ *coincide in* $\pi_1(\tilde{\Sigma})$.

This result follows from the main theorem of [5] once the following is noted. By the uniform positivity of $f_n(D)$ and condition (4) we have that

$$C_2 \text{ Area }_G(f_n(D)) < \text{Area }_{\mathbb{G}}(f_n(D)) < C_1,$$

and so we have an a priori area bound for the area of the discs $f_n(D)$ with respect to the Riemannian metric G. This is all that is required to apply the referenced theorem.

The map f_∞ is called a J-holomorphic bubble-disc, as the interior of $f_\infty(D)$ consists of $k + 1$ discs whose boundaries are joined by the k points $f_\infty(C)$. The condition that $\pi_2(\mathbb{M}) = 0$ ensures that no holomorphic spheres bubble off.

5.2. **Convergence of the flow to a $\mathbb{J}$-holomorphic bubble-disc in $\mathbf{TS}^2$.** The main ingredient in convergence of the evolution is an a priori area bound with respect to the indefinite metric:

Proposition 36. *Let* $f : D \to TS^2$ *be a positive disc with boundary lying in* $\tilde{\Sigma}$*, a linear perturbation of a Lagrangian surface, as given in section 4.7. Then the area of* $f(D)$ *has the a priori bound*

$$Area_{\mathbb{G}}(f(D)) \leq \pi C_0.$$

Proof. Recall the expression for the area form of the induced metric

$$\text{Area}_{\mathbb{G}}(f(D)) = \iint_D \sqrt{\lambda^2 - \sigma\bar{\sigma}}\ \frac{d\xi d\bar{\xi}}{(1 + \xi\bar{\xi})^2},$$

where

$$\lambda = \mathbb{I}\text{m } (1 + \xi\bar{\xi})^2 \partial \left(\frac{F}{(1 + \xi\bar{\xi})^2} \right) \qquad\qquad \sigma = -\partial\bar{F}.$$

Thus

$$\text{Area}_{\mathbb{G}}(f(D)) \leq \iint_D |\lambda|\ \frac{d\xi d\bar{\xi}}{(1 + \xi\bar{\xi})^2} = -\iint_{D_s} \Omega,$$

where Ω is the symplectic form on TS^2. Since this 2-form is closed and $f(\partial D) \subset \tilde{\Sigma}$ we have

$$\iint_D \Omega - \iint_{\Sigma_0'} \Omega = 0,$$

where Σ_0' is the disc in $\tilde{\Sigma}$ which is contained within the boundary of $f(D)$. In fact, the symplectic form is exact

$$\Omega = d\Theta \qquad\qquad \text{where} \qquad\qquad \Theta = \frac{2(\eta d\bar{\xi} + \bar{\eta} d\xi)}{(1 + \xi\bar{\xi})^2},$$

and so

$$\mathrm{Area}_{\mathbb{G}}(f(D)) \leq -\int_{\partial\Sigma_0'} \Theta.$$

But a straight-forward computation shows that, when $\tilde{\Sigma}$ is a linear holomorphic deformation of a Lagrangian disc, in terms of polar coordinates (R, ϑ), $\xi = Re^{i\vartheta}$:

$$\int_{\partial\Sigma_0'} \Theta = -\int_{\partial\Sigma_0'} \frac{2C_0R^2}{(1+R^2)^2} d\vartheta.$$

Thus the area can be estimated by the winding number of the boundary about the origin

$$\mathrm{Area}_{\mathbb{G}}(f(D)) \leq \int_{\partial\Sigma_0'} \frac{4C_0R^2}{(1+R^2)^2} d\vartheta \leq \left(\mathrm{Max}\frac{2C_0R^2}{(1+R^2)^2} - \mathrm{Min}\frac{2C_0R^2}{(1+R^2)^2}\right)\int_{\partial\Sigma_0'} d\vartheta \leq \pi C_0.$$

□

We now apply the above and Theorem 5 to the case of TS^2:

Theorem 6. *Let $\tilde{\Sigma}$ be a linear holomorphic perturbation of a Lagrangian surface Σ, as in equation (4.9). Let Σ_0 be an initial disc of radius R_0, as established in Proposition 35.*

Then for C_0 big enough and R_0 small enough, the mean curvature flow **P.B.V.P.** *converges to $f_\infty(D)$, a $\mathbb{J}$-holomorphic bubble-disc, with continuous boundary enclosing the isolated complex point on $\tilde{\Sigma}$. In particular, the Keller-Maslov index $\mu(f_\infty, \tilde{\Sigma})$ is well defined.*

Proof. In section 4 we established long-time existence of a solution f_t of the **P.B.V.P.** under the above assumptions. Moreover, we established that the flow is asymptotically holomorphic in Theorem 4.

Let f_n be a sequence of immersions extracted from the evolving immersion f_t. Then by Proposition 25, $f_n(D)$ lies in a fixed compact set of $\mathbb{M}$ for all n. For n large enough, we can chose a sequence of almost complex structures J_n, converging in $C^{k,\alpha}$ to $\mathbb{J}$, such that f_n is J_n-holomorphic. By our gradient estimates in Proposition 27 the sequence $f_n(D)$ is uniformly positive and in the previous Proposition we have established an a priori area bound for $f_n(D)$.

Since we are working in an open subset U of TS^2, and $\pi_2(U) = 0$. We have thus met all of the conditions of Theorem 5, which we can therefore apply to the sequence. The result then follows from properties (a)-(e) of this Theorem.

□

6. Concluding Remarks

6.1. **Background.** The affirmation of such a venerable conjecture is deserving of some general remarks on the methods employed. Previous efforts at proving the Carathéodory conjecture have also been focused on the Loewner conjecture, although these efforts have almost exclusively assumed that the surface is real analytic. This has involved the use of ODE analysis, dynamics of the principal curvature foliation and the combinatorics of its singularities. Rather than review the history of these attempts, which in any case are well-documented, we will discuss our proof of the C^3 case in the context of PDE methods.

Our proof brings together two currents in contemporary PDE: parabolic flows and the rigidity of J-holomorphic curves, both in a new geometric setting. That these techniques are just sufficient to prove the Carathéodory conjecture gives an indication of the depth at which it lies. The geometric setting (that of neutral Kähler surfaces) is sufficiently new that results from both of these fields require modification. In what follows we sketch the salient features of the proof, highlighting the key points and offering an enlarged perspective on the work.

The basic outline of the proof is as follows. The Loewner conjecture is essentially a hyperbolic problem. By transferring to the enlarged geometric setting of the space of oriented lines, the neutral Kähler metric allows us to deform it to an elliptic boundary value problem. The mean curvature flow then allows us not only to solve this problem, but to get to the most rigid of elliptic objects: holomorphic curves. The proof of the Conjecture then follows from a generalization of the argument principal for holomorphic functions.

6.2. **Mean curvature flow.** In the case of the mean curvature flow, we must establish a priori gradient estimates for long-time existence in the indefinite setting with higher codimension. In this instance the indefinite signature of the metric assists the analysis: as long as the flowing submanifold remains in a compact set, a mild curvature assumption ensures that singularity formation does not occur. Thus many of the difficulties associated with flowing in the definite case are avoided and a general result on mean curvature flow of spacelike surfaces in indefinite manifolds is established.

However, the higher codimension significantly complicates the gradient estimates required for long-time existence. In addition, rather than working with the more usual case of compact submanifolds, we must consider the flow with mixed Dirichlet and Neumann boundary conditions. These conditions must be chosen so that the boundary of the disc does not cross the complex point on the boundary surface, while still remaining parabolic.

6.3. **J-holomorphic curves.** For TS^2 with neutral metric a holomorphic curve is maximal [8], and perhaps the most surprising aspect of this flow is that, under suitable circumstances, we show that it converges to a holomorphic disc. To establish this we first prove that the flow is asymptotically holomorphic and then utilize compactness results for J-holomorphic discs with boundary lying on a totally real surface. Once again, existing results must be modified, as our symplectic structure is not tamed by the complex structure, and our area bound is with respect to a non-Hermitian metric.

In this regard, the a priori area bound enjoyed by positive discs with boundary lying in the linear holomorphic perturbation of a Lagrangian surface must be seen as a key element of our proof. Without this, not only would it be hard to prove convergence, but it is easy to construct examples where convergence fails. Indeed, this is the only point at which the Lagrangian condition is exploited.

The Keller-Maslov index estimates of the type used originate in symplectic geometry and topology. It is worth noting that the Loewner conjecture is a stronger local statement than would be required by the Carathéodory conjecture: an umbilic point of index $3/2$ would disprove the Loewner but not the Carathéodory conjecture. Here we have an example of the well-known phenomenon in symplectic rigidity of gaining a half-integer more than is strictly necessary globally.

6.4. **Neutral Kähler surfaces.** Let us now turn to the geometric setting. The metric employed in the proof was first noted, as far as the authors are aware, by Study [12]. As shown in [7], it admits a neutral Kähler structure and is canonical in the sense that it is the unique metric on the space of oriented lines (up to addition of a spherical element) that is invariant under the action of the Euclidean group [11]. Such an invariant metric exists on the space of oriented geodesics of any 3-dimensional space form and so our method may well extend to a proof the Carathéodory conjecture in the 3-sphere and hyperbolic 3-space. These would require ambient curvature conditions, an a priori area bound, confinement of the flow to a compact region and asymptotic holomorphicity - issues that we postpone to a future paper.

In fact, minor modifications of our argument establish: the index of an isolated umbilic point on a spacelike surface in Lorentz 3-space must be less than or equal to one [9]. Given our reformulation, it might also be conjectured that the index of an isolated complex point on a Lagrangian surface in any neutral Kähler surface must have index less than or equal to 1.

Perhaps some insight into the difficulty of the Conjecture in Euclidean 3-space is afforded by the following observation: our Kähler metric is not Kähler-Einstein and so mean curvature flow does not preserve the Lagrangian condition. Thus, viewed in $\mathbb{E}^3$, our flow twists the normal lines and we lose the orthogonal surface. It might be that any proof of the conjecture must follow from the argument principle in some setting, and that, in order to approach these holomorphic objects the orthogonal surface must be destroyed in the process. Hence, any proof that remains exclusively in the space of surfaces in $\mathbb{E}^3$ will run into serious difficulties.

It is worth noting that our proof does not require arbitrarily small initial discs for convergence. A careful study of the estimates reveals that the proof works on any disc of radius less than one quarter. This yields:

The winding number of the prinicpal foliation of a convex surface in $\mathbb{E}^3$ *along a closed curve contained in a disc whose Gauss image has radius* $1/4$ *must be less than or equal to one.*

Since these estimates are not necessarily sharp, it may be possible to improve on them and to establish convergence on discs of radius less than one. This would then imply strong global rigidity results for isolated umbilic points on closed convex surfaces in $\mathbb{E}^3$.

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