

# The center conjecture for thick spherical buildings

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## Abstract

We prove that a convex subcomplex of a spherical building of type  $E_7$  or  $E_8$  is a subbuilding or the group of building automorphisms preserving the subcomplex has a fixed point in it. Together with previous results of Mühlherr-Tits, and Leeb and the author, this completes the proof of Tits' Center Conjecture for thick spherical buildings.

## 1 Introduction

The Center Conjecture was first proposed by J. Tits in the 50's and in its actual version it reads as follows (compare [MT06] and [Se05, Conjecture 2.8]).

**Conjecture 1.1 (Center Conjecture).** *Suppose that  $B$  is a spherical building and that  $K \subseteq B$  is a convex subcomplex. Then  $K$  is a subbuilding or the action  $Stab_{Aut(B)}(K) \curvearrowright K$  of the automorphisms of  $B$  preserving  $K$  has a fixed point.*

A building automorphism is an isometry, which preserves the polyhedral structure of the building. In particular, it induces an isometry of the model Weyl chamber, which may be nontrivial. If it is trivial, the automorphism is type preserving. The isometries of the model Weyl chamber can be identified with the symmetries of the Dynkin diagram.

A fixed point of the action  $Stab_{Aut(B)}(K) \curvearrowright K$  is called a *center* of the subcomplex  $K$ .

Apparently, the first motivation of Tits for considering the Center Conjecture was to prove a result associating a parabolic subgroup  $P$  to a unipotent subgroup  $U$  of a reductive algebraic group  $G$  [Ti62, Lemma 1.2]. This result is a direct consequence of the Center Conjecture. The desired parabolic subgroup is obtained as the center of the fixed point set of the action  $U \curvearrowright B$ , where  $B$  is the building associated to the group  $G$ . This result was later obtained by Borel and Tits in [BT71] using other methods.

In Geometric Invariant Theory a special case of the Center Conjecture is used to find parabolic subgroups that are *most* responsible for the instability of a point (see [Mu65]). This special case was proven by Rousseau [Rou78] and Kempf [Ke78].

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From the point of view of metric geometry and CAT(1) spaces, a natural generalization of Conjecture 1.1 is to drop the assumption of  $K$  being a subcomplex and consider arbitrary closed convex subsets  $C \subset B$ . Such a subset  $C$  is a CAT(1) space itself. We can also forget the ambient building and look for fixed points for the whole group of isometries  $Isom(C)$ .

**Conjecture 1.2.** *If  $C$  is a closed convex subset of a spherical building  $B$ , then  $C$  is a subbuilding or the action  $Isom(C) \curvearrowright C$  has a fixed point.*

Conjecture 1.2 was answered positively in [BL05] for the case  $dim(C) \leq 2$ . The strategy of their proof is basically to consider a smallest  $Isom(C)$ -invariant closed convex subset  $Y \subset C$  and then prove that if  $Y$  is not a subbuilding, it has intrinsic radius  $\leq \frac{\pi}{2}$  (by intrinsic radius of  $Y$  we denote the infimum of the radii of balls centered at  $Y$  and containing  $Y$ ). If a CAT(1) space  $X$  has intrinsic radius  $\leq \frac{\pi}{2}$ , it was also shown in [BL05] that the set  $Z$  of circumcenters of  $X$  is not empty and has radius  $< \frac{\pi}{2}$ , in particular,  $Z \subset X$  has a unique circumcenter and it is fixed by  $Isom(X)$ . It follows that  $Isom(C)$  fixes a point in  $Y \subset C$ .

If  $C \subset B$  has intrinsic radius  $\pi$  then it must be a building (see [BL06]) and if it has intrinsic radius  $\leq \frac{\pi}{2}$ , it satisfies the fixed point property asserted in Conjecture 1.2. It is natural to ask if there are closed convex subsets between these two possibilities or if Conjecture 1.2 is just a consequence of a more general “gap phenomenon” (cf. [KL06, Question 1.5]).

**Conjecture 1.3.** *If  $C$  is a closed convex subset of a spherical building  $B$ , then  $C$  is a subbuilding or  $rad_C(C) \leq \frac{\pi}{2}$ .*

If  $dim(C) \leq 1$  then it is easy to see that Conjecture 1.3 holds, namely, a one-dimensional convex subset is a building or a tree of radius  $\leq \frac{\pi}{2}$ . Another easy case is when the building  $B$  is just a spherical Coxeter complex, i.e.  $B$  is a round sphere with curvature  $\equiv 1$ , then  $C$  is a sphere or has intrinsic radius  $\leq \frac{\pi}{2}$ .

Unfortunately, we do not know more positive results for the Conjectures 1.2 and 1.3, other than those mentioned above. Notice that we have the implications  $1.3 \Rightarrow 1.2 \Rightarrow 1.1$ .

If  $K$  is a convex subcomplex of a reducible building  $B = B_1 \circ \cdots \circ B_k$ , then  $K$  decomposes as a spherical join  $K = K_1 \circ \cdots \circ K_k$  where  $K_i \subset B_i$  is a convex subcomplex for  $i = 1, \dots, k$ . Thus, the Center Conjecture easily reduces to the case of irreducible buildings. For irreducible buildings of classical type (i.e.  $A_n$ ,  $B_n$  and  $D_n$ ) the Center Conjecture was shown in [MT06]. The proof uses the incidence-geometric realizations of the corresponding different types of buildings.

Our approach is of differential-geometric nature, using methods of the theory of metric spaces with curvature bounded above. The cases of buildings of type  $F_4$  and  $E_6$  are settled in [LR09].

The main result in this paper is:

**Theorem 1.4.** *The Center Conjecture 1.1 holds for spherical buildings of type  $E_7$  and  $E_8$ .*

The case of buildings of type  $H_3$  can be easily treated with our methods or just be considered as a consequence of the result in [BL05]. Hence we have the following result.

**Corollary 1.5.** *The Center Conjecture 1.1 holds for spherical buildings without factors of type  $H_4$ .*

While any spherical Coxeter complex is a spherical building, not all spherical Coxeter complexes occur as Coxeter complexes for thick spherical buildings ([Ti77]), namely, there are no thick spherical buildings of type  $H_3$  ( $\overset{1}{\bullet} \xrightarrow{s} \overset{2}{\bullet} \xrightarrow{3}$ ) and  $H_4$  ( $\overset{1}{\bullet} \xrightarrow{s} \overset{2}{\bullet} \xrightarrow{3} \overset{4}{\bullet}$ ). Any spherical building has a canonical (depending only on its isometry type) thick structure resulting by restricting to a subgroup of the Weyl group ([Sch87], [KL98, Sec. 3.7]). The polyhedral structure so obtained is coarser. The Center Conjecture is more natural when posed for thick spherical buildings, because in this way  $K$  is a subcomplex of the natural polyhedral structure of  $B$ . In this case we have:

**Corollary 1.6.** *The Center Conjecture 1.1 holds for all thick spherical buildings.*

A completely different approach for spherical buildings  $B$  associated to algebraic groups  $G$  and the special case of subcomplexes  $K$  which are fixed point sets of the action of a subgroup  $H \subset G$  can be found in [BMR09]. They show that such a subcomplex is a subbuilding or the action  $Stab_G(K) \curvearrowright K$  fixes a point. In [BMRT09] this result is extended to the action  $Stab_{Aut(G)}(K) \curvearrowright K$ .

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## 2 Preliminaries

### 2.1 CAT(1) spaces

Recall that a CAT(1) space is a complete  $\pi$ -geodesic metric space where geodesic triangles are not *thicker* than those in the round sphere with curvature  $\equiv 1$ .

For points  $x, y$  in a CAT(1) space  $X$  at distance  $< \pi$ , we denote by  $xy$  the unique segment connecting both points. Let  $m(x, y)$  denote the midpoint of the segment  $xy$ .

A subset  $C$  of a CAT(1) space is called convex, if for any  $x, y \in C$  at distance  $< \pi$  the segment  $xy$  is contained in  $C$ . A closed convex subset of a CAT(1) space is itself CAT(1).

Let  $A$  be a subset of a CAT(1) space  $X$  and let  $x \in X$ . The radius of  $A$  with respect to  $x$  is defined as  $rad(x, A) := \sup\{d(x, y) | y \in A\}$  and the circumradius of  $A$  in  $X$  is  $rad_X(A) = \inf\{rad(x, A) | x \in X\}$ . A point  $x \in X$ , such that  $rad(x, A) = rad_X(A)$  is called a *circumcenter*.

For more information on CAT(1) spaces we refer to [BH99].

### 2.2 Coxeter complexes

A spherical Coxeter complex  $(S, W)$  is a pair consisting in a round sphere  $S$  with curvature  $\equiv 1$  together with a finite group of isometries  $W$ , called the *Weyl group*, generated by reflections on great spheres of codimension 1.

There is a structure of spherical polyhedral complex on  $S$  induced by  $W$ . The spheres of codimension 1, that are the fixed point sets of the reflections in  $W$  are called the *walls*. The *Weyl chambers* are the closures of the connected components of  $S$  minus the union of the walls.

A *root* is a top-dimensional hemisphere bounded by a wall. A *singular* sphere is an intersection of walls.

The geometry of a spherical Coxeter complex can be encoded in a graph, the so-called Dynkin diagram. A labelling by an index set  $I$  of the vertices of the Dynkin diagram induces a labelling of the vertices of  $S$ . We say that a vertex in  $S$  is an  $i$ -vertex for  $i \in I$ , if it has label  $i$ .

We refer to [GB71] and [KL98, Sec. 3.1, 3.3] for further information on spherical Coxeter complexes.

## 2.3 Spherical buildings

We refer to [AB08], [KL98] and [Ti74] for more information on spherical buildings. We will consider spherical buildings from the point of view of CAT(1) spaces as presented in [KL98].

A spherical building  $B$  modelled in a spherical Coxeter complex  $(S, W)$  is a CAT(1) space together with an atlas  $\mathcal{A}$  of isometric embeddings  $S \hookrightarrow B$  (the images of these embeddings are called *apartments*) with the following properties: any two points in  $B$  are contained in a common apartment, the atlas  $\mathcal{A}$  is closed under precomposition with isometries in  $W$  and the coordinate changes are restrictions of isometries in  $W$ . The empty set is considered to be a building.

The polyhedral structure of  $(S, W)$  induces a polyhedral structure on the building  $B$ . The objects (walls, roots,...) defined for spherical Coxeter complexes can be defined for the building  $B$  as the corresponding images in  $B$ .

A *subbuilding* is a convex subcomplex  $K$  of a building, such that any two points in  $K$  are contained in a singular sphere  $s \subset K$  of the same dimension as  $K$ . A subbuilding carries a natural structure as a spherical building (cf. [LR09, Proposition 2.3]).

## 3 Spherical Coxeter complexes

This section contains some geometric properties of spherical Coxeter complexes.

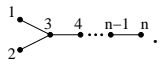
In our arguments later, we will need some information on singular spheres of codimension  $\leq 2$  in the different Coxeter complexes.

If the Coxeter complex  $(S, W)$  is irreducible and its Dynkin diagram has no weights on its edges, i.e. if it is of type  $A_n, D_n, E_6, E_7$  or  $E_8$ , then it is easy to see, that the Weyl group acts transitively on the set of roots ([GB71, Proposition 5.4.2]). In particular all walls (singular spheres of codimension 1) are equivalent modulo the action of  $W$ . If there is more than one orbit of roots, then we define the *type* of a wall as the type of the center of the corresponding root. Note that this definition is independent of which of both roots we take.

A singular sphere of codimension 2 is the intersection of two different walls. We define the *type* of a sphere of codimension 2 as the type of the circle spanned by the centers of the corresponding roots.

We gather in the next sections some of the geometric properties of the different Coxeter complexes. This information can be deduced from the data in the Appendix A.

### 3.1 The Coxeter complex of type $D_n$

For  $n \geq 4$  let  $(S, W_{D_n})$  be the spherical Coxeter complex of type  $D_n$  with Dynkin diagram . It has dimension  $n - 1$ .

The  $(n - 1)$ -vertices are the vertices of *root type*. All hemispheres bounded by walls are centered at a  $(n - 1)$ -vertex.

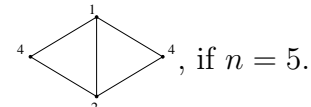
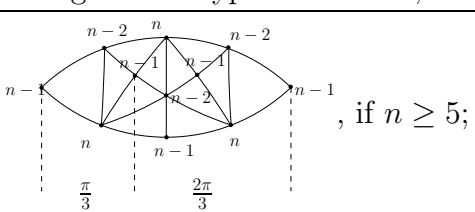
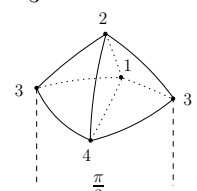
For  $n \geq 5$  the Dynkin diagram has one symmetry: it exchanges the vertices  $1 \leftrightarrow 2$  and fixes the others. This symmetry is induced by the canonical involution of the Weyl chamber  $\Delta_{mod}^{D_n}$  if  $n$  is odd. If  $n$  is even, then the canonical involution is trivial. For  $n = 4$  the Dynkin diagram has six symmetries, they permute the vertices 1, 2, 4 and fix the vertex 3.

We describe now the possible lengths and types (modulo the action of the Weyl group) of segments between vertices. We list only the ones that we will need later.

- Distances between two  $n$ -vertices  $x$  and  $x'$ :

Distance	Simplicial convex hull of segments $xx'$
$0, \pi$	
$\frac{\pi}{2}$	singular segment of type $n(n - 1)n$

- Distances between two  $(n - 1)$ -vertices  $x$  and  $x'$ :

Distance	Simplicial convex hull of segments $xx'$
$0, \pi$	
$\frac{\pi}{2}$	<p>singular segment of type <math>(n - 1)n(n - 1)</math> for <math>n \geq 4</math>/  singular segment of type <math>(n - 1)(n - 3)(n - 1)</math>, if <math>n \geq 6</math>;</p>  <p>, if <math>n = 5</math>.</p> <p>singular segment of type 313 or 323, if <math>n = 4</math>.</p>
$\frac{\pi}{3}$ ( $\frac{2\pi}{3}$ )	 <p>, if <math>n \geq 5</math>;</p> <p>if <math>n=4</math>, the simplicial convex hull of a segment <math>xx'</math> of length <math>\frac{\pi}{3}</math> is 3-dimensional:</p>  <p>A segment <math>xx'</math> of length <math>\frac{2\pi}{3}</math> consists of two segments of length <math>\frac{\pi}{3}</math> as above.</p>

- Distances between two 1- (2)-vertices  $x$  and  $x'$ :

Distance	Simplicial convex hull of segments $xx'$
$\arccos(\frac{n-4k}{n})$ for $k = 0, 1, \dots, \lfloor \frac{n}{2} \rfloor$	singular segment of type $1(2k+1)1, (2(2k+1)2$ resp.)

- Distances between a 1- (2)-vertex  $x$  and a  $n$ -vertex  $y$ :

Distance	Simplicial convex hull of segments $xy$
$\arccos(\frac{1}{\sqrt{n}})$	singular segment of type $1n, (2n$ resp.)
$\arccos(-\frac{1}{\sqrt{n}})$	singular segment of type $12n, (21n$ resp.)

The following properties of singular spheres in  $D_n$  can be easily seen in the vector space realization of the Coxeter complex presented in Appendix A.

A wall in  $S$  contains a singular sphere of codimension 1 spanned by  $n-2$  pairwise orthogonal  $n$ -vertices.

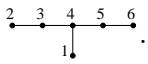
The convex hull of  $n-1$  pairwise orthogonal  $n$ -vertices and their antipodes is a  $(n-2)$ -sphere, but it is not a wall, in particular, it is not a subcomplex. Its simplicial convex hull is  $S$ .

If  $n \geq 5$  ( $n = 4$ ) there are three (four) types of singular spheres of codimension 2. They correspond to the two (three) types of segments connecting two  $(n-1)$ -vertices at distance  $\frac{\pi}{2}$  and the unique type of segments connecting two  $(n-1)$ -vertices at distance  $\frac{\pi}{3}$ . We say that a sphere of the last type is a  $(n-3)$ -sphere of type  $\frac{\pi}{3}$ .

A singular sphere of codimension 2 always contains a singular  $(n-5)$ -sphere spanned by  $n-4$  pairwise orthogonal  $n$ -vertices.

Let  $h$  be a singular hemisphere of codimension 1 bounded by a singular  $(n-3)$ -sphere  $s$ . It is the intersection of a wall and a root bounded by a different wall. If  $n \geq 6$  and  $s$  is of type  $(n-1)n(n-1)$  (or  $(n-1)(n-3)(n-1)$ ), then  $h$  is centered at a  $(n-1)$ -vertex  $x$ . The link  $\Sigma_x h$  in the Coxeter complex  $\Sigma_x S$  of type  $D_{n-2} \circ A_1$  is a wall of type  $n$  (or  $(n-3)$ ). If  $n \geq 5$  and  $s$  is of type  $\frac{\pi}{3}$ , then  $h$  is centered at a point contained in a singular segment of type  $n(n-2)$ , it is the midpoint of two  $(n-1)$ -vertices at distance  $\frac{\pi}{3}$ .

### 3.2 The Coxeter complex of type $E_6$

Let  $(S, W_{E_6})$  be the spherical Coxeter complex of type  $E_6$  with Dynkin diagram . It has dimension 5.

The 1-vertices are the vertices of *root type*. All hemispheres bounded by walls are centered at a 1-vertex.

The Dynkin diagram has one symmetry, namely, the one that exchanges the vertices  $2 \leftrightarrow 6$ ,  $3 \leftrightarrow 5$  and fixes the 1- and 4-vertices. It corresponds to the canonical involution of the Weyl chamber  $\Delta_{mod}^{E_6}$ . Therefore, the properties of  $i$ - and  $(8-i)$ -vertices for  $i = 2, 3, 5, 6$ , are *dual* to each other.

We describe now the possible lengths and types (modulo the action of the Weyl group) of segments between vertices. We list only the ones that we will need later.

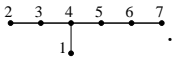
- Distances between two 2- (6)-vertices  $x$  and  $x'$ :

Distance	Simplicial convex hull of segments $xx'$
0	
$\arccos(\frac{1}{4})$	singular segment of type 232 (656)
$\frac{2\pi}{3}$	singular segment of type 262 (626)

- Distances between a 2-vertex  $x$  and a 6-vertex  $y$ :

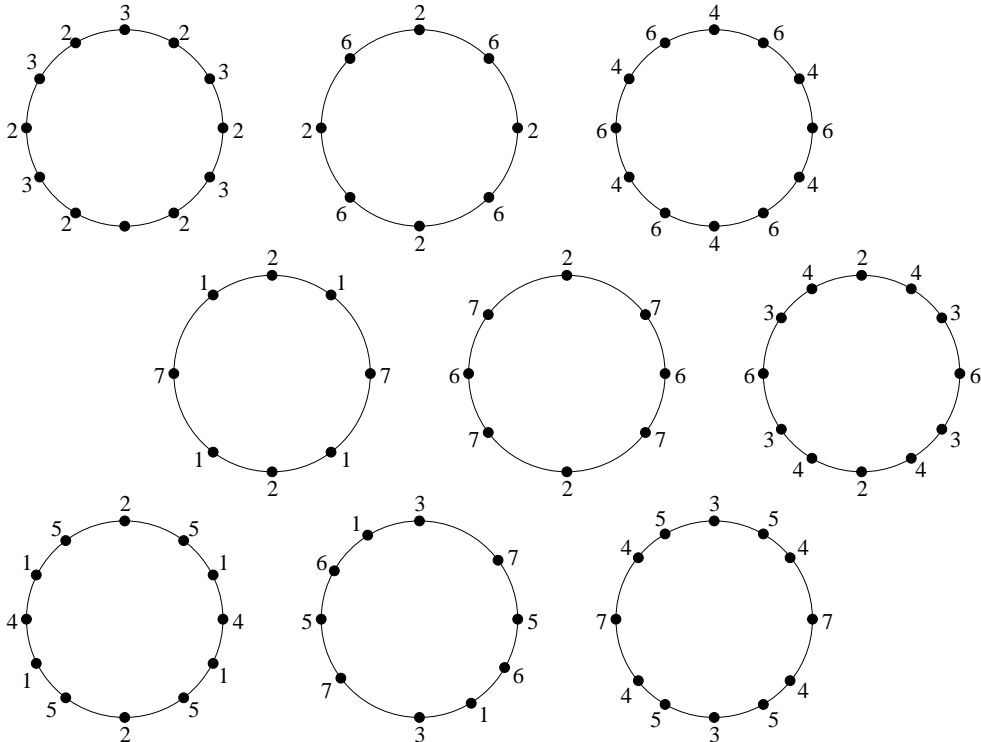
Distance	Simplicial convex hull of segments $xx'$
$\pi$	
$\arccos(-\frac{1}{4})$	singular segment of type 216
$\frac{\pi}{3}$	singular segment of type 26

### 3.3 The Coxeter complex of type $E_7$

Let  $(S, W_{E_7})$  be the spherical Coxeter complex of type  $E_7$  with Dynkin diagram . It has dimension 6.

The Dynkin diagram for  $E_7$  has no symmetries, therefore all automorphisms of  $(S, W_{E_7})$  are *type preserving*.

These are the one dimensional singular spheres in  $(S, W_{E_7})$ :



The 2-vertices are the vertices of *root type*. All hemispheres bounded by walls are centered at a 2-vertex.

We describe now the possible lengths and types (modulo the action of the Weyl group) of segments between vertices. We list only the ones that we will need later.

- Distances between two 2-vertices  $x$  and  $x'$ :



Distance	Simplicial convex hull of segments $xx'$
$0, \pi$	
$\frac{\pi}{3}$	singular segment of type 232
$\frac{\pi}{2}$	singular segment of type 262
$\frac{2\pi}{3}$	singular segment of type 23232

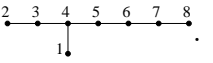
- Distances between two 7-vertices  $x$  and  $x'$ :

Distance	Simplicial convex hull of segments $xx'$
$0, \pi$	
$\arccos(\frac{1}{3})$	singular segment of type 767
$\arccos(-\frac{1}{3})$	singular segment of type 727

- Distances between a 2-vertex  $x$  and a 7-vertex  $y$ :

Distance	Simplicial convex hull of segments $xy$
$\arccos(\frac{1}{\sqrt{3}})$	singular segment of type 27
$\frac{\pi}{2}$	singular segment of type 217
$\arccos(-\frac{1}{\sqrt{3}})$	singular segment of type 2767

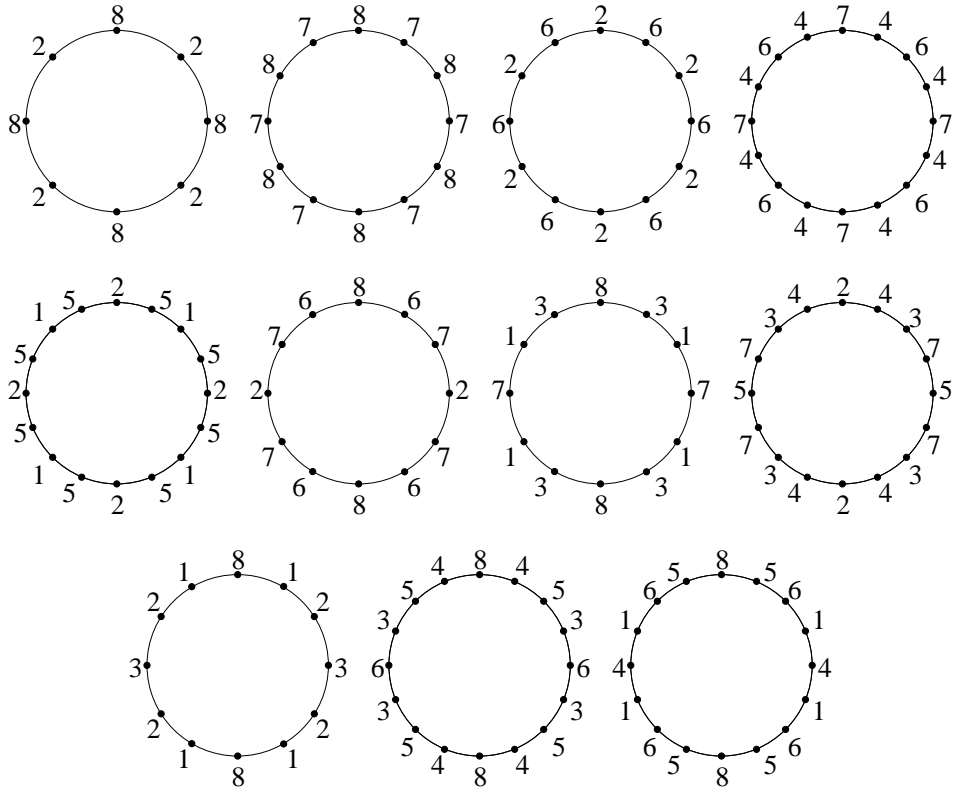
### 3.4 The Coxeter complex of type $E_8$

Let  $(S, W_{E_8})$  be the spherical Coxeter complex of type  $E_8$  with Dynkin diagram . It has dimension 7.

The Dynkin diagram for  $E_8$  has no symmetries, therefore all automorphisms of  $(S, W_{E_8})$  are *type preserving*.

The 8-vertices are the vertices of *root type*. All hemispheres bounded by walls are centered at an 8-vertex.

These are the one dimensional singular spheres in  $(S, W_{E_8})$ :



We describe now the possible lengths and types (modulo the action of the Weyl group) of segments between vertices. We list only the ones that we will need later.

- Distances between two 8-vertices  $x$  and  $x'$ :

Distance	Simplicial convex hull of segments $xx'$
$0, \pi$	
$\frac{\pi}{3}$	singular segment of type 878
$\frac{\pi}{2}$	singular segment of type 828
$\frac{2\pi}{3}$	singular segment of type 87878

- Distances between two 2-vertices  $x$  and  $x'$ :

Distance	Simplicial convex hull of segments $xx'$
$0, \pi$	
$\arccos(\frac{3}{4})$	singular segment of type 232
$\frac{\pi}{3}$	singular segment of type 262
$\arccos(\frac{1}{4}), \arccos(-\frac{1}{4})$	
$\pi/2$	singular segment of type 282 / singular segment of type 25152
$\frac{2\pi}{3}$	singular segment of type 26262
$\arccos(-\frac{3}{4})$	singular segment of type 21812

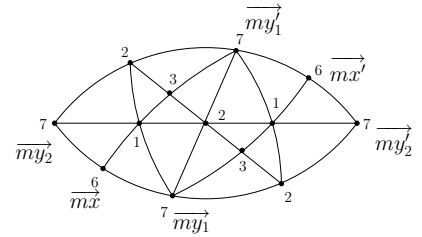
- The possible distances between two 7-vertices  $x$  and  $x'$  are  $\arccos(\frac{k}{6})$  for integer  $-6 \leq k \leq 6$ . Here we will just need to describe the following segments:

Distance	Simplicial convex hull of segments $xx'$	Comments
$\arccos(-\frac{1}{6})$		<p>There are two types of segments <math>xx'</math>. The simplicial convex hull <math>C</math> of <math>xx'</math> is 2- or 3-dimensional, resp.</p> <p>For the case <math>\dim(C) = 3</math>, we present two perspectives from the <i>front</i> and from <i>behind</i> of a larger polyhedron <math>C'</math>. It is the simplicial convex hull of <math>xx' \cup \{y_2, y'_2\}</math>. We describe <math>\Sigma_m C'</math> below. (†)</p>
$\arccos(-\frac{1}{3})$	<p>singular segment of type 76867 /</p>	
$\arccos(-\frac{2}{3})$	<p>singular segment of type 7342437</p>	<p>We present here only the segment <math>xx'</math>, such that the simplex containing <math>\overrightarrow{xx'}</math> does not contain 2- or 8-vertices.</p>

(†)

For a detailed description of the 3-dimensional spherical polyhedra  $C$  and  $C'$  we refer to Appendix A.4, p.49.

$\Sigma_m C'$ :



The possible lengths of segments  $xx'$ , such that  $\pi > d(x, x') > \frac{\pi}{2}$  and the simplex containing the direction  $\overrightarrow{xx'}$  in its interior does not contain a 2- or 8-vertex, are only  $\arccos(-\frac{1}{3})$  and  $\arccos(-\frac{2}{3})$ .

- Distances  $> \frac{\pi}{2}$  and  $< \pi$  between two 1-vertices  $x$  and  $x'$ , such that the simplex containing  $\overrightarrow{xx'}$  in its interior has no 2-, 7- or 8-vertex:

Distance	Simplicial convex hull of segments $xx'$
$\arccos(-\frac{3}{8})$	
$\frac{2\pi}{3}$	singular segment of type 13831
$\arccos(-\frac{5}{8})$	
$\arccos(-\frac{7}{8})$	singular segment of type 1658561

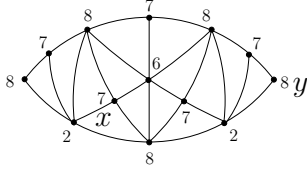
- Distances  $> \frac{\pi}{2}$  and  $< \pi$  between two 6-vertices  $x$  and  $x'$ , such that the simplex containing  $\overrightarrow{xx'}$  in its interior has no 1-, 2-, 7- or 8-vertices:

Distance	Simplicial convex hull of segments $xx'$
$\arccos(-\frac{1}{4})$	singular segment of type 65856
$\frac{2\pi}{3}$	
$\arccos(-\frac{3}{4})$	

- Distances between a 2-vertex  $x$  and an 8-vertex  $y$ :

Distance	Simplicial convex hull of segments $xy$
$\frac{\pi}{4}$	singular segment of type 28
$\arccos(\frac{1}{2\sqrt{2}})$	singular segment of type 218
$\frac{\pi}{2}$	singular segment of type 2768
$\arccos(-\frac{1}{2\sqrt{2}})$	singular segment of type 23218
$\frac{3\pi}{4}$	singular segment of type 2828

- Distances  $> \frac{\pi}{2}$  between a 7-vertex  $x$  and an 8-vertex  $y$ :

Distance	Simplicial convex hull of segments $xy$
$\frac{5\pi}{6}$	singular segment of type 787878
$\arccos(-\frac{1}{\sqrt{3}})$	singular segment of type 72768
$\arccos(-\frac{1}{2\sqrt{3}})$	

## 4 Convex subcomplexes

In this section we will describe some general properties of convex subcomplexes of buildings, as well as some results for buildings of specific types. These will be needed later in the proof of the Center Conjecture.

Let  $K$  be a convex subcomplex of a spherical building  $B$ .

Let  $v \in \Sigma_x K$ . We say that  $v$  is  $d$ -extendable, if there is a segment  $xy \subset K$  of length  $d$  and so that  $v = \overrightarrow{xy}$ .

We say that a point  $x \in K$  is *interior in  $K$* , if the link  $\Sigma_x K$  is a subbuilding of  $\Sigma_x B$ .

**Lemma 4.1.** *Let  $x_1 x_2 \subset K$  be a segment. Suppose  $z$  is a point in the interior of the simplicial convex hull of  $x_1 x_2$ , which has an antipode  $\hat{z} \in K$ . Then  $x_i$  has also an antipode in  $K$ .*

*Proof.* Let  $C$  be the simplicial convex hull of  $x_1 x_2$ . Notice that  $C$  is contained in an apartment and  $\Sigma_z C$  is a sphere. Let  $\gamma_i \subset K$  for  $i = 1, 2$  be the geodesic connecting  $z$  and  $\hat{z}$ , such that the initial direction of  $\gamma_i$  at  $z$  is the antipode in  $\Sigma_z C$  of  $\overrightarrow{zx_i}$ . Then  $x_i z \cup \gamma_i$  is a geodesic of length  $> \pi$ . It is clear that  $\gamma_i$  contains an antipode of  $x_i$ .  $\square$

The following results give us conditions, under which  $K$  satisfies the conclusions of the Center Conjecture 1.1.

The next Lemma puts together the results [LR09, Prop. 2.4, Lemma 2.5]. Compare also [Se05, Thm. 2.2] and [KL98, Prop. 3.10.3].

**Lemma 4.2.** *The following assertions are equivalent:*

- (i)  $K$  is a subbuilding of  $B$ ,
- (ii) every vertex of  $K$  has an antipode in  $K$ ,
- (iii)  $K$  contains a sphere of dimension equal to the dimension of  $K$ .

The following result was stated in [LR09, Cor. 2.10] for convex subcomplexes, but the proof works also for closed convex subsets. In [BL05] a more general result is shown, namely, for an arbitrary CAT(1) space  $C$  of finite dimension and the action  $Isom(C) \curvearrowright C$ .

**Lemma 4.3.** *Let  $C \subset B$  be a closed convex subset. Suppose that  $rad_C(C) \leq \frac{\pi}{2}$ , then the action  $Stab_{Aut(B)}(C) \curvearrowright C$  has a fixed point.*

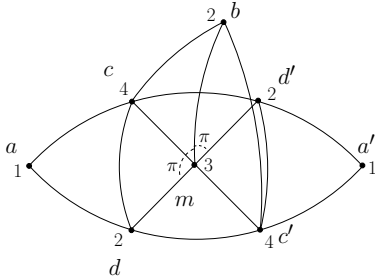
**Lemma 4.4 ([LR09, Cor. 2.12]).** *If  $K$  contains a singular sphere of dimension  $\dim(K) - 1$ , then  $K$  is a subbuilding or  $\text{Stab}_{\text{Aut}(B)}(K) \curvearrowright K$  has a fixed point.*

## 4.1 Convex subcomplexes of buildings of type $D_n$

In this section let  $L \subset B$  be a convex subcomplex of a building of type  $D_n$  for  $n \geq 4$ . We use the following labelling of the Dynkin diagram  $\begin{matrix} 1 \\ \swarrow \searrow \\ 2 \end{matrix} \text{---} 3 \text{---} 4 \text{---} \dots \text{---} n-1 \text{---} n$ .

**Lemma 4.5.** *Let  $n = 4$ , i.e.  $B$  is of type  $D_4$  and suppose that  $L$  contains a pair of antipodal  $i$ -vertices and a pair of antipodal  $j$ -vertices for  $i \neq j$  and  $i, j \in \{1, 2, 4\}$ . Then it contains a singular circle of type 1241241.*

*Proof.* By the symmetry of the Dynkin diagram of type  $D_4$ , we may assume w.l.o.g. that  $i = 1$  and  $j = 2$ . Let  $a, a' \in L$  be the antipodal 1-vertices and let  $b, b' \in L$  be the antipodal 2-vertices. If  $b$  lies on a geodesic  $\gamma$  connecting  $a$  and  $a'$ , then  $\gamma$  is of type 1421. The convex hull of  $b'$  and a small neighborhood of  $b$  in  $\gamma$  is the desired circle.



Let us suppose then, that  $d(a, b) + d(b, a') > \pi$ . The segments  $ba$  and  $ba'$  are of type 241. Let  $c, c'$  be the 4-vertices on the segments  $ba$  and  $ba'$ , respectively. Let  $d, d'$  be the 2-vertices on the segments  $ac'$  and  $a'c$ , respectively. Since  $c, c'$  are adjacent to  $b$ , it follows that the segment  $cc'$  is of type 434. Let  $m$  be the 3-vertex  $m(c, c')$ , then the segment  $mb'$  is of type 3232. This implies that  $\overrightarrow{mb'}$  must be antipodal to  $\overrightarrow{md}$  or  $\overrightarrow{md'}$ . In particular

$b'$  is antipodal to  $d$  or  $d'$ . Either way, we find the desired circle, as above.  $\square$

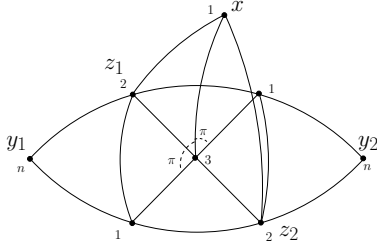
**Remark 4.6.** The proof of Lemma 4.5 shows that we can choose the circle in  $L$  to contain the two antipodal  $i$ -vertices or the two antipodal  $j$ -vertices.

**Lemma 4.7.** *If  $L$  contains a singular  $(n - 2)$ -sphere  $S$  (i.e.  $S$  is a wall) and  $x \in L$  is a 1-, 2- or  $n$ -vertex without antipodes in  $S$ , then  $\Sigma_x L$  contains an apartment. In particular,  $x$  is an interior vertex in  $L$ .*

*Proof.* Let first  $x$  be an  $n$ -vertex. The sphere  $S$  contains  $n - 2$  pairwise orthogonal  $n$ -vertices and their antipodes. They span a singular  $(n - 3)$ -sphere  $S' \subset S$ . Since  $x$  has no antipodes in  $S$ , then it must have distance  $\frac{\pi}{2}$  to all these  $n$ -vertices, and  $h := CH(S', x)$  is a  $(n - 2)$ -dimensional hemisphere centered at  $x$ . Put  $D_3 := A_3$ . The link  $\Sigma_x B$  has type  $D_{n-1}$ .  $\Sigma_x h$  is a  $(n - 3)$ -sphere spanned by  $n - 2$  pairwise orthogonal  $(n - 1)$ -vertices. This  $(n - 3)$ -sphere is not a subcomplex, its simplicial convex hull is an apartment contained in  $\Sigma_x L$ .

We may now assume w.l.o.g. that  $x$  is a 1-vertex. We prove the assertion by induction on  $n$ . Let  $B$  be of type  $D_3$  with Dynkin diagram  $\begin{matrix} 1 \\ \swarrow \searrow \\ 2 \end{matrix} \text{---} 3$ . In this case the 1-dimensional sphere  $S$  contained in  $L \subset B$  is a circle of type 1312321. Since the 1-vertex  $x$  has no antipodes in  $S$ , it must be adjacent to the 2-vertices in  $S$  and therefore it is also adjacent to the 3-vertex between them. It follows that the convex hull  $CH(S, x)$  is a 2-dimensional hemisphere with  $x$  in its interior.  $\Sigma_x CH(S, x)$  is an apartment in  $\Sigma_x L$ .

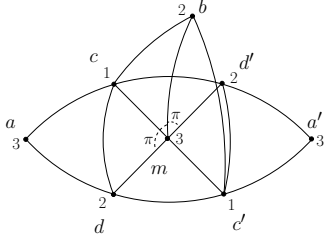
Let now  $B$  be of type  $D_n$  for  $n \geq 4$ . Let  $y_1, y_2 \in S$  be two antipodal  $n$ -vertices. If  $x$  lies on a geodesic of length  $\pi$  connecting  $y_1$  and  $y_2$ , then the geodesic  $y_1xy_2$  is of type  $n21n$ . The link  $\Sigma_{y_1}L$  is of type  $D_{n-1}$ . By induction it follows that  $\Sigma_{\overline{y_1x}}\Sigma_{y_1}L$  contains an apartment, and therefore,  $\Sigma_xL$  contains also an apartment.



On the other hand, if  $d(x, y_1) + d(x, y_2) > \pi$ , then the segments  $xy_i$  are of type  $12n$ . Let  $z_i$  be the 2-vertex on the segment  $xy_i$ . Since  $z_i$  is adjacent to  $y_i$  we deduce that  $z_1 \neq z_2$ . Since the link  $\Sigma_xL$  has type  $A_{n-1}$ , it follows that the segment  $\overline{xz_1}\overline{xz_2} \subset \Sigma_xL$  is of type  $232$ . Again by the induction hypothesis,  $\Sigma_{\overline{y_iz_i}}\Sigma_{y_i}L$  contains an apartment, which in turn implies that  $\Sigma_{\overline{xz_i}}\Sigma_xL$  contains an apartment. In particular the 2-vertices  $\overline{xz_i}$  are interior vertices in  $\Sigma_xL$ . Thus, we can extend the segment  $\overline{xz_1}\overline{xz_2}$  to a geodesic in  $\Sigma_xL$  of length  $\pi$  and type  $232n$ . The convex hull of a small neighborhood in  $\Sigma_xL$  of the interior vertex  $\overline{xz_1}$  and an antipode contains the desired apartment in  $\Sigma_xL$ .  $\square$

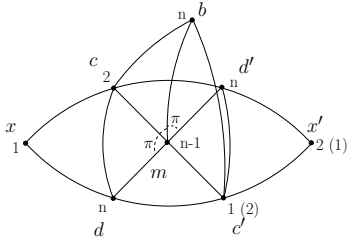
**Lemma 4.8.** *Let  $n \geq k \geq 3$ . Suppose that  $L$  contains a singular  $(n - k)$ -sphere  $S$  spanned by  $n - k + 1$  pairwise orthogonal  $n$ -vertices. Assume also that  $L$  contains a 1-vertex  $x$  and an antipode of  $x$  (of type 1 or 2 depending on the parity of  $n$ ). Then  $L$  contains a singular  $(n - k + 1)$ -sphere spanned by a simplex of type  $1k(k + 1) \dots (n - 1)n$ .*

*Proof.* We prove this again by induction on  $n$ . Let  $B$  be of type  $D_3 := A_3$  with Dynkin diagram  $\begin{array}{c} 1 \\ \vdots \\ 3 \\ \vdots \\ 2 \end{array}$ , that is  $n = k = 3$ .



The hypothesis in this case is that  $L$  contains a pair of antipodal 3-vertices  $a, a'$  and a pair of antipodal 1- and 2-vertices  $b', b$ , respectively. If  $b$  lies on a geodesic connecting  $a$  and  $a'$ , then we find a circle of type  $2321312$  (compare with the proof of Lemma 4.5). Otherwise,  $d(a, b) + d(b, a') > \pi$ . The segments  $ba$  and  $ba'$  are of type  $213$ . Let  $c, c'$  be the 1-vertices on these segments. It is clear that  $c \neq c'$  and the segment connecting them must be of type  $131$ . Let  $m := m(c, c')$ . Since  $c$  and  $c'$  are adjacent to  $b$ , it follows that  $m$  is also adjacent to  $b$ . Let  $d, d'$  be the 2-vertices in the segments of type  $321$   $ac'$  and  $a'c$ . By considering the spherical triangles  $CH(a, c, c')$  and  $CH(a', c, c')$ , we see that  $d$  and  $d'$  are adjacent to  $m$ . The segment  $mb'$  is of type  $321$ . It follows that  $b'$  must be antipodal to  $d$  or  $d'$  (either  $b'md$  or  $b'md'$  is a geodesic of length  $\pi$ ) and we find again a circle in  $L$  spanned by a simplex of type  $13$ .

The argument for the induction step is very similar. Let  $n \geq 4$ . Let  $b, b'$  be a pair of antipodal  $n$ -vertices in the  $(n - k)$ -sphere  $S \subset L$  and let  $x'$  be an antipode in  $L$  of the 1-vertex  $x$ . If  $b$  lies on a geodesic connecting  $x$  and  $x'$ , then this geodesic is of type  $1n21$ ,  $1n12$ ,  $12n2$  or  $12n1$  depending on the parity of  $n$  and if  $b$  is adjacent to  $x$  or  $x'$ . It follows that  $\Sigma_bL$  or  $\Sigma_{b'}L$  contains a 1-vertex and an antipode of it.



If  $d(x, b) + d(b, x') > \pi$ , then the segment  $bx$  is of type  $n21$  and the segment  $bx'$  is of type  $n12$  or  $n21$ . Let  $c, c'$  be the vertices in the interior of the segments  $bx, bx'$  and let  $d, d'$  be the  $n$ -vertices on the segments  $c'x$  and  $cx'$ . Since  $c$  and  $d$  are adjacent to  $x$ , then they are adjacent or  $cx'd$  is a segment. In this last case,  $c$  and  $c'$  must be antipodal, but this cannot happen, because they are adjacent to  $b$ .

So  $c$  and  $d$  are adjacent. This implies that the segment  $cc'$  is of type  $2(n-1)1$  or  $2(n-1)2$ . The  $(n-1)$ -vertex  $m := m(d, d') = m(c, c')$  is adjacent to  $b$ . It follows that the segment  $mb'$  is of type  $(n-1)n(n-1)n$ . Again we conclude that  $b'$  is antipodal to  $d$  or  $d'$ . This implies that  $b'$  lies in a circle in  $L$  of type  $n21n21n$  or  $n21n12n$ . In particular  $\Sigma_{b'}L$  or  $\Sigma_bL$  contains a 1-vertex and an antipode of it. Suppose w.l.o.g. that it holds for  $\Sigma_bL$ . It follows, that  $L$  contains a circle spanned by a simplex of type  $1n$ . So, if  $k = n$ , we are done. Suppose then, that  $k \leq n-1$ .

We have seen that the link  $\Sigma_bL$  of type  $D_{n-1}$  contains a 1-vertex and an antipode of it. It also contains the singular  $(n-1-k)$ -sphere  $\Sigma_bS$  spanned by  $n-k$  pairwise orthogonal  $(n-1)$ -vertices. By the induction assumption,  $\Sigma_bL$  contains a singular  $(n-k)$ -sphere spanned by a simplex of type  $1k(k+1)\dots(n-1)$ . Hence,  $L$  contains a singular  $(n-k+1)$ -sphere spanned by a simplex of type  $1k(k+1)\dots(n-1)n$ .  $\square$

**Remark 4.9.** Lemma 4.5 is just the special version of Lemma 4.8 where  $n = k = 4$ . If  $k = 3$  in Lemma 4.8, then the conclusion is that  $L$  contains a wall.

**Remark 4.10.** The proof of Lemma 4.8 shows that if  $n = k$ , we can choose the 1-sphere in  $L$  to contain the 1-vertex  $x$  (this is true in general, but it is less obvious from the proof).

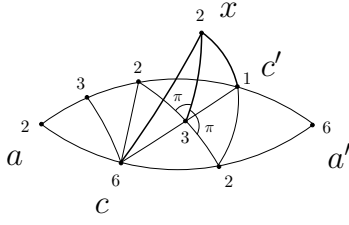
## 4.2 Convex subcomplexes of buildings of type $E_6$

In this section let  $L \subset B$  be a convex subcomplex of a building of type  $E_6$ . We use the following labelling of the Dynkin diagram  $\overset{2}{\bullet} \overset{3}{\bullet} \overset{4}{\bullet} \overset{5}{\bullet} \overset{6}{\bullet}$ .

**Lemma 4.11.** *If  $L$  contains a singular 4-sphere  $S$  (i.e  $S$  is a wall) and  $x \in L$  is a 2 or 6-vertex without antipodes in  $S$ , then  $\Sigma_xL$  contains an apartment. In particular,  $x$  is an interior vertex in  $L$ .*

*Proof.* By the symmetry of the Dynkin diagram for  $E_6$  it suffices to show it for a 2-vertex  $x \in L$ . The wall  $S$  contains a pair of antipodal 2- and 6-vertices  $a$  and  $a'$ , respectively. The link  $\Sigma_aB$  ( $\Sigma_{a'}B$ ) is of type  $D_5$  and Dynkin diagram  $\overset{3}{\bullet} \overset{4}{\bullet} \overset{5}{\bullet} \overset{6}{\bullet}$  ( $\overset{2}{\bullet} \overset{3}{\bullet} \overset{4}{\bullet} \overset{5}{\bullet} \overset{6}{\bullet}$ ).  $\Sigma_aL$  and  $\Sigma_{a'}L$  contain a singular 3-sphere  $\Sigma_aS$ , respectively  $\Sigma_{a'}S$ . Suppose first that  $x$  lies on a geodesic  $\gamma$  connecting  $a$  and  $a'$ .  $\gamma$  is of type  $23216$  or  $2626$ . Since  $x$  has no antipodes in  $S$ , the vertex  $\overrightarrow{ax}$  of type 3 or 6 has no antipodes in  $\Sigma_aS$ . It follows from Lemma 4.7, that  $\Sigma_{\overrightarrow{ax}}\Sigma_aL$  contains an apartment and this implies in turn, that  $\Sigma_{\overrightarrow{ax}}\Sigma_xL$  contains also an apartment. Since  $\overrightarrow{xa'} \in \Sigma_xL$  is antipodal to  $\overrightarrow{xa}$ , this implies that  $\Sigma_xL$  contains an apartment.





On the other hand, if  $d(x, a) + d(x, a') > \pi$ , then the segments  $xa$  and  $xa'$  are of type 262 and 216. Let  $c$  be the 6-vertex on  $xa$  and let  $c'$  be the 1-vertex on  $xa'$ .  $c$  is adjacent to  $a$  and  $c'$  is adjacent to  $a'$ , therefore  $c$  and  $c'$  cannot be adjacent and since both are adjacent to  $x$ , it follows that the segment  $\overrightarrow{xcx'}$  is of type 631. It follows again from Lemma 4.7, that  $\Sigma_{\overrightarrow{ax}}\Sigma_a L$  and  $\Sigma_{\overrightarrow{a'x}}\Sigma_{a'} L$  contain

a 3-sphere. This implies that  $\Sigma_{\overrightarrow{xc}}\Sigma_x L$  and  $\Sigma_{\overrightarrow{xc'}}\Sigma_x L$  contain a 3-sphere, in particular,  $\overrightarrow{xc}$  and  $\overrightarrow{xc'}$  are interior vertices in  $\Sigma_x L$ . The segment  $\overrightarrow{xcx'}$  is of type 631 and since  $\overrightarrow{xc'}$  is interior, it can be extended in  $\Sigma_x L$  to a segment of type 6316. This means that  $\overrightarrow{xc}$  has an antipode in  $\Sigma_x L$  implying that  $\Sigma_x L$  contains a 4-sphere as desired.  $\square$

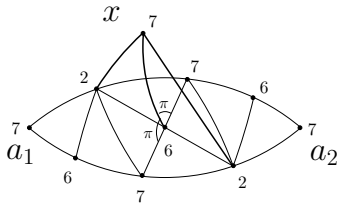
### 4.3 Convex subcomplexes of buildings of type $E_7$

In this section let  $L \subset B$  be a convex subcomplex of a building of type  $E_7$ . We use the following labelling of the Dynkin diagram  $\overset{2}{\bullet} \overset{3}{\bullet} \overset{4}{\bullet} \overset{5}{\bullet} \overset{6}{\bullet} \overset{7}{\bullet}$ .

**Lemma 4.12.** *If  $L$  contains a singular 5-sphere  $S$  (i.e.  $S$  is a wall) and  $x \in L$  is a 7-vertex without antipodes in  $S$ , then  $\Sigma_x L$  contains an apartment. In particular,  $x$  is an interior vertex in  $L$ .*

*Proof.* The wall  $S$  contains a pair of antipodal 7-vertices  $a_1, a_2$ . The link  $\Sigma_{a_i} B$  is of type  $E_6$  with Dynkin diagram  $\overset{2}{\bullet} \overset{3}{\bullet} \overset{4}{\bullet} \overset{5}{\bullet} \overset{6}{\bullet}$ .  $\Sigma_{a_i} L$  contains the wall  $\Sigma_{a_i} S$ .

Suppose w.l.o.g. that  $d(x, a_1) = \arccos(-\frac{1}{3})$ . Then the segment  $xa_1$  is of type 727. Since  $x$  has no antipodes in  $S$  it follows that the 2-vertex  $\overrightarrow{a_1 x}$  has no antipodes in  $\Sigma_{a_1} S$ . We apply now Lemma 4.11 to deduce that  $\Sigma_{\overrightarrow{a_1 x}}\Sigma_{a_1} L$  contains an apartment. This implies in turn, that  $\Sigma_{\overrightarrow{xa_1}}\Sigma_x L$  contains an apartment. Therefore, if we find an antipode in  $\Sigma_x L$  of  $\overrightarrow{xa_1}$ , we are done. This is trivial if  $x$  lies on a geodesic connecting  $a_1$  and  $a_2$ .



Otherwise also  $d(x, a_2) = \arccos(-\frac{1}{3})$ . We may argue as above and conclude that  $\Sigma_{\overrightarrow{xa_2}}\Sigma_x L$  contains an apartment. In particular  $\overrightarrow{xa_2}$  is an interior vertex in  $\Sigma_x L$ . Notice that the segment connecting  $m(x, a_i)$  for  $i = 1, 2$  cannot be of type 232, otherwise we find a curve of length  $< \pi$  connecting  $a_1$  and  $a_2$ . Therefore, the segment  $\overrightarrow{xa_1 xa_2}$  is of type 262. Since  $\overrightarrow{xa_2}$  is interior, we can extend the segment  $\overrightarrow{xa_1 xa_2}$  to a segment of type 2626 and length  $\pi$  in  $\Sigma_x L$ . We have found an antipode of  $\overrightarrow{xa_1}$ .  $\square$

## 5 The Center Conjecture

Let  $B$  be a spherical building and  $K \subset B$  a convex subcomplex. We say that  $K$  is a *counterexample* to the Center Conjecture, if  $K$  is not a subbuilding and  $G := \text{Stab}_{\text{Aut}(B)}(K)$  has no fixed points in  $K$ .

From the Lemmata 4.2, 4.3 and 4.4 we can deduce some general properties of convex subcomplexes  $K \subset B$ , which are counterexamples to the Center Conjecture:

1. If  $x \in K$  and  $y \in CH(G \cdot x)$ , then there exists  $x' \in G \cdot x$ , such that  $d(y, x') > \frac{\pi}{2}$ . This is just Lemma 4.3 applied to  $CH(G \cdot x)$ . In particular, if  $x \in K$ , then there exists  $x' \in G \cdot x$ , such that  $d(x, x') > \frac{\pi}{2}$ .

Another way to look at this is the following. If  $P$  is a property of vertices in  $K$  invariant under the action of  $G$ , then for every point  $y$  in the convex hull of the  $P$ -vertices, we can find a  $P$ -vertex  $x$  with  $d(x, y) > \frac{\pi}{2}$ .

2.  $K$  contains no sphere of dimension  $\dim(K) - 1$ .
3. If  $K$  has dimension  $\leq 1$  and is not a subbuilding, then by Lemma 4.2, it contains no circles. It follows that  $K$  is a (bounded) tree and it has a unique circumcenter, which is fixed by  $Isom(K)$ . Hence, a counterexample  $K$  has dimension  $\geq 2$ . By the main result in [BL05] mentioned in the introduction, a counterexample has actually dimension  $\geq 3$ , but we do not use this fact in our proof.

Let  $A$  be the property of a point in  $K$  of not having antipodes in  $K$ . Let  $I$  be the property of a point in  $x \in K$  of being interior, i.e.  $\Sigma_x K$  is a subbuilding of  $\Sigma_x B$ , or equivalently,  $\Sigma_x K$  contains a singular sphere of dimension  $\dim(K) - 1$ .

Notice that an interior point in  $K$  cannot have antipodes in  $K$ , that is,  $I \Rightarrow A$ . Otherwise  $K$  would contain a singular sphere of dimension  $\dim(K)$  and  $K$  would be a subbuilding.

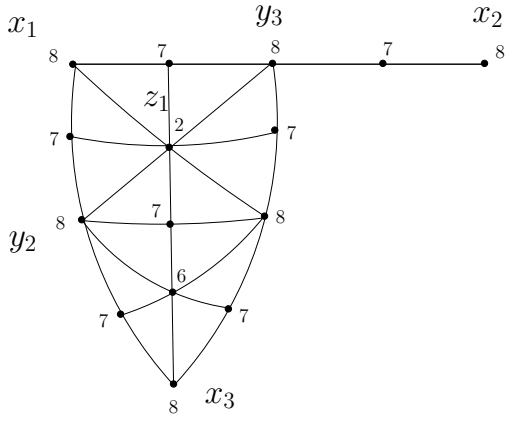
## 5.1 The $E_8$ -case

Let  $K$  be a convex subcomplex of a spherical building  $B$  of type  $E_8$ , which is a counterexample to the center conjecture.

Our strategy is as follows. We focus our attention mainly on the vertices of type 2 and 8. The 8-vertices are the vertices of root type and there are few possibilities for the types of segments between 8-vertices. The 2-vertices have the second smallest orbit (after the 8-vertices) under the action of the Weyl group in the Coxeter complex of type  $E_8$ . This implies that the types of the segments between 2-vertices are still manageable. Another reason to consider 2-vertices is that their links have a relatively simple geometry, they are buildings of type  $D_7$ . In these buildings, there is only one type of segments between two distinct non-antipodal 8-vertices, namely 878, and it has length  $\frac{\pi}{2}$ . First we want to prove that  $K$  cannot contain 2- or 8-vertices, whose links contain spheres of large dimension. This is achieved in the Lemmata 5.1-5.9. Then under the assumption of existence of 8A-vertices, we find 2- and 8-vertices in  $K$ , with links containing spheres of larger and larger dimensions. This allows us to conclude that all 8-vertices in  $K$  have antipodes in  $K$  (Corollary 5.17). At this point the hard work is already done. Finally we show that all other vertices in  $K$  must also have antipodes in  $K$ . This contradicts Lemma 4.2 and the assumption that  $K$  is not a subbuilding.

We describe first some configurations of points of  $K$ , which will be used several times during the argument.

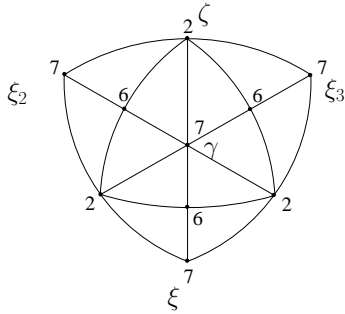
Let  $P$  be a property of 8-vertices implying  $A$  (the property of not having antipodes in  $K$ ) and suppose there are 8P-vertices in  $K$ .



Since  $K$  is a counterexample, there are  $8P$ -vertices  $x_1, x_2 \in K$  at distance  $> \frac{\pi}{2}$ . Since they do not have antipodes, it follows that  $d(x_1, x_2) = \frac{2\pi}{3}$ . Let  $y_3 := m(x_1, x_2)$ , it is an  $8A$ -vertex by Lemma 4.1. Again there is an  $8P$ -vertex  $x_3 \in K$ , such that  $d(y_3, x_3) = \frac{2\pi}{3}$  because  $y_3$  lies in the convex hull of the  $8P$ -vertices in  $K$ . Notice that, since  $x_i$  are  $8A$ -vertices,  $0 < \angle_{y_3}(x_3, x_i) < \pi$  for  $i = 1, 2$ . We may assume w.l.o.g. that  $\angle_{y_3}(x_3, x_1) \geq \frac{\pi}{2}$ . The link  $\Sigma_{y_3}B$  is a building of type  $E_7$  and with Dynkin diagram  $\begin{array}{c} 2-3-4-5-6-7 \\ | \\ 1 \end{array}$ . It follows that  $\angle_{y_3}(x_3, x_1) =$

$\arccos(-\frac{1}{3})$  and this angle is of type 727, i.e. the segment  $\overrightarrow{y_3x_1}\overrightarrow{y_3x_3} \subset \Sigma_{y_3}K$  is singular of type 727. The convex hulls  $CH(x_3, y_3, m(x_1, y_3))$  and  $CH(x_3, x_1, m(x_1, y_3))$  are spherical triangles, because  $y_3$  and  $m(x_1, y_3)$  ( $x_1$  and  $m(x_1, y_3)$ , respectively) are contained in a common Weyl chamber and therefore  $x_3, y_3$  and  $m(x_1, y_3)$  ( $x_3, x_1$  and  $m(x_1, y_3)$ , respectively) lie in a common apartment. The segment  $m(x_1, y_3)x_3$  is of type 72768. Since  $\Sigma_{m(x_1, y_3)}B$  is of type  $E_6 \circ A_1$  with Dynkin diagram  $\begin{array}{c} 2-3-4-5-6-8 \\ | \\ 1 \end{array}$ , it follows that  $\angle_{m(x_1, y_3)}(x_1, x_3) = \angle_{m(x_1, y_3)}(x_1, x_3) = \frac{\pi}{2}$ . Hence, the convex hull  $CH(x_1, y_3, x_3)$  is the union of  $CH(x_3, y_3, m(x_1, y_3))$  and  $CH(x_3, x_1, m(x_1, y_3))$ , and it is an isosceles spherical triangle with sides of type 878, 87878 and 87878. Let  $y_2 := m(x_1, x_3)$  and  $z_1 := (y_2, y_3)$ .

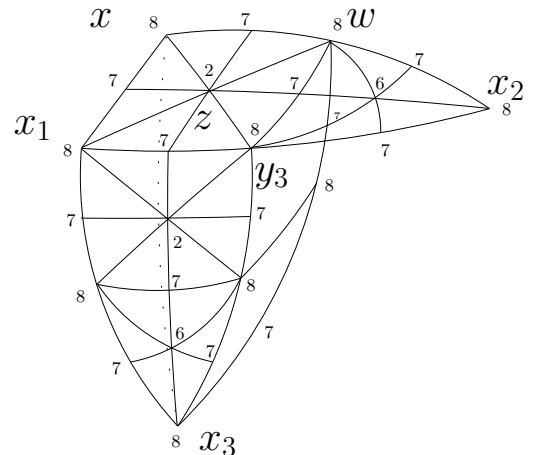
We refer to this configuration of  $8P$ -vertices as configuration  $*$ .

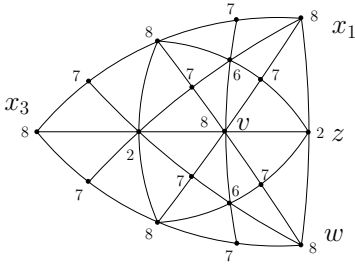


Let now  $\xi_i := \overrightarrow{x_1x_i}$  for  $i = 2, 3$  and  $\zeta := \overrightarrow{x_1z_1}$ . Suppose there is an  $8$ -vertex  $x$  at distance  $\frac{\pi}{3}$  to  $x_1$ , let  $\xi := \overrightarrow{x_1x}$ . Assume furthermore that  $d(\zeta, \xi) = \arccos(-\frac{1}{\sqrt{3}})$ , then the segment  $\xi\zeta$  is of type 7672. Recall that  $\xi_i$  is  $\frac{2\pi}{3}$ -extendable to  $8A$ -vertices and  $\xi$  is  $\frac{\pi}{3}$ -extendable. Thus,  $d(\xi, \xi_i) < \pi$  for  $i = 2, 3$ . It follows that  $\angle_{\zeta}(\xi, \xi_i) = \frac{\pi}{2}$  and  $d(\xi, \xi_i) = \arccos(-\frac{1}{3})$  for  $i = 2, 3$ . Hence, the convex hull  $CH(\xi, \xi_2, \xi_3)$  is the union of the spherical triangles  $CH(\xi, \zeta, \xi_i)$  for  $i = 2, 3$ . It is an equilateral spherical triangle with sides of type

727. Let  $\gamma$  be the 7-vertex at the center of this triangle.

From  $d(\xi, \xi_i) = \arccos(-\frac{1}{3})$ , it follows for  $i = 2, 3$  that  $d(x, x_i) = \frac{2\pi}{3}$  and the convex hulls  $CH(x_1, x, x_i)$  are isosceles spherical triangles (compare with the spherical triangle  $CH(x_1, y_3, x_3)$  above). Let  $w := m(x, x_2)$  be the  $8A$ -vertex between  $x$  and  $x_2$ . Then by considering the triangle  $CH(x_1, x, x_2)$ , we see that  $\omega := \overrightarrow{x_1w} = m(\xi, \xi_2)$ . Let  $z := m(x_1, w)$  be the 2-vertex between  $x_1$  and  $w$ , then  $\overrightarrow{x_1z} = m(\xi, \xi_2)$ . The angle  $\angle_{x_1}(z, x_3) = \arccos(-\frac{1}{\sqrt{3}})$  is of type 2767 (compare with the triangle  $CH(\xi, \xi_2, \xi_3)$ ). Notice that  $CH(z, x_1, x_3)$  is a spherical triangle, this implies that  $d(z, x_3) = \frac{3\pi}{4}$ .





The segment  $zx_3$  is of type 2828. Let  $v$  be the  $8A$ -vertex on the segment  $zx_3$  adjacent to  $z$ . Recall that  $x_3$  is an  $8A$ -vertex. Then  $x_3$  cannot be antipodal to  $w$ , thus  $d(x_3, w) = \frac{2\pi}{3}$  and  $\angle_z(x_3, x_1) = \angle_z(x_3, w) = \frac{\pi}{2}$ . Recall also that  $d(x_3, y_3) = d(x_3, x) = \frac{2\pi}{3}$ , therefore  $\angle_z(x_3, y_3) = \angle_z(x_3, x) = \frac{\pi}{2}$ . The convex hulls  $CH(x_3, x_1, w)$  and  $CH(x_3, y_3, x)$  are isosceles spherical triangles with sides of type 87878, 87878 and 828.

The convex hull in  $\Sigma_z K$  of the 8-vertices  $\overrightarrow{zx}$ ,  $\overrightarrow{zx_1}$ ,  $\overrightarrow{zy_3}$ ,  $\overrightarrow{zw}$  and  $\overrightarrow{zv}$  is a 2-dimensional singular hemisphere  $h$  centered at  $\overrightarrow{zv}$ . Let  $s \subset \Sigma_z B$  be a singular 2-sphere containing  $h$  and let  $\widehat{x}_3$  be an 8-vertex in  $B$ , such that it is adjacent to  $z$  and  $\overrightarrow{zx_3}$  is the antipode of  $\overrightarrow{zv}$  in  $s$ . It follows that  $\widehat{x}_3$  is antipodal to  $x_3$  in  $B$ . The convex hull in  $B$  of  $x_3, \widehat{x}_3, x, x_1, y_3, w$  is a 3-dimensional spherical bigon connecting  $x_3$  and  $\widehat{x}_3$ , with edges  $x_3\alpha\widehat{x}_3$  for  $\alpha \in \{x, x_1, y_3, w\}$  of type 8787878. It follows that the convex hull  $CH(x_1, x, w, y_3, x_3)$  is a (3-dimensional) spherical convex polyhedron in  $K$  obtained by truncating this spherical bigon. Notice that the 7-vertex  $\gamma$  at the center of the triangle  $CH(\xi, \xi_2, \xi_3) \subset \Sigma_{x_1} K$  is  $\frac{2\pi}{3}$ -extendable in  $K$  to the 8-vertex  $m(x_3, w)$ .

We refer to this configuration in  $K$  as configuration \*\*.

**Lemma 5.1.**  *$K$  contains no  $8I$ -vertices.*

*Proof.* Suppose the contrary. There are  $8I$ -vertices  $x_1, x_2 \in K$  with distance  $> \frac{\pi}{2}$ . Clearly  $I \Rightarrow A$ , therefore,  $d(x_1, x_2) = \frac{2\pi}{3}$  and the segment  $x_1x_2$  is of type 87878. Since  $x_i$  are interior vertices, we can find 7-vertices  $y_i \in K$  adjacent to  $x_i$  and such that  $y_1x_1x_2y_2$  is a geodesic of length  $\pi$  and type 7878787. The direction  $\overrightarrow{y_i x_i}$  is an interior 8-vertex in  $\Sigma_{y_i} K$ . Note that  $\Sigma_{y_i} B$  is a building of type  $E_6 \circ A_1$  and with Dynkin diagram  $\begin{array}{cccccc} 2 & 3 & 4 & 5 & 6 & 8 \\ & & \downarrow & & & \end{array}$ . It follows that  $\Sigma_{y_i} K$  contains a top-dimensional hemisphere centered at  $\overrightarrow{y_i x_i}$ . This implies that  $K$  contains a hemisphere of dimension  $\dim(K)$ . A contradiction to the properties of a counterexample.  $\square$

**Lemma 5.2.**  *$K$  contains no 8-vertices  $x$ , such that  $\Sigma_x K$  contains a singular 5-sphere, i.e. a wall.*

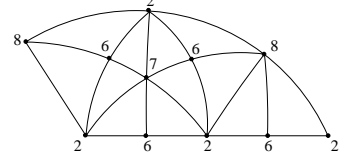
*Proof.* Let  $x_1$  be an 8-vertex, such that  $\Sigma_{x_1} K$  contains a singular 5-sphere  $S_1$ . Clearly, by Lemma 4.4,  $x_1$  is an  $8A$ -vertex. Let  $x_2 \in G \cdot x_1$  be at distance  $\frac{2\pi}{3}$  to  $x_1$ .  $\Sigma_{x_2} K$  contains a singular 5-sphere  $S_2$ . If  $\overrightarrow{x_i x_{3-i}}$  has an antipode in  $S_i$  for  $i = 1, 2$ , then there are 7-vertices  $y_i \in K$  adjacent to  $x_i$ , such that  $y_1x_1x_2y_2$  is a geodesic of length  $\pi$ . The midpoint  $z := m(x_1, x_2)$  is again an  $8A$ -vertex and it is the center of a 6-dimensional hemisphere  $h \subset K$  (cf. proof of Lemma 5.1). In particular,  $\Sigma_z K$  contains the 5-sphere  $\Sigma_z h$  and the 7-vertices in this sphere are all  $\frac{\pi}{2}$ -extendable. Let  $z' \in G \cdot z$  be at distance  $\frac{2\pi}{3}$  to  $z$ . Since  $z'$  is an  $8A$ -vertex and the 7-vertices in  $\Sigma_z h$  are  $\frac{\pi}{2}$ -extendable, we deduce that  $zz'$  has no antipodes in  $\Sigma_z h$ . It follows from Lemma 4.12 that  $\Sigma_{zz'} \Sigma_z K$  contains an apartment and that  $\Sigma_{\overrightarrow{wz}} \Sigma_w K$  contains an apartment for the 8-vertex  $w := m(z, z')$ . It follows that  $\Sigma_w K$  contains also an apartment, contradicting Lemma 5.1. We may therefore assume w.l.o.g. that  $\overrightarrow{x_1 x_2}$  has no antipodes in  $S_1$ . Using again Lemma 4.12 we conclude that  $\Sigma_z K$  contains an apartment. Again a contradiction.  $\square$

**Lemma 5.3.**  *$K$  contains no 2-vertices  $x$ , such that  $\Sigma_x K$  contains an apartment.*

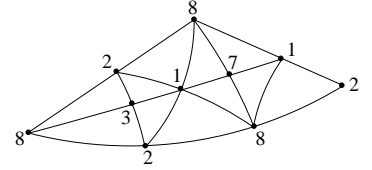
*Proof.* Let  $x$  be such a 2-vertex in  $K$ . Then there is another 2-vertex  $x' \in G \cdot x$  at distance  $> \frac{\pi}{2}$  to  $x$ . Notice that  $x, x'$  are interior vertices in  $K$ .

Case 1:  $d(x, x') = \arccos(-\frac{3}{4})$ . The segment  $xx'$  is of type 21812. Since  $x$  is interior, the direction  $\overrightarrow{xx'}$  is also interior in  $\Sigma_x K$ . It follows that the 8-vertex  $m(x, x')$  must be interior in  $K$ , contradicting Lemma 5.1.

Case 2:  $d(x, x') = \frac{2\pi}{3}$ . The segment  $xx'$  is of type 26262. Recall that  $\Sigma_x B$  is of type  $D_7$  with Dynkin diagram  $\begin{array}{c} 3 \bullet \rightarrow 4 \bullet \rightarrow 5 \bullet \rightarrow 6 \bullet \rightarrow 7 \bullet \rightarrow 8 \bullet \\ 1 \bullet \end{array}$ . Since  $x$  is interior and  $K$  is top-dimensional, then  $\Sigma_x K$  is a building of type  $D_7$  and we can find an 8-vertex  $y \in K$  adjacent to  $x$  and such that  $\angle_x(y, x') > \frac{\pi}{2}$ . Then  $\angle_x(y, x') = \arccos(-\frac{1}{\sqrt{3}})$  and it must be of type 8676. Since the triangle  $CH(y, x, x')$  is spherical, it follows that  $d(y, x') = \frac{3\pi}{4}$  and the segment  $yx'$  is of type 2828. The 8-vertex in the interior of this segment must be an interior vertex. A contradiction to Lemma 5.1.



Case 3:  $d(x, x') = \arccos(-\frac{1}{4})$ . The simplicial convex hull of the segment  $xx'$  is 2-dimensional and contains 8-vertices  $y, y' \in K$  adjacent to  $x, x'$ . Let  $z \in K$  be an 8-vertex adjacent to  $x$ , such that  $zxy$  is a segment of type 828. Then  $d(z, x') = \frac{3\pi}{4}$ . Again a contradiction as in Case 2 above.  $\square$



**Lemma 5.4.**  *$K$  contains no 7-vertices  $x$ , such that  $\Sigma_x K$  contains an apartment.*

*Proof.* Suppose there is such a 7-vertex  $x \in K$  and let  $y \in K$  be an 8-vertex. If  $d(x, y) = \frac{5\pi}{6}$ , then the segment  $xy$  is of type 787878 and we would find interior 8-vertices in  $K$ . If  $d(x, y) = \arccos(-\frac{1}{\sqrt{3}})$ , then the segment  $xy$  is of type 72768 and we would find interior 2-vertices contradicting Lemma 5.3. So,  $d(x, y) \leq \arccos(-\frac{1}{2\sqrt{3}})$ .

Let  $y_1, y_2 \in K$  be 8-vertices adjacent to  $x$ , such that  $y_1xy_2$  is a segment of type 878. Let  $x' \in G \cdot x$  with  $d(x, x') > \frac{\pi}{2}$ . Then  $d(x', y_i) \leq \arccos(-\frac{1}{2\sqrt{3}})$  and triangle comparison with the triangle  $(x', y_1, y_2)$  implies that  $d(x, x') \leq \arccos(-\frac{1}{3})$ .

Case 1:  $d(x, x') = \arccos(-\frac{1}{3})$ . If the segment  $xx'$  is singular of type 76867, then the 8-vertex  $m(x, x')$  is interior, contradiction. If the segment  $xx'$  has 2-dimensional simplicial convex hull  $C$ , then there is an 8-vertex  $y \in C$  adjacent to  $x$  or  $x'$ . Since  $x, x'$  are in the same  $G$ -orbit, we may suppose w.l.o.g. that  $y$  is adjacent to  $x$ . Let  $y' \in K$  be another 8-vertex adjacent to  $x$  and such that  $xyy'$  is a segment of type 878. Then  $d(x', y') = \arccos(-\frac{1}{\sqrt{3}})$  and this case cannot occur by the above.

Case 2:  $d(x, x') = \arccos(-\frac{1}{6})$ . Let  $C$  be the simplicial convex hull of the segment  $xx'$ . If  $C$  is 2-dimensional, there are 8-vertices  $y, y' \in C \subset K$  adjacent to  $x$  and  $x'$  respectively. Let  $z \in K$  be an 8-vertex adjacent to  $x$  and such that  $zxy$  is a segment of type 878. Define  $z'$  analogously. Then  $d(x', z)$  or  $d(x, z') = \arccos(-\frac{1}{\sqrt{3}})$ , which is not possible.

If  $C$  is 3-dimensional, there is an 8-vertex  $m \in C$ , such that the segments  $mx$  and  $mx'$  are of type 867 and  $\angle_m(x, x') = \arccos(-\frac{3}{4})$ . Since  $x, x'$  are interior vertices, there exist 2-



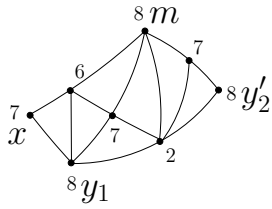
If  $d(x, y) = \arccos(-\frac{1}{\sqrt{3}})$ , then the segment  $xy$  is of type 72768. Lemma 4.11 implies that  $\overrightarrow{xy}$  has an antipode in  $S$  or  $\Sigma_{\overrightarrow{xy}}\Sigma_x K$  contains an apartment. In both cases the link  $\Sigma_z K$  of the 2-vertex  $z$  on the segment  $xy$  adjacent to  $x$  contains a wall. A contradiction to Lemma 5.5. So,  $d(x, y) \leq \arccos(-\frac{1}{2\sqrt{3}})$ .

Let  $y_1, y_2 \in K$  be 8-vertices adjacent to  $x$ , such that  $y_1xy_2$  is a segment of type 878. Let  $x' \in G \cdot x$  with  $d(x, x') > \frac{\pi}{2}$ . Then  $d(x', y_i) \leq \arccos(-\frac{1}{2\sqrt{3}})$  and triangle comparison with the triangle  $(x', y_1, y_2)$  implies that  $d(x, x') \leq \arccos(-\frac{1}{3})$ .

Case 1:  $d(x, x') = \arccos(-\frac{1}{3})$ . If the segment  $xx'$  is singular of type 76867, then Lemma 4.11 implies that the 6-vertex  $\overrightarrow{xx'}$  has an antipode in  $S$  or  $\Sigma_{\overrightarrow{xx'}}\Sigma_x K$  contains an apartment. Either way, the link in  $K$  of the 8-vertex  $m(x, x')$  contains a wall, which is not possible by Lemma 5.2. The case, where the segment  $xx'$  has 2-dimensional simplicial convex hull  $C$ , follows as in the proof of Lemma 5.4.

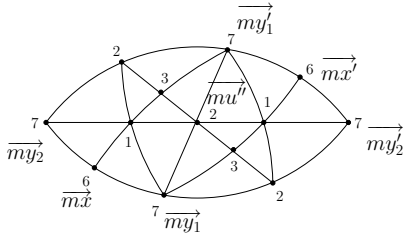
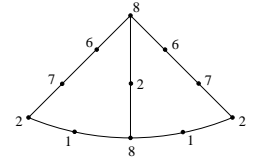
Case 2:  $d(x, x') = \arccos(-\frac{1}{6})$ . Let  $C$  be the simplicial convex hull of the segment  $xx'$ . If  $C$  is 2-dimensional, we argue as in the proof of Lemma 5.4.

If  $C$  is 3-dimensional (see Section 3.4 for a description of  $C$ ), there is an 8-vertex  $m \in C$ , such that the segments  $mx$  and  $mx'$  are of type 867 and  $\angle_m(x, x') = \arccos(-\frac{3}{4})$ .  $C$  contains also 8-vertices  $y_1, y'_1$  adjacent to  $x, x'$  respectively. Let  $y_2 \in K$  be an 8-vertex adjacent to  $x$  and such that  $y_2xy_1$  is a segment of type 878. Define  $y'_2$  analogously. Then the angle  $\angle_m(x, y'_2)$  is of type 6727 (compare with  $\Sigma_m C'$  in Section 3.4). This implies that  $d(x, y'_2) = \arccos(-\frac{1}{2\sqrt{3}})$ .



If the 6-vertex  $\overrightarrow{xm}$  has no antipodes in  $S$ , then it follows from Lemma 4.11 that  $\Sigma_{\overrightarrow{xm}}\Sigma_x K$  contains an apartment, i.e.  $\overrightarrow{xm}$  is interior in  $\Sigma_x K$ . In particular the link  $\Sigma_w K$  of the 7-vertex  $w$  in the interior of the simplicial convex hull of  $xy'_2$  contains an apartment. A contradiction to Lemma 5.4. It follows that  $\overrightarrow{xm}, \overrightarrow{x'm}$  have antipodes in the walls  $S \subset \Sigma_x K$ , respectively  $S' \subset \Sigma_{x'} K$ . Therefore, there exist 2-vertices  $u, u' \in K$ , such that  $mxu$  and  $mx'u'$  are segments of length  $\frac{\pi}{2}$  and of type 8672.  $\angle_m(x, x') = \arccos(-\frac{3}{4})$  implies that  $\pi > d(u, u') \geq \arccos(-\frac{3}{4})$ . Hence  $d(u, u') = \arccos(-\frac{3}{4})$ .

It follows that the segment  $uu'$  is of type 21812 and  $CH(m, u, u')$  is a (non-simplicial) spherical triangle. The segment  $mm(u, u')$  has length  $\frac{\pi}{2}$  and therefore it has type 828. The 2-vertex  $u'' := m(m, m(u, u'))$  lies in the interior of the spherical triangle  $CH(m, u, u')$ .



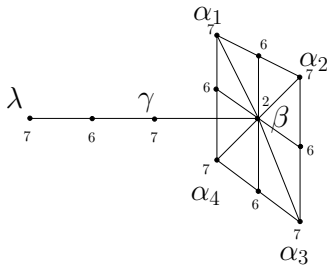
Consider the link of  $m$ . Since  $\overrightarrow{xm}$  has an antipode in the wall  $S \subset \Sigma_x K$ , it follows that  $\Sigma_{\overrightarrow{m\hat{x}}}\Sigma_m K$  contains a wall. The link  $\Sigma_{\overrightarrow{m\hat{x}}}\Sigma_m B$  is of type  $D_5 \circ A_1$ . The wall in  $\Sigma_{\overrightarrow{m\hat{x}}}\Sigma_m K$  contains a wall in the  $D_5$ -factor. The direction  $\xi := \overrightarrow{m\hat{x}my_1}$  is a 1-vertex in  $\Sigma_{\overrightarrow{m\hat{x}}}\Sigma_m K$ . By Lemma 4.7 we conclude that the  $A_4$ -factor of  $\Sigma_\xi \Sigma_{\overrightarrow{m\hat{x}}}\Sigma_m K$  contains at least a wall. Taking spherical join with the directions to the 7-vertices  $\overrightarrow{my_2}$  and  $\overrightarrow{my_1}$  we find a wall in  $\Sigma_\xi \Sigma_{\overrightarrow{m\hat{x}}}\Sigma_m K$ . This implies that  $\Sigma_{\overrightarrow{mu''}}\Sigma_m K$  contains at least a wall. Since  $\overrightarrow{mu''}$  is extendable, it follows that  $\Sigma_{u''} K$  contains a wall. But this contradicts Lemma 5.5.  $\square$

In a special case we can also exclude 8-vertices, whose links contain a 3-sphere:

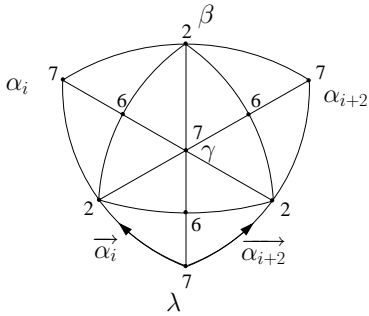
**Lemma 5.7.**  *$K$  contains no 8A-vertices  $x$ , such that  $\Sigma_x K$  contains a singular 3-sphere  $S$  with the following properties:  $S$  contains a pair of antipodal 2-vertices  $\xi_1, \xi_2$ , such that  $\Sigma_{\xi_i} S$  is a singular 2-sphere spanned by three pairwise orthogonal 7-vertices. Furthermore, all 7-vertices in  $S$  are  $\frac{\pi}{3}$ -extendable to 8A-vertices.*

Notice that all 7-vertices in  $S$  are adjacent to  $\xi_i$  for some  $i = 1, 2$ . Indeed, a segment in  $\Sigma_x K$  (of type  $E_7$ ) connecting a 2- and a 7-vertex at distance  $\leq \frac{\pi}{2}$  is of type 27 or 217. This last segment cannot occur between  $\xi_i$  and a 7-vertex in  $S$  because  $\Sigma_{\xi_i} S$  does not contain 1-vertices. Observe also, that the link  $\Sigma_\lambda S$  of a 7-vertex  $\lambda \in S$  contains a singular circle of type 2626262: suppose w.l.o.g. that  $\lambda$  is adjacent to  $\xi_1$ , then  $\overrightarrow{\xi_1 \lambda}$  is contained in a circle in  $\Sigma_{\xi_1} S$  of type 767676767. In particular  $\Sigma_{\overrightarrow{\lambda \xi_1}} \Sigma_\lambda S$  contains a pair of antipodal 6-vertices. It follows that the antipodal directions  $\overrightarrow{\lambda \xi_1}$  and  $\overrightarrow{\lambda \xi_2}$  are contained in a singular circle in  $\Sigma_\lambda S$  of type 2626262.

*Proof of Lemma 5.7.* Suppose there are such 8A-vertices. Let  $x_1, x_2, x_3 \in K$  be such 8A-vertices as in configuration  $*$ , and let  $S_{x_i} \subset \Sigma_{x_i} K$  denote the corresponding 3-spheres in their links. Let  $y_3, z_1 \in K$  be as in the notation of the configuration  $*$ . Suppose that there is a 7-vertex  $\xi \in S_{x_1} \subset \Sigma_{x_1} K$ , such that  $d(\xi, \zeta) = \arccos(-\frac{1}{\sqrt{3}})$  for  $\zeta := \overrightarrow{x_1 z_1}$ . The segment  $\xi \zeta$  is of type 7672. By assumption, there exists an 8A-vertex  $x \in K$ , such that  $d(x_1, x) = \frac{\pi}{3}$  and  $\overrightarrow{x_1 x} = \xi$ . Under these circumstances we obtain the configuration  $**$ . We use the same notation as in the configuration  $**$ . Let  $\alpha_i \in \Sigma_{x_3} K$  for  $i = 1, \dots, 4$  be the directions  $\overrightarrow{x_3 x_1}, \overrightarrow{x_3 x}, \overrightarrow{x_3 w}$  and  $\overrightarrow{x_3 y_3}$ . Let  $\beta := \overrightarrow{x_3 z}$ . Then the 7-vertices  $\alpha_i$  are adjacent to the 2-vertex  $\beta$ . And the directions  $\overrightarrow{\beta \alpha_i}$  lie on a circle  $\kappa$  of type 767676767 contained in  $\Sigma_\beta \Sigma_{x_3} K$ .



Suppose again that there is a 7-vertex  $\lambda$  in the 3-sphere  $S_{x_3} \subset \Sigma_{x_3} K$ , such that  $d(\beta, \lambda) = \arccos(-\frac{1}{\sqrt{3}})$ . So the segment  $\beta \lambda$  is of type 2767. Recall that the 7-vertices  $\alpha_i$  are  $\frac{2\pi}{3}$ -extendable and  $\lambda$  is  $\frac{\pi}{3}$ -extendable to an 8A-vertex, so they cannot be antipodal. It follows that  $\angle_\beta(\lambda, \alpha_i) = \frac{\pi}{2}$  and  $d(\alpha_i, \lambda) = \arccos(-\frac{1}{3})$ . The segments  $\alpha_i \lambda$  are of type 727. Let  $\gamma \in \Sigma_{x_3} K$  be the 7-vertex on the interior of the segment  $\beta \lambda$ . Then  $\gamma$  is the center of an equilateral spherical triangle  $CH(\lambda, \alpha_1, \alpha_3)$  with sides of type 727. We are now in the situation of the configuration  $**$  (compare with the triangle  $CH(\xi, \xi_2, \xi_3)$  in the definition of the configuration  $**$ ). It follows that  $\gamma$  is  $\frac{2\pi}{3}$ -extendable.

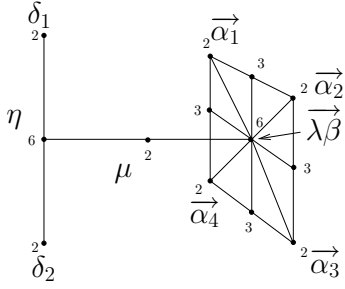


The convex hull  $CH(\kappa, \overrightarrow{\beta \lambda})$  is a 2-dimensional hemisphere centered at  $\overrightarrow{\beta \lambda}$ . Hence,  $\Sigma_{\overrightarrow{\beta \lambda}} \Sigma_\beta \Sigma_{x_3} K$  (of type  $\overset{3}{1} \rightarrow \overset{4}{5} \leftarrow \overset{6}{6}$ ) contains a circle of type 656565656. This is equivalent to  $\Sigma_{\overrightarrow{\lambda \beta}} \Sigma_\lambda \Sigma_{x_3} K$  (of type  $\overset{2}{2} \leftarrow \overset{3}{3} \rightarrow \overset{4}{5} \leftarrow \overset{5}{1}$ ) containing a circle of type 232323232. Note that the 2-vertices on this circle correspond to the 2-vertices  $m(\lambda, \alpha_i) \in \Sigma_{x_3} K$  (consider the equilateral spherical triangles  $CH(\lambda, \alpha_i, \alpha_{i+2})$  with sides of type 727). Let  $\overrightarrow{\alpha_i} := \overrightarrow{\lambda \alpha_i} \in \Sigma_\lambda \Sigma_{x_3} K$ .

Recall that the link  $\Sigma_\lambda S_{x_3}$  contains a circle  $c$  of type 2626262 and notice that  $\overrightarrow{\lambda \beta}$  cannot be antipodal to any of the 2-vertices on this circle: otherwise, we find a 7-vertex in the 3-sphere  $S_{x_3}$  antipodal to  $\gamma$ . This cannot happen, because  $\gamma$  is  $\frac{2\pi}{3}$ -extendable and the 7-vertices in  $S_{x_3}$



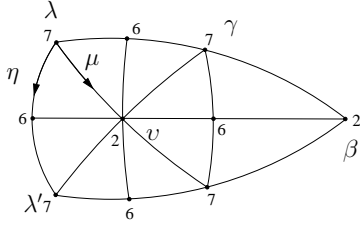
are  $\frac{\pi}{3}$ -extendable to 8A-vertices. It is also clear that  $\overrightarrow{\lambda\beta}$  cannot have distance  $< \frac{\pi}{2}$  to the three 6-vertices on the circle  $c$ , otherwise  $c$  would be contained in a ball centered at  $\overrightarrow{\lambda\beta}$  with radius  $< \frac{\pi}{2}$ , but this is not possible since  $\text{diam}(c) = \pi$ .



Therefore we can find a 6-vertex  $\eta$  on the circle  $c \subset \Sigma_\lambda S_{x_3}$  such that  $d(\eta, \overrightarrow{\lambda\beta}) \geq \frac{\pi}{2}$ . Hence,  $d(\eta, \overrightarrow{\lambda\beta}) = \frac{2\pi}{3}$  and the segment  $\eta\lambda\beta$  is of type 626. Let  $\mu := m(\eta, \overrightarrow{\lambda\beta})$ . Let also  $\delta_1, \delta_2$  be the two 2-vertices in the circle  $c \subset \Sigma_\lambda \Sigma_{x_3} K$  adjacent to  $\eta$ .

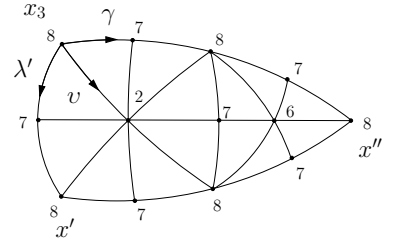
We have already seen, that  $\overrightarrow{\lambda\beta}$  cannot be antipodal to  $\delta_i$ . This implies that  $\angle_\eta(\delta_i, \mu) = \frac{\pi}{2}$  and these angles are of type 232. It follows that  $\Sigma_{\overrightarrow{\eta\mu}} \Sigma_\eta \Sigma_\lambda \Sigma_{x_3} K$  (of type  $D_4$ :  $\begin{matrix} 3 & 4 & 5 \\ & \downarrow & \\ & 1 & \end{matrix}$ ) contains a pair of

antipodal 3-vertices. On the other hand, if  $\eta$  is antipodal to some  $\overrightarrow{\alpha_i}$ , then  $\alpha_i \in \Sigma_{x_3} K$  has an antipode in  $S_{x_3}$ , but this cannot happen either, because  $\alpha_i$  is  $\frac{2\pi}{3}$ -extendable in  $K$ . Therefore  $\angle_{\overrightarrow{\lambda\beta}}(\mu, \overrightarrow{\alpha_i}) = \frac{\pi}{2}$  and these angles are of type 232. It follows that  $\Sigma_{\overrightarrow{\lambda\beta\mu}} \Sigma_{\overrightarrow{\lambda\beta}} \Sigma_\lambda \Sigma_{x_3} K$  contains a singular circle of type 343434343. This in turn implies, that  $\Sigma_{\overrightarrow{\eta\mu}} \Sigma_\eta \Sigma_\lambda \Sigma_{x_3} K$  contains a singular circle of type 141414141, because the antipode of a 3- (4)-vertex in  $\Sigma_\mu \Sigma_\lambda \Sigma_{x_3} K$ , of type  $\begin{matrix} 3 & 4 & 5 & 6 \\ & \downarrow & & \\ 1 & & & \end{matrix}$ , adjacent to  $\overrightarrow{\mu\lambda\beta}$  is a 1- (4)-vertex adjacent to  $\overrightarrow{\mu\eta}$ . We apply now Lemma 4.8 to conclude that  $\Sigma_{\overrightarrow{\eta\mu}} \Sigma_\eta \Sigma_\lambda \Sigma_{x_3} K$  contains a wall. Hence  $\Sigma_\mu \Sigma_\lambda \Sigma_{x_3} K$  contains a wall.



Let  $\lambda' \in S_{x_3}$  be the 7-vertex at distance  $\arccos(\frac{1}{3})$  to  $\lambda$ , so that  $\overrightarrow{\lambda\lambda'} = \eta$ . By considering the spherical triangle  $CH(\lambda, \lambda', \beta) \subset \Sigma_{x_3} K$  we deduce that  $\mu$  is  $\arccos(-\frac{1}{3})$ -extendable in  $\Sigma_{x_3} K$ . Let  $v$  be the 2-vertex in  $\Sigma_{x_3} K$  adjacent to  $\lambda$  with  $\overrightarrow{\lambda v} = \mu$ . It follows that  $\Sigma_v \Sigma_{x_3} K$  contains a wall.

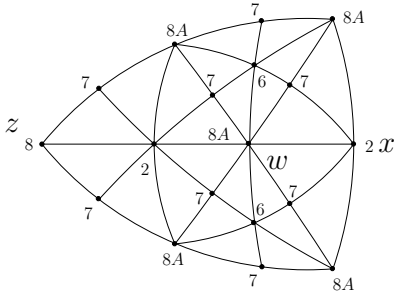
Recall that  $\gamma$  is  $\frac{2\pi}{3}$ -extendable and let  $x'' \in K$  be an 8-vertex with  $d(x_3, x'') = \frac{2\pi}{3}$  and  $\overrightarrow{x_3 x''} = \gamma$ . Since  $\lambda' \in S_{x_3}$ , it is  $\frac{\pi}{3}$ -extendable. Let  $x' \in K$  be an 8-vertex, so that  $d(x_3, x') = \frac{\pi}{3}$  and  $\overrightarrow{x_3 x'} = \lambda'$ . Consider the spherical triangle  $CH(x_3, x'', x')$ . One sees that  $v$  is  $\frac{\pi}{2}$ -extendable in  $K$ , thus we have found a 2-vertex in  $K$ , whose link contains a wall, contradicting Lemma 5.5.



So it follows that  $d(\beta, \lambda) \leq \frac{\pi}{2}$  for all 7-vertices  $\lambda \in S_{x_3}$ . Since  $S_{x_3}$  is the convex hull of the 7-vertices contained in it, this implies that  $d(\beta, S_{x_3}) \equiv \frac{\pi}{2}$  and  $s := \Sigma_\beta CH(\beta, S_{x_3})$  is a 3-sphere. Let  $\theta \in S_{x_3} \subset \Sigma_{x_3} K$  be a 2-vertex, so that  $\Sigma_\theta S_{x_3}$  is a 2-sphere spanned by three pairwise orthogonal 7-vertices (compare with the description of the 3-sphere  $S_{x_3}$ ). The segment  $\theta\beta$  is of type 262.

Notice that  $d(\beta, S_{x_3}) \equiv \frac{\pi}{2}$  implies that  $d(\overrightarrow{\theta\beta}, \Sigma_\theta S_{x_3}) \equiv \frac{\pi}{2}$ . It follows that  $\Sigma_{\overrightarrow{\theta\beta}} \Sigma_\theta CH(\beta, S_{x_3})$  (subset of a building of type  $\begin{matrix} 3 & 4 & 5 & 7 \\ & \downarrow & & \\ & 1 & & \end{matrix}$ ) is a 2-sphere. Notice that in the building  $\Sigma_\theta \Sigma_{x_3} B$  of type  $\begin{matrix} 3 & 4 & 5 & 6 & 7 \\ & \downarrow & & & \\ 1 & & & & \end{matrix}$ ; two 7-, 6-vertices at distance  $\frac{\pi}{2}$  are joined by a segment of type 756. This implies that  $\Sigma_{\overrightarrow{\theta\beta}} \Sigma_\theta CH(\beta, S_{x_3})$  is spanned by three pairwise orthogonal 5-vertices. Such a 2-sphere in





For the second assertion, suppose there is an 8-vertex  $z \in K$  with  $d(x, z) = \frac{3\pi}{4}$ . Since all 8-vertices in  $S$  correspond to 8A-vertices in  $K$ , the 8-vertex  $\overrightarrow{xz}$  must be orthogonal to the 8-vertices in  $S$ . In both cases (of type 757 or  $\frac{\pi}{3}$ ),  $S$  contains a singular 2-sphere spanned by three pairwise orthogonal 8-vertices (cf. Section 3.1). This implies that  $\Sigma_{\overrightarrow{xz}}\Sigma_x K$  contains a 2-sphere spanned by three pairwise orthogonal 7-vertices. Let  $w$  be the 8-vertex in  $xz$  adjacent to  $x$ . Recall that  $x$  is a 2A-vertex,

therefore  $w$  is an 8A-vertex. Then  $\Sigma_w K$  contains a 3-sphere as described in the statement of Lemma 5.7. It also follows that the 7-vertices in this sphere are  $\frac{\pi}{3}$ -extendable to 8A-vertices, contradicting Lemma 5.7.  $\square$

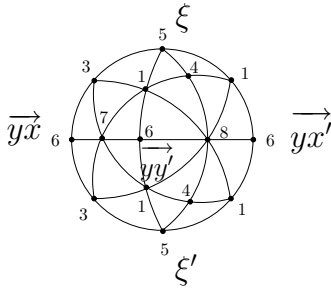
**Lemma 5.9.**  $K$  contains no 2-vertices  $x$ , such that  $\Sigma_x K$  contains a singular 4-sphere  $S$  of type  $\frac{\pi}{3}$ .

*Proof.* Let  $x$  be such a 2-vertex. It follows from Lemma 5.8 that  $x$  is a 2A-vertex and  $\text{rad}(x, 8\text{-vert. in } K) \leq \arccos(-\frac{1}{2\sqrt{2}})$ . As in the proof of Lemma 5.5, we deduce that  $\text{diam}(G \cdot x) \leq \frac{2\pi}{3}$ . Let  $x' \in G \cdot x$  with  $d(x, x') = \text{diam}(G \cdot x)$ .

Case 1:  $\text{diam}(G \cdot x) = \frac{2\pi}{3}$ . The segment  $xx'$  is of type 26262. As in the proof of Lemma 5.3 we deduce that the 6-vertex  $\overrightarrow{xx'}$  has distance  $\frac{\pi}{2}$  to the 8-vertices in  $S$ . If  $S' \subset S$  is the 3-sphere spanned by the 8-vertices in  $S$ , then  $d(\overrightarrow{xx'}, S') \equiv \frac{\pi}{2}$ . It follows that  $\Sigma_{\overrightarrow{xx'}}\Sigma_x K$  (of type  $\begin{matrix} 3 & 4 & 5 \\ & \downarrow & \\ & 1 & \end{matrix} \begin{matrix} 7 \\ \leftarrow \end{matrix} \begin{matrix} 8 \end{matrix}$ ) contains a 3-sphere spanned by four pairwise orthogonal 5-vertices, this sphere is an apartment in the  $D_4$ -factor. Let  $y := m(x, x')$ . Then the link  $\Sigma_{\overrightarrow{yx}}\Sigma_y K$  (again of type  $\begin{matrix} 3 & 4 & 5 \\ & \downarrow & \\ & 1 & \end{matrix} \begin{matrix} 7 \\ \leftarrow \end{matrix} \begin{matrix} 8 \end{matrix}$ ) contains also an apartment in the  $D_4$ -factor. This is a 3-sphere spanned by a simplex of type 1345. This implies that the link  $\Sigma_y K$  contains a singular 4-sphere  $S_y$  spanned by a simplex of type 13456. Hence  $S_y$  is of type  $\frac{\pi}{3}$  and the 6-vertices  $\overrightarrow{yx}, \overrightarrow{yx'}$  are orthogonal to the 3-sphere  $S'_y \subset S_y$  spanned by the 8-vertices in  $S_y$ . To see this consider the vector space model of the Coxeter complex of type  $D_7$  introduced in the Appendix A. The sphere  $S_y$  can be identified with the sphere  $\{x_5 = x_6 = x_7\} \cap S^6 \subset \mathbb{R}^7$  and  $S'_y$ , with the sphere  $\{x_5 = x_6 = x_7 = 0\} \cap S^6$ . A 5-vertex in  $S'_y$  is of the form  $(\pm 1, \dots, \pm 1, 0, 0, 0)$  and a 6-vertex orthogonal to this sphere must be of the form  $(0, \dots, 0, \pm 1, \pm 1, \pm 1)$ . Hence, a 5-vertex in  $S'_y$  and a 6-vertex orthogonal to  $S'_y$  are connected by a segment of type 536 or 516.

As in the beginning of the proof, we obtain that  $\text{rad}(y, 8\text{-vert. in } K) \leq \arccos(-\frac{1}{2\sqrt{2}})$  and  $\text{diam}(G \cdot y) \leq \frac{2\pi}{3}$ . We assume again that  $\text{diam}(G \cdot y) = \frac{2\pi}{3}$  and let  $y' \in G \cdot y$  have distance  $\frac{2\pi}{3}$  to  $y$ . It follows as above, that  $\Sigma_{\overrightarrow{yy'}}\Sigma_y K$  contains an apartment in the  $D_4$ -factor.

Let  $\xi, \xi' \in S'_y$  be antipodal 5-vertices. The vertices  $\overrightarrow{yx}, \overrightarrow{yx'}, \xi$  and  $\xi'$  lie on a singular circle of type 635161536 contained in  $S_y$ . The link  $\Sigma_\xi \Sigma_y K$  is of type  $A_3 \circ A_3$  and has Dynkin diagram  $\begin{matrix} 3 & 4 & 1 & 6 & 7 & 8 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{matrix}$ . Notice that  $\Sigma_\xi S'_y$  is an apartment in the second  $A_3$ -factor. Therefore the second factor in the spherical join splitting of  $\Sigma_\xi \Sigma_y K$  is a subbuilding. Since  $\text{rad}(y, 8\text{-vert. in } K) \leq \arccos(-\frac{1}{2\sqrt{2}})$ , this implies as above that  $d(\overrightarrow{yy'}, S'_y) \equiv \frac{\pi}{2}$ . In particular,  $d(\overrightarrow{yy'}, \xi) = \frac{\pi}{2}$  and the direction  $\overrightarrow{\xi yy'}$  must be orthogonal to the 2-sphere  $\Sigma_\xi S'_y$ . Recall that this sphere is an apartment in the second  $A_3$ -factor. Thus  $\overrightarrow{\xi yy'}$  must lie on the  $\begin{matrix} 3 & 4 & 1 \\ \vdots & \vdots & \vdots \end{matrix}$ -factor of  $\Sigma_\xi \Sigma_y K$ .



It follows from this that the segments  $\xi\vec{yy'}$  and  $\xi'\vec{yy'}$  must be of type 536 or 516. Further, since  $d(\xi, \vec{yy'}) + d(\xi', \vec{yy'}) = d(\xi, \xi') = \pi$ , the segments are of the same type. Observe also, that  $\vec{yy'}$  cannot be antipodal to  $\vec{y\bar{x}}$  or  $\vec{yx'}$ , otherwise the 2A-vertex  $y'$  would be antipodal to  $x$  or  $x'$ . Suppose w.l.o.g. that the segments  $\xi\vec{yy'}\xi'$  and  $\xi\vec{yx'}\xi'$  are of type 51615. This implies that the segment  $\xi\vec{y\bar{x}}\xi'$  is of type 53635.

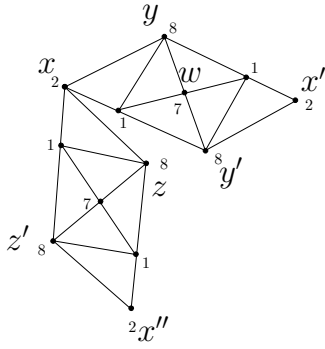
Since  $\vec{yy'}$  is not antipodal to  $\vec{y\bar{x}}$ , then the directions  $\vec{\xi\vec{y\bar{x}}}$  and  $\vec{\xi\vec{yy'}}$  of type 3 and 1, respectively, cannot be antipodal, thus, they are adjacent (recall that these directions lie in a building of type  $\mathfrak{3}\text{--}\mathfrak{4}\text{--}\mathfrak{4}$ ). This implies that the segment  $\vec{y\bar{x}}\vec{yy'}$  has length  $\arccos(\frac{1}{3})$  and is of type 676. It also follows that  $\vec{yy'}$  lies on a segment of length  $\pi$  and type 67686 connecting  $\vec{y\bar{x}}$  and  $\vec{yx'}$ . Therefore, the segment  $\vec{yx'}\vec{yy'}$  has length  $\arccos(-\frac{1}{3})$  and is of type 686. Hence,  $\Sigma_{\vec{yy'}}\Sigma_y K$  contains antipodal 7- and 8-vertices, that is, it contains a wall in the  $A_2$ -factor. Together with the apartment in the  $D_4$ -factor (compare with the beginning of Case 1), this implies that the link  $\Sigma_{\vec{yy'}}\Sigma_y K$  contains a wall. It follows that the link in  $K$  of the 2-vertex  $m(y, y')$  contains a wall, contradicting Lemma 5.5.

Thus,  $\text{diam}(G \cdot y) = \arccos(-\frac{1}{4})$  and by relabeling  $y$  by  $x$  we have reduced the possibilities to the following case.

Case 2:  $\text{diam}(G \cdot x) = \arccos(-\frac{1}{4})$ . The simplicial convex hull  $C$  of  $xx'$  is 2-dimensional. Let  $y, y' \in C$  be the 8-vertices adjacent to  $x$  and  $x'$ , respectively. If  $\vec{x\bar{y}}$  has an antipode in  $\Sigma_x K$ , then there would be an 8-vertex in  $K$  at distance  $\frac{3\pi}{4}$  to  $x'$ , but this is not possible (cf. proof of Lemma 5.3). It follows that  $d(\vec{x\bar{y}}, S') \equiv \frac{\pi}{2}$ , where  $S' \subset S$  is the 3-sphere spanned by the 8-vertices in  $S$ .  $\Sigma_{\vec{x\bar{y}}}CH(\vec{x\bar{y}}, S')$  is a 3-sphere spanned by four pairwise orthogonal 7-vertices.

Let  $w \in C$  be the 7-vertex  $m(x, x')$  and let  $x'' \in G \cdot x$  with  $d(w, x'') > \frac{\pi}{2}$ . The possible distances between 2- and 7-vertices in the Coxeter complex of type  $E_8$  are of the form  $\arccos(-\frac{k}{2\sqrt{6}})$  for  $k$  an integer (this can be deduced from the table of 2- and 7-vertices in Appendix A.4). Notice that  $d(x, w) = d(w, x') = \arccos(\frac{3}{2\sqrt{6}})$ . Triangle comparison for the triangle  $(x, x', x'')$  and  $\text{diam}(G \cdot x) \leq \arccos(-\frac{1}{4})$  imply that  $d(x'', w) = d(x'', m(x, x')) \leq \arccos(-\frac{1}{\sqrt{6}})$ . If  $d(w, x'') = \arccos(-\frac{1}{\sqrt{6}})$ , then by rigidity,  $CH(x, x', x'')$  is an equilateral spherical triangle with side lengths  $\arccos(-\frac{1}{4})$ . In particular  $d(x, x'') = \arccos(-\frac{1}{4})$  and  $\angle_x(x', x'') > \frac{\pi}{2}$ .

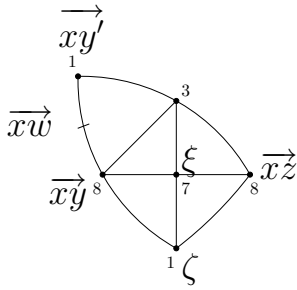
If  $d(w, x'') = \arccos(-\frac{1}{2\sqrt{6}})$ , we may assume w.l.o.g. that  $\angle_w(x, x'') \geq \frac{\pi}{2}$ . This implies that  $d(x, x'') \geq \arccos(-\frac{1}{8})$ , i.e.  $d(x, x'') = \arccos(-\frac{1}{4})$ . Again by triangle comparison and  $\angle_w(x, x'') \geq \frac{\pi}{2}$  we want to see that  $CH(x, w, x'')$  must be a spherical triangle: let  $\tilde{x}, \tilde{x}''$  be 2-vertices and let  $\tilde{w}$  be a 7-vertex in the Coxeter complex of type  $E_8$ , such that  $d(\tilde{x}, \tilde{w}) = d(x, w) = \arccos(\frac{3}{2\sqrt{6}})$ ,  $d(\tilde{w}, \tilde{x}'') = d(w, x'') = \arccos(-\frac{1}{2\sqrt{6}})$  and  $\angle_w(x, x'') = \angle_{\tilde{w}}(\tilde{x}, \tilde{x}'')$ . By triangle comparison,  $d(\tilde{x}, \tilde{x}'') \leq d(x, x'') = \arccos(-\frac{1}{4})$ , but since the angle  $\angle_{\tilde{w}}(\tilde{x}, \tilde{x}'') = \angle_w(x, x'') \geq \frac{\pi}{2}$ , then  $d(\tilde{x}, \tilde{x}'') > \frac{\pi}{2}$ . It follows that  $d(\tilde{x}, \tilde{x}'') = \arccos(-\frac{1}{4}) = d(x, x'')$  and by rigidity  $CH(x, w, x'')$  is a spherical triangle. We can now compute that  $\angle_x(x', x'') = \arccos(-\frac{1}{15}) > \frac{\pi}{2}$ .



Let  $C'$  be the 2-dimensional simplicial convex hull of  $xx''$  and let  $z, z' \in C'$  be the 8-vertices adjacent to  $x$  and  $x''$ . By considering the spherical triangle  $CH(x, x', y)$ , we can compute  $\angle_x(y, x') = \arccos(\frac{3}{\sqrt{15}}) < \frac{\pi}{4}$ . Then we can see that, if  $\vec{xy} = \vec{xz}$ , it follows  $\angle_x(x', x'') < \frac{\pi}{2}$ , thus  $\vec{xy} \neq \vec{xz}$ . They cannot be antipodal either, because  $\vec{xy}$  has no antipodes in  $\Sigma_x K$  (compare with the beginning of Case 2). Hence, the segment  $\vec{xy}\vec{xz}$  has length  $\frac{\pi}{2}$  and is of type 878.

Let  $\xi \in \Sigma_x K$  be the 7-vertex  $m(\vec{xy}, \vec{xz})$ . Notice that as for  $\vec{xy}$ , it also holds  $d(\vec{xz}, S') \equiv \frac{\pi}{2}$ . This implies that the convex hull of  $S'$  and the segment  $\vec{xy}\vec{xz}$  is isometric to the spherical join  $S' \circ \vec{xy}\vec{xz}$ . In particular,  $d(\xi, S') \equiv \frac{\pi}{2}$ . Notice that in a building of type  $D_7$  with Dynkin diagram  $\overset{3}{\bullet} \rightarrow \overset{4}{\bullet} \rightarrow \overset{5}{\bullet} \rightarrow \overset{6}{\bullet} \rightarrow \overset{7}{\bullet} \rightarrow \overset{8}{\bullet}$ , a 7- and an 8-vertex at distance  $\frac{\pi}{2}$  are joined by a segment of type 768. It follows that  $\Sigma_\xi \Sigma_x K$  (of type  $\overset{3}{\bullet} \rightarrow \overset{4}{\bullet} \rightarrow \overset{5}{\bullet} \rightarrow \overset{6}{\bullet} \rightarrow \overset{8}{\bullet}$ ) contains a 3-sphere spanned by four pairwise orthogonal 6-vertices. This 3-sphere is not simplicial, and its simplicial convex hull is an apartment in the  $D_5$ -factor of  $\Sigma_\xi \Sigma_x K$ . Since  $\{\vec{xy}, \vec{xz}\}$  is an apartment in the  $A_1$ -factor of  $\Sigma_\xi \Sigma_x K$ , it follows that  $\Sigma_\xi \Sigma_x K$  contains an apartment. In particular  $\xi$  is an interior 7-vertex in  $\Sigma_x K$ .

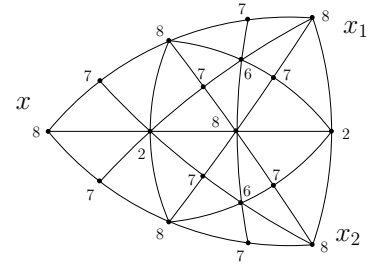
We can also see, that if both 1-vertices  $\vec{xy}'$  and  $\vec{xz}'$  are adjacent to  $\xi$ , then  $\angle_x(x', x'') < \frac{\pi}{2}$ , because in this case  $d(\xi, \vec{xw}) = d(\xi, \vec{xx}'') = \arccos(\frac{2\sqrt{2}}{\sqrt{15}}) < \frac{\pi}{4}$  (just consider the spherical triangle  $CH(\vec{xy}, \vec{xw}, \xi)$  with sides  $d(\vec{xy}, \vec{xw}) = \arccos(\frac{3}{\sqrt{15}})$ ,  $d(\vec{xy}, \xi) = \frac{\pi}{4}$  and angle  $\angle_{\vec{xy}}(\vec{xw}, \xi) = \arccos(\frac{1}{\sqrt{6}})$ ).



Therefore w.l.o.g.  $\vec{xy}'$  is not adjacent to  $\xi$ , but since both are adjacent to  $\vec{xy}$ , the angle  $\angle_{\vec{xy}}(\xi, \vec{xy}')$  must be of type 731, because  $\Sigma_{\vec{xy}} \Sigma_x B$  is of type  $D_6$  with Dynkin diagram  $\overset{3}{\bullet} \rightarrow \overset{4}{\bullet} \rightarrow \overset{5}{\bullet} \rightarrow \overset{6}{\bullet} \rightarrow \overset{7}{\bullet}$ . Now recall that  $\xi$  is an interior vertex in  $\Sigma_x K$ , this implies that we can find a 1-vertex  $\zeta \in \Sigma_x K$ , so that  $\vec{xy}'\vec{xz}'\zeta$  is a segment of type 181. Thus, the link  $\Sigma_{\vec{xy}} \Sigma_x K$  (of type  $D_6$ ) contains a pair of antipodal 1-vertices and a 3-sphere spanned by four pairwise orthogonal 7-vertices (compare with the beginning of Case 2).

We can apply Lemma 4.8 to see that  $\Sigma_{\vec{xy}} \Sigma_x K$  contains a wall. By Lemma 4.7,  $\Sigma_{\vec{xy}\vec{xw}} \Sigma_{\vec{xy}} \Sigma_x K$  contains at least a wall. This implies that  $\Sigma_{\vec{xw}} \Sigma_x K$  contains a wall and  $\Sigma_w K$  contains a wall of type 1, contradicting Lemma 5.6.  $\square$

Let  $x \in K$  be an 8A-vertex. We say that  $x$  has the *property T*, if there is no spherical triangle in  $K$  with 8A-vertices  $x, x_1$  and 8-vertex  $x_2$ , with side lengths  $d(x, x_i) = \frac{2\pi}{3}$ ,  $d(x_1, x_2) = \frac{\pi}{2}$ , and such that the direction  $\vec{xx_2}$  is  $\frac{2\pi}{3}$ -extendable to an 8A-vertex in  $K$ . This last assumption is fulfilled if e.g.  $x_2$  is also an 8A-vertex.

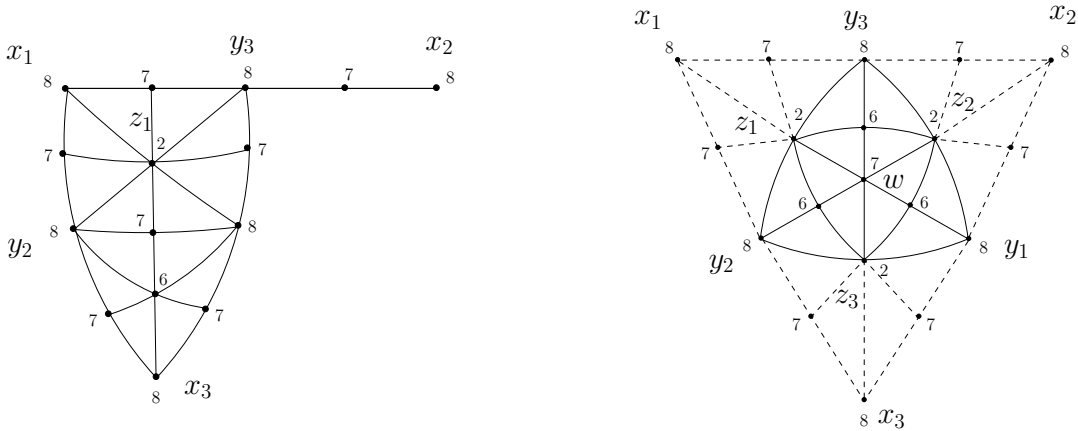


Let  $x_1, x_2, x_3 \in K$  be 8T-vertices as in configuration \*. If  $\angle_{y_3}(x_3, x_2) = \arccos(\frac{1}{3})$ , then the simplicial convex hull of  $y_3, x_3, m(y_3, x_2)$  is a spherical triangle with vertices  $x_3, y_3, x'_2$  and sides  $y_3x_3, x_3x'_2$  and  $x'_2y_3$  of type 87878, 828 and 878, respectively, and  $m(y_3, x_2) = m(y_3, x'_2)$ . It follows that the simplicial convex hull of  $x_1, m(y_3, x_2), x_3$  is a spherical triangle in  $K$  as ruled out by the property *T*, hence the property *T* implies that  $\angle_{y_3}(x_3, x_i) = \arccos(-\frac{1}{3})$  and

$d(x_3, x_i) = \frac{2\pi}{3}$  for  $i = 1, 2$ . Thus,  $\angle_{x_i}(x_{i-1}, x_{i+1}) = \arccos(-\frac{1}{3})$  for  $i = 1, 2$  (the indices to be understood modulo 3) and these angles are of type 727. Let  $y_1 := m(x_2, x_3)$  and  $y_2 := m(x_1, x_3)$ . Then it also follows that  $d(x_i, y_i) = \frac{2\pi}{3}$  for  $i = 1, 2$ . Consider the vertices  $x_1, x_3, x_2, y_2$ , then we are again in the situation of the configuration  $*$  (just exchange the indices  $2 \leftrightarrow 3$ ). It follows as above that  $\angle_{y_2}(x_2, x_3) = \arccos(-\frac{1}{3})$  because  $x_1$  is an  $8T$ -vertex. This implies that  $\angle_{x_3}(x_1, x_2) = \arccos(-\frac{1}{3})$  as well, and this angle is of type 727.

The convex hulls  $CH(x_i, y_j, x_j)$  for distinct  $i, j = 1, 2, 3$  are isosceles spherical triangles with sides of type 87878, 87878 and 878. This implies  $d(y_i, y_{i+1}) = \frac{\pi}{2}$  and the segments  $y_i y_{i+1}$  are of type 828. The intersection  $CH(x_i, y_{i+1}, x_{i+1}) \cap CH(x_i, y_{i-1}, x_{i-1})$  is the spherical triangle  $CH(x_i, y_{i-1}, y_{i+1})$  with sides of type 878, 878 and 828. In particular the 8-vertices  $m(x_i, y_i)$  are pairwise distinct.

Observe that the 2-vertices  $\overrightarrow{y_3 y_2}, \overrightarrow{y_3 y_1} \in \Sigma_{y_3} K$  are adjacent to the antipodal 7-vertices  $\overrightarrow{y_3 x_1}, \overrightarrow{y_3 x_2}$ , respectively. This implies that  $d(\overrightarrow{y_3 y_2}, \overrightarrow{y_3 y_1}) \geq \arccos(\frac{1}{3}) > \frac{\pi}{3}$ , thus  $d(\overrightarrow{y_3 y_2}, \overrightarrow{y_3 y_1}) \geq \frac{\pi}{2}$ . On the other hand, triangle comparison for the triangle  $(y_1, y_2, y_3)$  implies  $d(\overrightarrow{y_3 y_2}, \overrightarrow{y_3 y_1}) \leq \frac{\pi}{2}$  and it follows that this triangle is rigid, i.e. the convex hull  $CH(y_1, y_2, y_3)$  is an equilateral spherical triangle with sides of type 828. Let  $z_i := m(y_i, y_{i-1})$ . Notice that  $z_i$  does not lie on the segment  $x_i y_i$  of type 87878. Let  $w$  be the 7-vertex at the center of the triangle  $CH(y_1, y_2, y_3)$  and consider the spherical triangles  $CH(x_i, z_i, y_i)$  for  $i = 1, 2, 3$  with sides of type 82, 2768 and 87878. Notice that  $w$  is the 7-vertex on the segments  $z_i y_i$ . It follows that  $w$  is adjacent to the 8A-vertices  $m(x_i, y_i)$  for  $i = 1, 2, 3$  and in particular,  $\Sigma_w K$  contains three pairwise antipodal 8-vertices.



We say that an  $8T$ -vertex  $x \in K$  has the *property  $T'$* , if  $\text{rad}(z_i, \{8\text{-vert. in } K\}) \leq \arccos(-\frac{1}{2\sqrt{2}})$  for  $i = 1, 2, 3$  and for any such configuration of vertices  $x_1, x_2, x_3 \in G \cdot x$ .

**Lemma 5.10.**  *$K$  contains no  $8T'$ -vertices.*

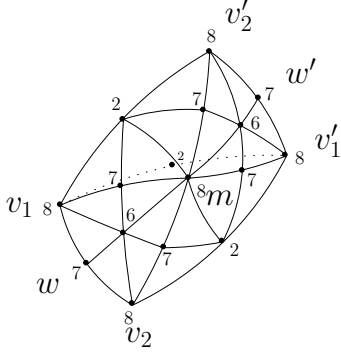
*Proof.* Suppose there are  $8T'$ -vertices. We use the notation as in the definition of the property  $T'$ . Let  $w$  be the center of the triangle  $CH(y_1, y_2, y_3)$ .

Let  $u \in K$  be an 8-vertex. Then for some  $i = 1, 2, 3$ ,  $\angle_w(z_i, u) \geq \frac{\pi}{2}$ . Suppose w.l.o.g. that it holds for  $i = 1$ . If  $d(w, u) = \frac{5\pi}{6}$ , then  $\overrightarrow{w u}$  is an 8-vertex and  $\angle_w(z_1, u) = \frac{\pi}{2}$ . It follows that  $d(u, z_1) = \frac{3\pi}{4}$ , but this contradicts the definition of the property  $T'$ . If  $d(w, u) = \arccos(-\frac{1}{\sqrt{3}})$ , then  $\overrightarrow{w u}$  is a 2-vertex and  $\angle_w(z_1, u) = \frac{2\pi}{3}$ . It follows again that  $d(u, z_1) = \frac{3\pi}{4}$ . Hence,  $d(w, u) \leq$

$\arccos(-\frac{1}{2\sqrt{3}})$  for all 8-vertices  $u \in K$  and as in the beginning of the proof of Lemma 5.4 we deduce by triangle comparison that if  $w' \in G \cdot w$ , then  $d(w, w') \leq \arccos(-\frac{1}{3})$ . We may also choose  $w'$ , so that  $d(w, w') > \frac{\pi}{2}$ .

Case 1:  $d(w, w') = \arccos(-\frac{1}{3})$ . If the segment  $ww'$  is singular of type 76867, then for some  $i = 1, 2, 3$ ,  $\angle_w(y_i, w') = \frac{2\pi}{3}$  and this angle is of type 626. It follows that  $d(w', y_i) = \arccos(-\frac{1}{\sqrt{3}})$ , a contradiction. If the simplicial convex hull of  $ww'$  is 2-dimensional, we can argue as in the proof of Lemma 5.4 (Case 1) to see that this case is not possible either.

Case 2:  $d(w, w') = \arccos(-\frac{1}{6})$ . The argument in the proof of Lemma 5.4 (Case 2) rules out the case where  $ww'$  has a 2-dimensional simplicial convex hull.

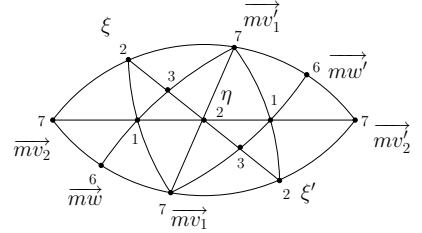


It remains to show that the case where the simplicial convex hull  $C$  of  $ww'$  is 3-dimensional is not possible either. Let  $v_1, v'_1 \in C$  be the 8-vertices adjacent to  $w$  and  $w'$ , respectively. Notice that they are 8A-vertices, otherwise an antipode of e.g.  $v_1$  in  $K$  would have distance  $\frac{5\pi}{6}$  to  $w$ ; but this cannot happen. Recall that there is an 8-vertex  $m \in C$ , such that  $mw$  and  $mw'$  are segments of type 867 and  $\angle_m(w, w') = \arccos(-\frac{3}{4})$ . Let  $v_2 \in K$  be an 8A-vertex adjacent to  $w$  and so that  $v_1wv_2$  is a segment of type 878. We can choose  $v_2$  to be one of the 8A-vertices  $m(x_i, y_i)$ . Define  $v'_2$  analogously. Then

the convex hulls  $CH(m, v_1, v_2)$  and  $CH(m, v'_1, v'_2)$  are equilateral spherical triangles with sides of type 878.

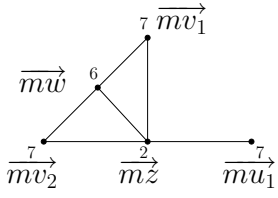
We want now to consider the convex hull  $C' := CH(C, v_2, v'_2)$ .

The link  $\Sigma_m C$  is a 2-dimensional spherical quadrilateral with vertices  $\overrightarrow{mw}$ ,  $\overrightarrow{mv_1}$ ,  $\overrightarrow{mw'}$  and  $\overrightarrow{mv'_1}$ . Notice that  $\overrightarrow{mv_2}\overrightarrow{mw}\overrightarrow{mv_1}$  and  $\overrightarrow{mv'_2}\overrightarrow{mw'}\overrightarrow{mv'_1}$  are segments of type 767. It follows that  $CH(\Sigma_m C, \overrightarrow{mv_2}, \overrightarrow{mv'_2})$  is a bigon connecting the antipodal 7-vertices  $\overrightarrow{mv_2}$  and  $\overrightarrow{mv'_2}$ . Then  $d(v_2, v'_2) = \frac{2\pi}{3}$  and  $m = m(v_2, v'_2)$ ,



in particular,  $m$  is an 8A-vertex. Let  $\xi, \xi' \in \Sigma_m C'$  be the 2-vertices  $m(\overrightarrow{mv'_1}, \overrightarrow{mv_2})$  and  $m(\overrightarrow{mv_1}, \overrightarrow{mv'_2})$ . Let  $\eta$  be the 2-vertex  $m(\overrightarrow{mv_1}, \overrightarrow{mv'_1})$ . The convex hulls  $CH(v_1, v_2, v'_2)$  and  $CH(v'_1, v'_2, v_2)$  are spherical triangles with sides of type 878, 87878 and 828.

Since  $m \in K$  is contained in the convex hull of the  $8T'$ -vertices, it is also contained in the convex hull of the  $8T$ -vertices. We can find another  $8T$ -vertex  $u_1 \in K$ , such that  $d(m, u_1) = \frac{2\pi}{3}$ . Notice that the 8A-vertex  $u_1$  cannot be antipodal to  $v_2$  or  $v'_2$ , in particular,  $\angle_m(u_1, v_2), \angle_m(u_1, v'_2) < \pi$ . Suppose w.l.o.g. that  $\angle_m(u_1, v_2) \geq \frac{\pi}{2}$ . Then  $\angle_m(u_1, v_2) = \arccos(-\frac{1}{3})$  and  $d(u_1, v_2) = \frac{2\pi}{3}$ .  $CH(v_2, m, u_1)$  is an isosceles spherical triangle (as in the configuration  $*$ ) with a 2-vertex  $z$  in its interior. Recall that  $d(w, u_1) \leq \arccos(-\frac{1}{2\sqrt{3}})$ . This implies that  $\angle_{v_2}(w, u_1) \leq \arccos(\frac{1}{3})$ . This angle cannot be 0, because  $\angle_{v_2}(m, w) = \arccos(\frac{1}{3})$  and  $\angle_{v_2}(m, u_1) = \arccos(-\frac{1}{3})$ . Thus  $\angle_{v_2}(w, u_1) = \arccos(\frac{1}{3})$  and it is of type 767.  $CH(\overrightarrow{v_2w}, \overrightarrow{v_2m}, \overrightarrow{v_2u_1})$  is then a spherical triangle with sides of type 767, 767 and 727. In particular  $w$  is adjacent to the 2-vertex  $z$ .

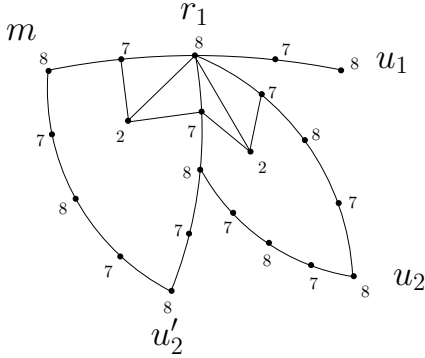


This consideration implies in the link  $\Sigma_m K$  that  $\overrightarrow{mz}$  and  $\overrightarrow{mw}$  are adjacent. Suppose that the segment  $\overrightarrow{mv_1}\overrightarrow{mu_1}$  is of type 727. This implies that the angle  $\angle_{\overrightarrow{mv_1}}(\overrightarrow{mu_1}, \xi')$  is of type 262. It follows that the segment  $\overrightarrow{mu_1}\xi'$  is of type 7672. Hence,  $d(u_1, m(v'_2, v_1)) = \frac{3\pi}{4}$  and  $CH(v_1, v'_2, u_1)$  is a spherical triangle with sides 87878, 87878 and 828. But this contradicts the definition of the property  $T$  for  $u_1$ . Therefore the segment  $\overrightarrow{mv_1}\overrightarrow{mu_1}$  is of type 767.

If  $\angle_m(u_1, v'_2) = \arccos(-\frac{1}{3})$  we argue analogously and conclude that the segment  $\overrightarrow{mv_1}\overrightarrow{mu_1}$  is of type 767. If  $\angle_m(u_1, v'_2) = \arccos(\frac{1}{3})$  we see as above that  $d(\overrightarrow{mu_1}, \xi) \leq \frac{\pi}{2}$ , otherwise we violate the property  $T$  for  $u_1$ . Using triangle comparison with the triangle  $(\xi, \overrightarrow{mu_1}, \overrightarrow{mv'_2})$  (or using the convexity of the ball centered at  $\overrightarrow{mu_1}$  with radius  $\frac{\pi}{2}$ ) we see that  $d(\overrightarrow{mv'_1}, \overrightarrow{mu_1}) \leq \arccos(\frac{1}{3})$ . Since  $\overrightarrow{mv_1}\overrightarrow{mu_1}$  is of type 767, then  $\overrightarrow{mu_1} \neq \overrightarrow{mv'_1}$ . Thus,  $d(\overrightarrow{mv'_1}, \overrightarrow{mu_1}) = \arccos(\frac{1}{3})$  and the segment  $\overrightarrow{mv'_1}\overrightarrow{mu_1}$  is of type 767 also in this case. It follows that  $CH(\overrightarrow{mv'_1}, \overrightarrow{mv_1}, \overrightarrow{mu_1})$  is a spherical triangle with sides 767, 767 and 727. In particular  $\overrightarrow{mu_1}$  is adjacent to  $\eta$ .

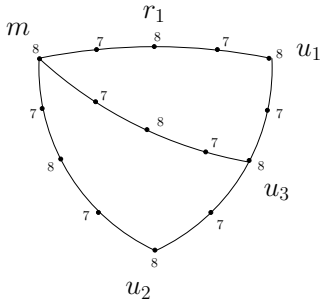
We have shown so far that any 7-vertex in  $\Sigma_m K$  that is  $\frac{2\pi}{3}$ -extendable to an  $8T$ -vertex in  $K$  must be adjacent to  $\eta$  and the segments connecting it with  $\overrightarrow{mv_1}$  and  $\overrightarrow{mv'_1}$  are of type 767.

Let  $r_1 := m(m, u_1) \in K$  and let  $u'_2 \in K$  be an  $8T$ -vertex with  $d(r_1, u'_2) = \frac{2\pi}{3}$ . Since  $u_1$  is an  $8T$ -vertex, the angle  $\angle_{r_1}(m, u'_2)$  cannot be of type 767. Hence, it is of type 727. If the angle  $\angle_{r_1}(u_1, u'_2)$  is also of type 727, then set  $u_2 := u'_2$ .



Otherwise, let  $u_2 \in K$  be another  $8T$ -vertex, so that  $d(u_2, m(r_1, u'_2)) = \frac{2\pi}{3}$ . Again, because  $u'_2$  is an  $8T$ -vertex, the angle  $\angle_{m(r_1, u'_2)}(r_1, u_2)$  is of type 727. In particular  $d(r_1, u_2) = \frac{2\pi}{3}$  and again  $\angle_{r_1}(m, u_2)$  is of type 727. We want to see now, that  $\angle_{r_1}(u_1, u_2)$  is also of type 727. Suppose that  $\angle_{r_1}(u_1, u_2)$  is of type 767. Then  $CH(\overrightarrow{r_1u_2}, \overrightarrow{r_1u_1}, \overrightarrow{r_1u'_2})$  is a spherical triangle with sides of type 767, 767 and 727. In particular  $\overrightarrow{r_1u_1}$  is adjacent to  $\delta := m(\overrightarrow{r_1u_2}, \overrightarrow{r_1u'_2})$ , this means that the segment  $\delta\overrightarrow{r_1m}$  is of type 2767. Notice that this is the configuration  $**$

for the vertices  $r_1, u'_2, u_2, m$ . This implies that  $CH(r_1, u_2, m(m, u'_2))$  is a spherical triangle with vertices of type  $8A$  and sides of type 87878, 87878 and 828 and  $u_2$  could not be an  $8T$ -vertex, a contradiction.

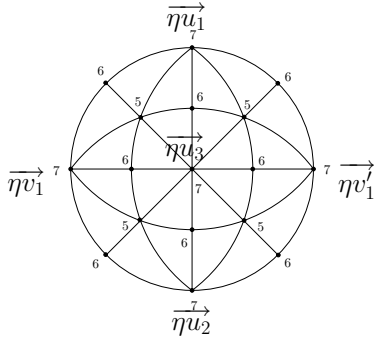


Thus  $\angle_{r_1}(u_1, u_2)$  is of type 727. This implies that  $d(u_1, u_2) = \frac{2\pi}{3}$  and  $\angle_{u_1}(m, u_2)$  is of type 727. Let  $u_3 \in K$  be the  $8A$ -vertex  $m(u_1, u_2)$ , then  $\angle_{u_1}(m, u_3)$  is of type 727 and this implies that  $d(m, u_3) = \frac{2\pi}{3}$ . Observe that  $u_3$  is not necessarily an  $8T$ -vertex. Notice that  $\overrightarrow{mu_1}\overrightarrow{mu_2}$  is of type 727 and recall that  $\overrightarrow{mu_i}$  is adjacent to  $\eta$  for  $i = 1, 2$ . It follows that  $\eta = m(\overrightarrow{mu_1}, \overrightarrow{mu_2})$ . In particular  $\eta$  is  $\frac{\pi}{2}$ -extendable in  $K$ . Consider the triangles  $(m, u_1, u_3)$  and  $(m, u_2, u_3)$ , then by triangle comparison, it follows that  $\angle_m(u_1, u_3), \angle_m(u_2, u_3) \leq \arccos(\frac{1}{3})$  and since  $\angle_m(u_1, u_2) = \arccos(-\frac{1}{3})$ , this implies that,  $\angle_m(u_1, u_i) = \arccos(\frac{1}{3})$  and  $\overrightarrow{mu_3}\overrightarrow{mu_i}$  is of type 767 for  $i = 1, 2$ . Hence,  $CH(\overrightarrow{mu_1}, \overrightarrow{mu_2}, \overrightarrow{mu_3})$  is a spherical triangle with sides of type 767, 767 and 727. In particular,  $\overrightarrow{mu_3}$  is adjacent to  $\eta$  as well.

Write  $\overrightarrow{\eta\star} := \overrightarrow{\eta m\star} \in \Sigma_\eta \Sigma_m K$ , where  $\star$  is any vertex in  $K$  adjacent to  $m$ , so that  $\overrightarrow{m\star} \in \Sigma_m K$



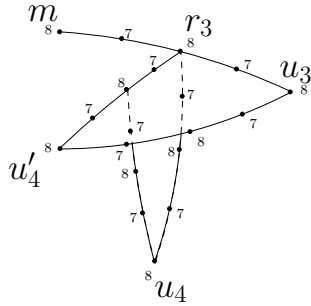
is adjacent to  $\eta$ .



The 7-vertices  $\vec{\eta v_1}$ ,  $\vec{\eta v_1}'$ ,  $\vec{\eta u_1}$  and  $\vec{\eta u_2}$  are the 7-vertices of a circle  $c \subset \Sigma_\eta \Sigma_m K$  of type 767676767, because as seen above,  $\vec{\eta u_i}$  for  $i = 1, 2$  is the midpoint of a geodesic of length  $\pi$  connecting  $\vec{\eta v_1}$  and  $\vec{\eta v_1}'$ , and  $\vec{\eta u_i}$  are antipodal for  $i = 1, 2$ . From the construction above we see that  $d(\vec{\eta u_3}, \vec{\eta u_i}) = \frac{\pi}{2}$  for  $i = 1, 2$  (the segments  $\vec{\eta u_3} \vec{\eta u_i}$  are of type 767). Suppose  $\vec{\eta u_3}$  is antipodal to  $\vec{\eta v_1}$ . This would imply that the segment  $\vec{m u_3} \vec{m w} \subset \Sigma_m K$  is of type 7316 and therefore  $d(\vec{m u_3}, \vec{m w}) > \frac{\pi}{2}$  (compare with the figure for  $\Sigma_m C'$  above). Consider now the triangle  $(w, m, u_3)$ ,

it has sides  $d(m, w) = \arccos(\frac{1}{\sqrt{3}})$ ,  $d(m, u_3) = \frac{2\pi}{3}$  and angle  $\angle_m(w, u_3) > \frac{\pi}{2}$ . It follows that  $d(w, u_3) > \arccos(-\frac{1}{2\sqrt{3}})$ , which is not possible. Hence,  $d(\vec{\eta u_3}, \vec{\eta v_1}) = d(\vec{\eta u_3}, \vec{\eta v_1}') = \frac{\pi}{2}$ . Therefore  $\vec{\eta u_3}$  is the center of a 2-dimensional hemisphere in  $\Sigma_\eta \Sigma_m K$  bounded by  $c$ .

Let  $r_3 := (m, u_3) \in K$  and let  $u'_4 \in K$  be another 8T-vertex, so that  $d(r_3, u'_4) = \frac{2\pi}{3}$ . Recall that  $u_3$  is not necessarily an 8T-vertex, therefore we cannot conclude directly that  $\angle_{r_3}(m, u'_4)$  is of type 727. If  $\angle_{r_3}(m, u'_4)$  is actually of type 727, then set  $u_4 := u'_4$ .

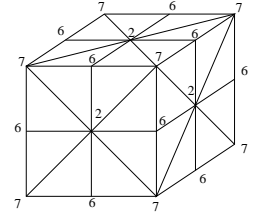


Otherwise (i.e. if  $\angle_{r_3}(m, u'_4)$  is of type 767), let  $u_4 \in K$  be an 8T-vertex, so that  $d(u_4, m(r_3, u'_4)) = \frac{2\pi}{3}$ . Then, since  $u'_4$  is an 8T-vertex, the angle  $\angle_{m(r_3, u'_4)}(r_3, u_4)$  must be of type 727. This implies that  $d(r_3, u_4) = \frac{2\pi}{3}$  and the angle  $\angle_{r_3}(u_4, u'_4)$  is of type 727. It follows that  $\angle_{r_3}(m, u_4)$  is of type 727, otherwise (as in the argument above for  $u_2$ ) we find the configuration \*\* and  $CH(u_3, u_4, m(r_3, u'_4))$  is a spherical triangle with sides of type 87878, 87878 and 828, contradicting the property  $T$  for  $u_4$ . From this we conclude that  $d(m, u_4) = \frac{2\pi}{3}$  and  $\angle_m(u_3, u_4)$  is of type 727. Recall that  $\vec{m u_4}$  must be adjacent to  $\eta$ . This implies that  $\vec{\eta u_4}$  is antipodal to  $\vec{\eta u_3}$ .

Thus,  $\Sigma_\eta \Sigma_m K$  (of type  $D_6$ ) contains a singular 2-sphere spanned by 3 pairwise orthogonal 7-vertices. Recall that it also contains a pair of antipodal 3-vertices  $\vec{\eta \xi}$  and  $\vec{\eta \xi}'$ . Lemma 4.8 implies that  $\Sigma_\eta \Sigma_m K$  contains a 3-sphere spanned by a simplex of type 1567. Since  $\eta$  is  $\frac{\pi}{2}$ -extendable in  $K$ , we have found a 2-vertex in  $K$ , whose link contains a 4-sphere spanned by a simplex of type 15678. This 4-sphere is of type  $\frac{\pi}{3}$  (this can be easily seen in the vector space realization of the Coxeter complex of type  $D_n$  presented in Appendix A). A contradiction to Lemma 5.9.  $\square$

Thus,  $\Sigma_\eta \Sigma_m K$  (of type  $D_6$ ) contains a singular 2-sphere spanned by 3 pairwise orthogonal 7-vertices. Recall that it also contains a pair of antipodal 3-vertices  $\vec{\eta \xi}$  and  $\vec{\eta \xi}'$ . Lemma 4.8 implies that  $\Sigma_\eta \Sigma_m K$  contains a 3-sphere spanned by a simplex of type 1567. Since  $\eta$  is  $\frac{\pi}{2}$ -extendable in  $K$ , we have found a 2-vertex in  $K$ , whose link contains a 4-sphere spanned by a simplex of type 15678. This 4-sphere is of type  $\frac{\pi}{3}$  (this can be easily seen in the vector space realization of the Coxeter complex of type  $D_n$  presented in Appendix A). A contradiction to Lemma 5.9.  $\square$

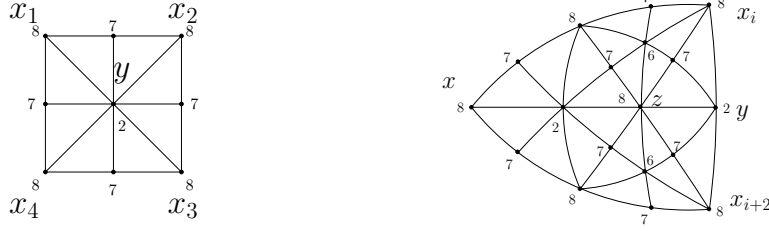
Let  $B_3$  be the property of an 8A-vertex  $x \in K$ , such that  $\Sigma_x K$  contains a singular 2-sphere with  $B_3$ -geometry  $\xrightarrow{7} \xrightarrow{6} \xrightarrow{2}$ , and such that all the 7-vertices in this sphere are  $\frac{\pi}{3}$ -extendable.



Consider the configuration \*\* and notice that the 8-vertex  $v$  on the segment  $z x_3$  (of type 2828) adjacent to  $z$  is an  $8B_3$ -vertex.

Another similar way of finding  $8B_3$ -vertices is the following. Let  $x_1, x_2, x_3, x_4 \in K$  be 8A-vertices adjacent to a 2-vertex  $y$ , so that  $CH(x_i)$  is a 2-dimensional spherical quadrilateral

with sides  $x_i x_{i+1}$  of type 878. Let  $x \in K$  be an 8-vertex at distance  $\frac{3\pi}{4}$  to  $y$ . Since the  $x_i$  are 8A-vertices, it follows that  $\angle_y(x, x_i) = \frac{\pi}{2}$ . This implies that  $\Sigma_{\overrightarrow{yx}} \Sigma_y K$  contains a singular circle of type 767676767. Let  $z$  be the 8-vertex in  $yx$  adjacent to  $y$ . Then  $\Sigma_z K$  contains a 2-sphere with  $B_3$ -geometry  $\overline{7-6-2}$ . Considering the spherical triangles  $CH(x, x_i, x_{i+2})$ , we see that the 7-vertices in this 2-sphere are  $\frac{\pi}{3}$ -extendable. Hence  $z$  is an  $8B_3$ -vertex.



Consider now the definition of the property  $T'$ . The 2-vertices  $z_i$  are centers of 2-dimensional spherical quadrilaterals as described above. In particular, if there are no  $8B_3$ -vertices in  $K$ , then it follows from the observation above, that  $rad(z_i, \{8\text{-vert. in } K\}) \leq \arccos(-\frac{1}{2\sqrt{2}})$  for  $i = 1, 2, 3$ . Hence, if  $K$  contains no  $8B_3$ -vertices, it follows that the property  $T$  implies the property  $T'$ .

Recall that our strategy is to find spheres of large dimension in the links of vertices of type 2 or 8. Notice that we have made the first step in this direction:

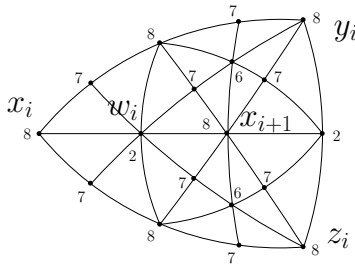
**Corollary 5.11.** *If  $K$  contains 8A-vertices, then it contains 8-vertices, whose links in  $K$  contain a singular circle.*

*Proof.* If  $K$  contains  $8B_3$ -vertices, we are done. Otherwise,  $8T \Rightarrow 8T'$ , and Lemma 5.10 implies that there are no  $8T$ -vertices in  $K$ . In particular, we find a spherical triangle in  $K$  with sides of type 87878, 87878 and 828. The link in  $K$  of the 8-vertex in the interior of this triangle contains a singular circle.  $\square$

Now we find 8-vertices, such that their links contain singular 2-spheres.

**Lemma 5.12.** *If  $K$  contains 8A-vertices, then it also contains  $8B_3$ -vertices.*

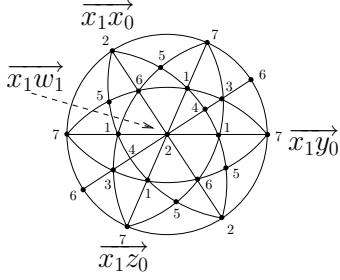
*Proof.* Suppose that  $K$  contains 8A-vertices but no  $8B_3$ -vertices. Then,  $8T \Rightarrow 8T'$  and Lemma 5.10 implies that there are no  $8T$ -vertices in  $K$ .



Hence, there are 8A-vertices  $x_0, y_0 \in K$  and an 8-vertex  $z_0 \in K$ , so that  $T_0 := CH(x_0, y_0, z_0)$  is a spherical triangle with sides of type 87878, 87878 and 828; where  $y_0 z_0$  is the side of type 828 (as in the definition of the property  $T$ ). Let  $x_1 \in K$  be the 8A-vertex on the segment  $x_0 m(y_0, z_0)$  (of type 8282) adjacent to the 2-vertex  $m(y_0, z_0)$ . Since  $x_1$  is not an  $8T$ -vertex, we can find 8-vertices  $y_1, z_1 \in K$  as vertices of a spherical triangle  $T_1 := CH(x_1, y_1, z_1)$  as above. Define  $x_i, y_i, z_i \in K$  and  $T_i \subset K$  inductively. Let  $w_i$  be the 2A-vertex  $m(x_i, x_{i+1})$ .

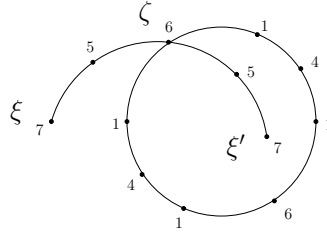
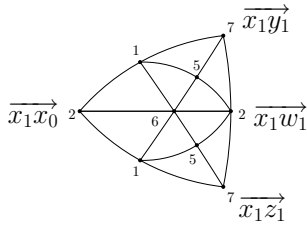
If  $\xi \in \Sigma_{x_i} K$  is a  $\frac{\pi}{3}$ -extendable 7-vertex and  $d(\xi, \overrightarrow{x_i x_{i+1}}) = \arccos(-\frac{1}{\sqrt{3}})$ , then we are in the setting of the configuration  $**$  because  $\overrightarrow{x_i y_i}$  and  $\overrightarrow{x_i z_i}$  are both  $\frac{2\pi}{3}$ -extendable to 8A-vertices (definition of the property  $T$ ). This implies that there are  $8B_3$ -vertices in  $K$ , contradicting our

assumption. Hence,  $\overrightarrow{x_i x_{i+1}}$  has distance  $\leq \frac{\pi}{2}$  to all  $\frac{\pi}{3}$ -extendable 7-vertices in  $\Sigma_{x_i} K$ . Notice also that  $d(\overrightarrow{x_i x_{i-1}}, \overrightarrow{x_i y_i})$  and  $d(\overrightarrow{x_i x_{i-1}}, \overrightarrow{x_i z_i})$  are both  $\leq \frac{\pi}{2}$ , otherwise  $w_{i-1}$  would have distance  $\frac{3\pi}{4}$  to the 8-vertex  $y_i$  or  $z_i$  and we would find an  $8B_3$ -vertex on the segment  $w_{i-1} y_i$  ( $w_{i-1} z_i$ ).



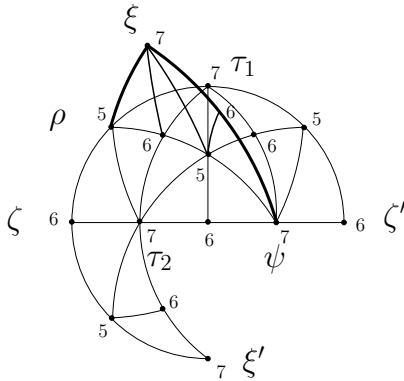
From these observations it follows, that  $\overrightarrow{x_1 w_1}$  has distance  $\equiv \frac{\pi}{2}$  to the circle  $\Sigma_{x_1} T_0$  of type 727672767. This implies that  $\Sigma_{\overrightarrow{x_1 w_1}} \Sigma_{x_1} K$  (of type  $\begin{smallmatrix} 3 & 4 & 5 & 6 & 7 \\ 1 & \nearrow & \rightarrow & \rightarrow & \rightarrow \end{smallmatrix}$ ) contains a singular circle of type 161416141. It also contains the pair of antipodal 7-vertices  $\xi := \overrightarrow{x_1 w_1 x_1 y_1}$  and  $\xi' := \overrightarrow{x_1 w_1 x_1 z_1}$ .

Since  $d(\overrightarrow{x_1 x_0}, \overrightarrow{x_1 y_1})$ ,  $d(\overrightarrow{x_1 x_0}, \overrightarrow{x_1 z_1}) \leq \frac{\pi}{2}$  and  $d(\overrightarrow{x_1 x_0}, \overrightarrow{x_1 w_1}) = \frac{\pi}{2}$ , it follows from triangle comparison that  $d(\overrightarrow{x_1 x_0}, \overrightarrow{x_1 y_1}) = d(\overrightarrow{x_1 x_0}, \overrightarrow{x_1 z_1}) = \frac{\pi}{2}$ , because the triangle  $(\overrightarrow{x_1 x_0}, \overrightarrow{x_1 y_1}, \overrightarrow{x_1 z_1})$  must be rigid. Let  $\zeta := \overrightarrow{x_1 w_1 x_1 x_0}$ . Then the segments  $\zeta \xi$  and  $\zeta \xi'$  have length  $\frac{\pi}{2}$  and are of type 657.



**Sublemma 5.13.**  $\Sigma_{\overrightarrow{x_1 w_1}} \Sigma_{x_1} K$  contains a singular circle of type 756575657. This circle contains the vertices  $\xi$ ,  $\xi'$  and  $\zeta$ .

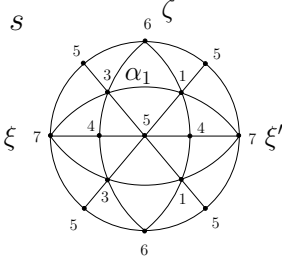
*Proof.* Let  $\zeta' \in \Sigma_{\overrightarrow{x_1 w_1}} \Sigma_{x_1} K$  be the 6-vertex in the circle of type 161416141 antipodal to  $\zeta$ . If  $d(\xi, \zeta') = \frac{\pi}{2}$ , then  $\zeta \xi \zeta'$  is a geodesic of type 65756. In particular,  $\overrightarrow{\xi \zeta}$  has an antipode in  $\Sigma_{\xi} \Sigma_{\overrightarrow{x_1 w_1}} \Sigma_{x_1} K$  and we find the desired circle. If  $d(\xi, \zeta') > \frac{\pi}{2}$ , then the segment  $\xi \zeta'$  is of type 7676.

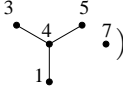


Let  $\rho$  be the 5-vertex on the segment  $\zeta \xi$  and let  $\psi$  be the 7-vertex on the segment  $\xi \zeta'$  adjacent to  $\zeta'$ . Consider the geodesics  $c_\rho$  and  $c_\psi$  of length  $\pi$  connecting  $\zeta$  and  $\zeta'$  through  $\rho$  and  $\psi$ . Let  $\tau_1$  be the 7-vertex at the center of  $c_\rho$  and  $\tau_2$  be the 7-vertex in  $c_\psi$  adjacent to  $\zeta$ . Then  $\rho$  and  $\tau_2$  are adjacent because  $\Sigma_{\zeta} \Sigma_{\overrightarrow{x_1 w_1}} \Sigma_{x_1} K$  is of type  $\begin{smallmatrix} 3 & 4 & 5 \\ 1 & \nearrow & \rightarrow \end{smallmatrix}$ .  $\xi$  cannot be adjacent to the 6-vertex at the center of  $c_\psi$ , otherwise it would have distance  $\frac{3\pi}{4}$  to  $\zeta$ . Thus, the intersection of the segments  $\xi \zeta'$  and  $c_\psi$  is the segment  $\psi \zeta'$ . Considering the spherical triangle  $CH(\rho, \xi, \psi)$  with sides of type 57, 767 and 7565, it follows that  $\xi$  is adjacent

to the 6-vertex  $m(\tau_1, \tau_2)$  on the segment  $\rho \psi$ . In particular,  $\xi'$  must be antipodal to at least one of  $\tau_1$  or  $\tau_2$ . Since  $\tau_2$  is adjacent to  $\zeta$  and  $d(\zeta, \xi') = \frac{\pi}{2}$ , then  $\xi'$  cannot be antipodal to  $\tau_2$ . It follows that  $\xi'$  and  $\tau_1$  are antipodal. Let finally  $c$  be the geodesic connecting  $\tau_1$  and  $\xi'$ , so that the initial direction coincides with  $\overrightarrow{\tau_1 \zeta'}$ . Then the initial direction of  $c$  at  $\xi'$  is antipodal to  $\overrightarrow{\xi' \zeta}$  and we can find the desired circle.  $\square$

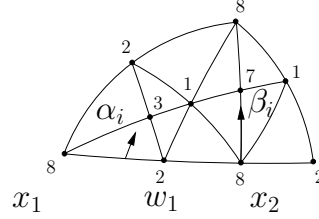
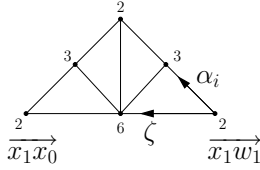
Continuation of proof of Lemma 5.12.



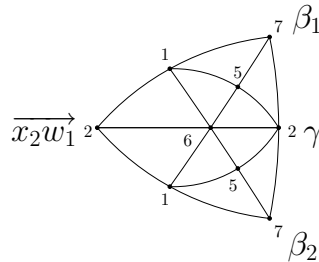
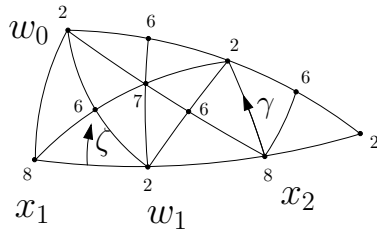
The link  $\Sigma_{\zeta}\Sigma_{\overrightarrow{x_1w_1}}\Sigma_{x_1}K$  (of type ) contains a pair of antipodal 5-vertices  $\overrightarrow{\zeta\xi}$  and  $\overrightarrow{\zeta\xi'}$  and a pair of antipodal 1-vertices. We apply Lemma 4.5 and Remark 4.6 to conclude that  $\Sigma_{\zeta}\Sigma_{\overrightarrow{x_1w_1}}\Sigma_{x_1}K$  contains a singular circle of type 5135135 with  $\overrightarrow{\zeta\xi}$  and  $\overrightarrow{\zeta\xi'}$  on it. It follows now from Sublemma 5.13 that  $\Sigma_{\overrightarrow{x_1w_1}}\Sigma_{x_1}K$  contains a singular 2-sphere  $s$  containing the vertices  $\zeta$ ,  $\xi$  and  $\xi'$ . Therefore  $\Sigma_{x_2}K$  contains a singular 3-sphere  $S$  containing the singular circle  $\Sigma_{x_2}T_1$ .

We investigate below which 7-vertices in  $S$  are  $\frac{\pi}{3}$ -extendable. Clearly the 7-vertices in  $\Sigma_{x_2}T_1 \subset S$  are  $\frac{\pi}{3}$ -extendable.

Let  $\alpha_1, \alpha_2 \in s$  be the 3-vertices adjacent to  $\zeta$  and recall that  $\zeta$  is  $\frac{\pi}{2}$ -extendable (to  $\overrightarrow{x_1x_0}$ ) in  $\Sigma_{x_1}K$ . This implies that  $\alpha_i$  is  $\frac{\pi}{3}$ -extendable to a segment of type 232 in  $\Sigma_{x_1}K$ . Therefore, we find 7-vertices  $\beta_1, \beta_2 \in S$  at distance  $\frac{\pi}{2}$  to  $\overrightarrow{x_2w_1}$  which are  $\frac{\pi}{3}$ -extendable in  $K$  (compare with the figure below).



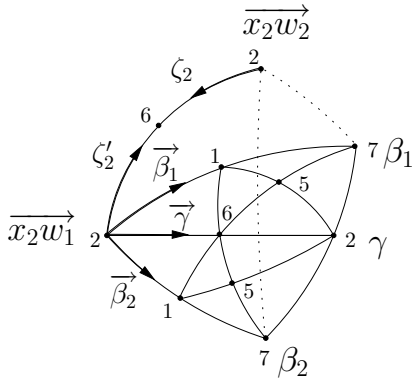
The segment  $\alpha_1\alpha_2 \subset \Sigma_{x_1w_1}\Sigma_{x_1}K$  is of type 363 with midpoint the 6-vertex  $\zeta$ , this implies that the angle  $\angle_{\overrightarrow{x_2w_1}}(\beta_1, \beta_2)$  is of type 161 and this implies in turn, that the segment  $\beta_1\beta_2 \subset \Sigma_{x_2}K$  is of type 727. Let  $\gamma \in S$  be the 2-vertex  $m(\beta_1, \beta_2)$ .



Let  $\zeta_2 := \overrightarrow{x_2w_2x_2x_1}$ . We can use the same argument as above to see that  $\Sigma_{\zeta_2}\Sigma_{\overrightarrow{x_2w_2}}\Sigma_{x_2}K$  contains a singular circle of type 5135135. We want to prove next that it also contains a pair of antipodal 7-vertices.

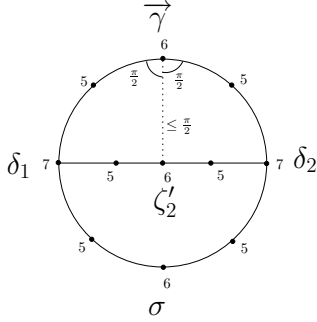
**Sublemma 5.14.** *The link  $\Sigma_{\zeta_2}\Sigma_{\overrightarrow{x_2w_2}}\Sigma_{x_2}K$  contains a pair of antipodal 7-vertices.*

*Proof.* Notice again that  $d(\overrightarrow{x_2w_2}, \Sigma_{x_2}T_1) \equiv \frac{\pi}{2}$ , in particular,  $d(\overrightarrow{x_2w_2}, \overrightarrow{x_2w_1}) = \frac{\pi}{2}$ .



Recall also from the beginning of the proof Lemma 5.12 that  $d(\overrightarrow{x_2 w_2}, \beta_i) \leq \frac{\pi}{2}$ , because  $\beta_i$  is  $\frac{\pi}{3}$ -extendable. This implies that  $d(\overrightarrow{x_2 w_2}, \gamma) \leq \frac{\pi}{2}$  and  $\angle_{\overrightarrow{x_2 w_1}}(\overrightarrow{x_2 w_2}, \gamma) \leq \frac{\pi}{2}$ . Let  $\zeta'_2 := \overrightarrow{x_2 w_1} \overrightarrow{x_2 w_2}$ ,  $\vec{\beta}_i := \overrightarrow{x_2 w_1} \beta_i$  and  $\vec{\gamma} := \overrightarrow{x_2 w_1} \gamma$ . We have already seen that  $d(\zeta'_2, \vec{\beta}_i) \leq \frac{\pi}{2}$  and  $d(\zeta'_2, \vec{\gamma}) \leq \frac{\pi}{2}$ . Furthermore, it follows from triangle comparison, that if  $d(\zeta'_2, \vec{\gamma}) = \frac{\pi}{2}$ , then  $d(\zeta'_2, \vec{\beta}_i) = \frac{\pi}{2}$  for  $i = 1, 2$ .

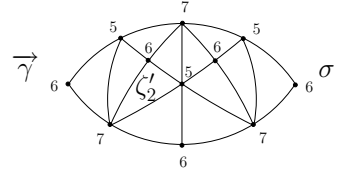
Notice that the link  $\Sigma_{\zeta_2} \Sigma_{\overrightarrow{x_2 w_2}} \Sigma_{x_2} K$  contains a pair of antipodal 7-vertices if and only if  $\Sigma_{\zeta'_2} \Sigma_{\overrightarrow{x_2 w_1}} \Sigma_{x_2} K$  contains a pair of antipodal 7-vertices. The latter is what we will show.



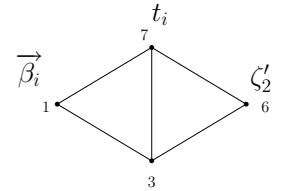
Let  $\delta_1, \delta_2 \in \Sigma_{\overrightarrow{x_2 w_1}} \Sigma_{x_2} K$  be the two 7-vertices in  $\Sigma_{\overrightarrow{x_2 w_1}} \Sigma_{x_2} T_1$  and recall that the 2-sphere  $\Sigma_{\overrightarrow{x_2 w_1}} S$  contains a singular circle of type 756575657 containing the vertices  $\delta_1, \delta_2$  and  $\vec{\gamma}$  (this is just the circle in  $\Sigma_{\overrightarrow{x_2 w_1}} \Sigma_{x_2} K$  corresponding to the circle in  $\Sigma_{\overrightarrow{x_1 w_1}} \Sigma_{x_1} K$  from the Sublemma 5.13 containing  $\xi, \xi'$  and  $\zeta$ ). Let  $\sigma$  be the 6-vertex in this circle antipodal to  $\vec{\gamma}$ . Further, we know that  $d(\zeta'_2, \delta_i) = \frac{\pi}{2}$ , because  $d(\overrightarrow{x_2 w_2}, \Sigma_{x_2} T_1) \equiv \frac{\pi}{2}$ . If  $\zeta'_2$  has an antipode in the 2-sphere  $\Sigma_{\overrightarrow{x_2 w_1}} S$ , then  $\overrightarrow{x_2 w_2}$  has an antipode in  $S$ . But this is impossible, since  $\overrightarrow{x_2 w_2} = \overrightarrow{x_2 m(y_2, z_2)}$  and  $m(y_2, z_2) \in K$  is a 2A-vertex at distance  $\frac{3\pi}{4}$  to  $x_2$ . Hence  $\frac{\pi}{2} \geq d(\zeta'_2, \vec{\gamma}) > 0$  and  $d(\zeta'_2, \sigma) < \pi$ .

Notice that  $\Sigma_{\overrightarrow{x_2 w_1}} \Sigma_{x_2} B$  is a building of type  $D_6$  and Dynkin diagram  $\begin{array}{c} 3 \\ | \\ 1-4-5-6-7 \end{array}$ . The distances between 6-vertices are  $0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}$  and  $\pi$ . The link  $\Sigma_{\zeta'_2} \Sigma_{\overrightarrow{x_2 w_1}} \Sigma_{x_2} K$  is of type  $\begin{array}{c} 3 \quad 4 \quad 5 \\ | \quad | \quad | \\ 1-4-5-6-7 \end{array}$ , thus two distinct 7-vertices in this link must be antipodal.

Case 1:  $d(\zeta'_2, \vec{\gamma}) = \frac{\pi}{3}$ . Since  $d(\zeta'_2, \sigma) < \pi$ , it follows that  $\vec{\gamma} \zeta'_2 \sigma$  is a geodesic of length  $\pi$ . Its simplicial convex hull is 2-dimensional and contains two 7-vertices adjacent to  $\zeta'_2$ . It follows that  $\Sigma_{\zeta'_2} \Sigma_{\overrightarrow{x_2 w_1}} \Sigma_{x_2} K$  contains a pair of antipodal 7-vertices.



Case 2:  $d(\zeta'_2, \vec{\gamma}) = \frac{\pi}{2}$  and the segment  $\zeta'_2 \vec{\gamma}$  is of type 646. In this case, we know that  $d(\zeta'_2, \vec{\beta}_i) = \frac{\pi}{2}$  for  $i = 1, 2$ . Thus,  $CH(\vec{\beta}_1, \vec{\beta}_2, \zeta'_2)$  is an isosceles spherical triangle with side lengths  $\frac{\pi}{2}, \frac{\pi}{2}$  and  $\arccos(-\frac{1}{3})$ . The simplicial convex hull of the segment  $\zeta'_2 \vec{\beta}_i$  contains a 7-vertex  $t_i$  adjacent to  $\zeta'_2$  and to  $\vec{\beta}_i$  for  $i = 1, 2$ . If  $t_1 = t_2$ , then  $t_1$  is adjacent to  $\vec{\beta}_i$  for  $i = 1, 2$ . It follows that  $t_1$  is also adjacent to  $\vec{\gamma} = m(\vec{\beta}_1, \vec{\beta}_2)$ . This means that  $d(\vec{\gamma}, t_1) = d(t_1, \zeta'_2) = \frac{\pi}{4}$ . Since  $d(\zeta'_2, \vec{\gamma}) = \frac{\pi}{2}$ ,  $\zeta'_2 t_1 \vec{\gamma}$  must be a geodesic. This contradicts the fact that the segment  $\zeta'_2 \vec{\gamma}$  is of type 646. Hence,  $t_1 \neq t_2$  and  $\Sigma_{\zeta'_2} \Sigma_{\overrightarrow{x_2 w_1}} \Sigma_{x_2} K$  contains a pair of antipodal 7-vertices.



Case 3:  $d(\zeta'_2, \vec{\gamma}) = \frac{\pi}{2}$  and the segment  $\zeta'_2 \vec{\gamma}$  is of type 676. If  $d(\zeta'_2, \sigma) = \frac{\pi}{2}$  then  $\vec{\gamma} \zeta'_2 \sigma$  is a geodesic of length  $\pi$  and of type 67676. If  $d(\zeta'_2, \sigma) = \frac{2\pi}{3}$ , then the segment  $\zeta'_2 \vec{\gamma}$  contains a 7-vertex adjacent to  $\zeta'_2$  at distance  $\frac{\pi}{4}$  to  $\vec{\gamma}$  and the simplicial convex hull of the segment  $\zeta'_2 \sigma$  contains a 7-vertex adjacent to  $\zeta'_2$  at distance  $\frac{\pi}{2}$  to  $\sigma$ . It follows that  $\zeta'_2$  is adjacent to two different 7-vertices. Thus,  $\Sigma_{\zeta'_2} \Sigma_{\overrightarrow{x_2 w_1}} \Sigma_{x_2} K$  contains a pair of antipodal 7-vertices.  $\square$

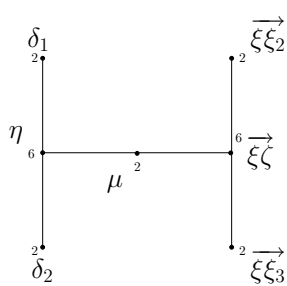
*End of proof of Lemma 5.12.* We know now that  $\Sigma_{\zeta_2} \Sigma_{x_2 w_2} \Sigma_{x_2} K$  (of type  $\begin{matrix} 3 & 4 & 5 \\ & \swarrow & \searrow \\ & 1 & \end{matrix}$ ) contains a singular circle of type 5135135 and a pair of antipodal 7-vertices. Hence, it contains a singular 2-sphere (the spherical join of the singular circle and the pair of antipodal 7-vertices). Since  $\zeta_2$  has an antipode in  $\Sigma_{x_2 w_2} \Sigma_{x_2} K$ , this implies that  $\Sigma_{x_2 w_2} \Sigma_{x_2} K$  contains a 3-sphere spanned by a simplex of type 1567. This in turn implies that  $\Sigma_{w_2} K$  contains a singular 4-sphere spanned by a simplex of type 15678. This sphere is of type  $\frac{\pi}{3}$  as can be verified by considering the vector space realization of the Coxeter complex of type  $D_n$  presented in Appendix A. We get a contradiction to Lemma 5.9 finishing the proof of the lemma.  $\square$

**Lemma 5.15.**  *$K$  contains no  $8B_3$ -vertices.*

*Proof.* We want to show first that an  $8B_3$ -vertex has the property  $T$ . Suppose  $x_1 \in K$  is an  $8B_3$ -vertex and let  $x_2, x_3 \in K$  be 8-vertices as in the configuration  $*$ . Suppose further, that  $x_3$  is an  $8A$ -vertex and that  $\overline{x_1 x_2}$  is  $\frac{2\pi}{3}$ -extendable to an  $8A$ -vertex. To prove that  $x_1$  has the property  $T$ , we have to show that  $CH(x_1, x_2, x_3)$  is not a spherical triangle. Let  $S \subset \Sigma_{x_1} K$  be the singular 2-sphere from the definition of the property  $B_3$ . Let  $\zeta := \overline{x_1 z_1}$  and  $\xi_i := \overline{x_1 x_i}$  for  $i = 2, 3$ , as in the notation of the configuration  $*$ .

Suppose there is a 7-vertex  $\xi \in S$ , such that  $d(\zeta, \xi) = \arccos(-\frac{1}{\sqrt{3}})$ . The segment  $\zeta\xi$  is of type 2767. Since  $\xi$  is  $\frac{\pi}{3}$ -extendable in  $K$  and  $\xi_i$  is  $\frac{2\pi}{3}$ -extendable to an  $8A$ -vertex,  $\xi$  is not antipodal to  $\xi_i$  for  $i = 1, 2$ . It follows that  $CH(\xi, \xi_2, \xi_3)$  is an equilateral spherical triangle sides of type 727. Let  $\gamma$  be the 7-vertex in  $\zeta\xi$  adjacent to  $\zeta$ .  $\gamma$  is the center of the spherical triangle  $CH(\xi, \xi_2, \xi_3)$ . It follows from the configuration  $**$ , that  $\gamma$  is  $\frac{2\pi}{3}$ -extendable to an  $8A$ -vertex in  $K$ .

$\Sigma_{\xi} S$  is a singular circle of type 2626262. Notice that  $\overline{\xi\zeta} = \overline{\xi\gamma}$  is not antipodal to any 2-vertex in this circle, otherwise we could find in  $S$  an antipodal 7-vertex to  $\gamma$ , but this is not possible, since  $\gamma$  is  $\frac{2\pi}{3}$ -extendable to an  $8A$ -vertex in  $K$ . On the other hand,  $\overline{\xi\zeta}$  cannot have distance  $< \frac{\pi}{2}$  to all the 6-vertices in this circle, so let  $\eta$  be a 6-vertex in  $\Sigma_{\xi} S$ , so that  $d(\eta, \overline{\xi\zeta}) = \frac{2\pi}{3}$  and let  $\delta_i \in \Sigma_{\xi} S$  be the 2-vertices adjacent to  $\eta$ . Let  $\mu := m(\eta, \overline{\xi\zeta})$ . (Compare with the configuration in the proof of Lemma 5.7.)

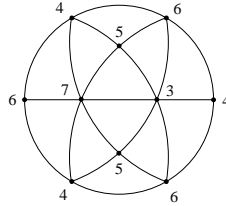


Since  $\overline{\xi\zeta}$  is not antipodal to  $\delta_i$ , it follows that  $\angle_{\eta}(\delta_i, \overline{\xi\zeta}) = \frac{\pi}{2}$  and these angles are of type 232. Therefore,  $\Sigma_{\overline{\mu\eta}} \Sigma_{\mu} \Sigma_{\xi} \Sigma_{x_1} K$  contains a pair of antipodal 3-vertices. Similarly, we see that  $\eta$  cannot be antipodal to  $\overline{\xi\xi_i}$  because  $\xi_i$  has no antipodes in  $S$ . Thus,  $\Sigma_{\overline{\mu\xi\xi_i}} \Sigma_{\mu} \Sigma_{\xi} \Sigma_{x_1} K$  contains a pair of antipodal 3-vertices. This implies in turn, that  $\Sigma_{\overline{\mu\eta}} \Sigma_{\mu} \Sigma_{\xi} \Sigma_{x_1} K$  contains a pair of antipodal 1-vertices. We apply now Lemma 4.5 to the building  $\Sigma_{\overline{\mu\eta}} \Sigma_{\mu} \Sigma_{\xi} \Sigma_{x_1} B$  of type  $D_4$  and conclude that  $\Sigma_{\overline{\mu\eta}} \Sigma_{\mu} \Sigma_{\xi} \Sigma_{x_1} K$  contains a singular circle of type 1351351. Therefore  $\Sigma_{\mu} \Sigma_{\xi} \Sigma_{x_1} K$  contains a singular 2-sphere spanned by a simplex of type 156. The same argument as in the proof of Lemma 5.7 (p. 25) shows that  $\mu$  is extendable in  $\Sigma_{x_1} K$  to a segment of type 727 and the 2-vertex on this segment is extendable in  $K$  to a segment of type 828 (this uses that  $\gamma \in \Sigma_{x_1} K$  is  $\frac{2\pi}{3}$ -extendable and the 7-vertices in  $S$  are  $\frac{\pi}{3}$ -extendable). This produces a 2-vertex in  $K$ , whose link contains a 4-sphere spanned by a simplex of type 15678. This singular 4-sphere is of type  $\frac{\pi}{3}$ , a contradiction to Lemma 5.9.

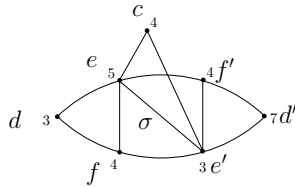
From this, it follows that  $\zeta$  has distance  $\leq \frac{\pi}{2}$  to all the 7-vertices in  $S$ . Since  $S$  is the convex hull of its 7-vertices, it follows that  $d(\zeta, S) \equiv \frac{\pi}{2}$ . Hence  $\Sigma_\zeta \Sigma_{x_1} K$  contains the 2-sphere  $s := \Sigma_\zeta CH(\zeta, S)$ . The segments connecting  $\zeta$  with the 2-vertices of  $S$  are of type 262, the segments connecting  $\zeta$  with the 7-vertices of  $S$  are of type 217 and since the 6-vertices in  $S$  are midpoints of segments of type 767 in  $S$ , this implies that the segments connecting  $\zeta$  with the 6-vertices of  $S$  are of type 2436. Since the sphere  $S$  has  $B_3$ -geometry  $\overrightarrow{7-6-2}$ , it follows that  $s$  has  $B_3$ -geometry  $\overleftarrow{4-6}$ .  $\Sigma_\zeta \Sigma_{x_1} K$  also contains the two antipodal 7-vertices  $\overleftrightarrow{\zeta x_1 x_i}$  for  $i = 2, 3$ .

**Sublemma 5.16.** *Let  $L \subset B$  be a convex subcomplex of a building of type  $D_6$  with Dynkin diagram  $\begin{array}{c} 3 \\ \nearrow \\ 1 \end{array} \begin{array}{c} 4 \\ \leftarrow \\ 5 \end{array} \begin{array}{c} 6 \\ \leftarrow \\ 7 \end{array}$ . Suppose  $L$  contains a singular 2-sphere  $S$  with  $B_3$ -geometry  $\overleftarrow{4-6}$  and also a pair of antipodal 7-vertices. Then  $L$  contains a 3-sphere spanned by a simplex of type 1467.*

*Proof.* Let  $a, a' \in L$  be the antipodal 7-vertices and let  $b, b'$  be antipodal 1-vertices in  $S \subset L$ . By Lemma 4.8 and Remark 4.10, it follows that  $L$  contains a circle of type 7317317 through  $b$  and  $b'$ . In particular  $\Sigma_b L$  contains a pair of antipodal 7- and 3-vertices.  $\Sigma_b S$  is a singular circle of type 6464646. So, it will suffice to show that under these circumstances  $\Sigma_b L$  contains a 2-sphere spanned by a simplex of type 467 (notice that such a sphere is also spanned by a simplex of type 346):



Let  $d, d' \in \Sigma_b L$  be the antipodal 3- and 7-vertices, respectively. Let  $c \in \Sigma_b S$  be a 4-vertex and let  $c'$  the 6-vertex in  $\Sigma_b S$  antipodal to  $c$ . If  $c$  is adjacent to  $d$ , then  $dcd'$  is a geodesic of type 3437 and  $\Sigma_c \Sigma_b L$  contains a pair of antipodal 3-vertices. If  $c$  is adjacent to  $d'$ , then  $dcd'$  is a geodesic of type 3547 and  $\Sigma_c \Sigma_b L$  contains a pair of antipodal 5- and 7-vertices.



Otherwise the segments  $cd$  and  $cd'$  are of type 453 and 437 respectively. Let  $e$  be the 5-vertex in  $cd$  and let  $e'$  be the 3-vertex in  $cd'$ . Let also  $f$  be the 4-vertex on the segment  $e'd$  (of type 343) and let  $f'$  be the 4-vertex on the segment  $ed'$  (of type 547). Notice that since  $e, e'$  are adjacent to  $c$ , then  $e$  is adjacent to  $e'$ . It follows that  $e$  is adjacent to  $f$  and  $e'$  is adjacent to  $f'$ . Let  $\sigma$  be the edge  $ee'$ . The link  $\Sigma_\sigma \Sigma_b B$  is of type  $\begin{array}{c} 4 \\ \leftarrow \\ 5 \end{array}$ ; and the direction  $\overrightarrow{\sigma c'}$  is of type 4. It follows that  $c'$  is antipodal to  $f$  or  $f'$  and  $c'$  is contained in a circle in  $\Sigma_b L$  of type 7673437 or 3657453. This implies that  $\Sigma_{c'} \Sigma_b L$  contains a pair of antipodal 7-vertices or a pair of antipodal 3- and 5-vertices. This means for  $\Sigma_c \Sigma_b L$ , that it contains a pair of antipodal 3-vertices or a pair of antipodal 5- and 7-vertices.

Recall that  $\Sigma_c \Sigma_b S$  consists of a pair of antipodal 6-vertices.  $\Sigma_c \Sigma_b B$  is of type  $A_1 \circ A_3$  with Dynkin diagram  $\begin{array}{c} 3 \\ \leftarrow \\ 1 \end{array} \begin{array}{c} 4 \\ \leftarrow \\ 5 \end{array} \begin{array}{c} 6 \\ \leftarrow \\ 7 \end{array}$ . If  $\Sigma_c \Sigma_b L$  contains a pair of antipodal 3-vertices, then it contains a circle of type 63636. This implies that  $\Sigma_b L$  contains a 2-sphere spanned by a simplex of type 364 as desired. If  $\Sigma_c \Sigma_b L$  contains a pair of antipodal 5- and 7-vertices, then we apply Lemma 4.8 (for  $n = k = 3$ ) to the  $A_3$ -factor of  $\Sigma_c \Sigma_b B$  and conclude that  $\Sigma_c \Sigma_b L$  contains a circle of type 7675657. We get again the 2-sphere in  $\Sigma_b L$  as we wanted.  $\square$

*End of proof of Lemma 5.15.* Sublemma 5.16 implies that  $\Sigma_\zeta \Sigma_{x_1} K$  contains a 3-sphere spanned by a simplex of type 1467. Recall the notation of the configuration  $*$ . Let  $u$  be the 8-vertex  $m(x_3, y_3)$ .  $x_1 z_1 u$  is a segment of type 828. Then, it follows that  $\Sigma_{z_1} K$  contains a singular 4-sphere spanned by a simplex of type 14678. This sphere is of type 757 (to verify this, one can consider the vector space realization of  $D_n$  in Appendix A). Lemma 5.8 implies that the segment  $x_1 u$  cannot be extended beyond  $u$  in  $K$ . This implies in turn, that  $CH(x_1, x_2, x_3)$  cannot be a spherical triangle. In particular  $x_1$  must be an  $8T$ -vertex. i.e.  $8B_3 \Rightarrow 8T$ .

Let now  $x_1, x_2, x_3$  be  $8B_3$ -vertices as in the definition of the property  $T'$ . Our argument above shows that  $\Sigma_{z_i} K$  contains a 4-sphere of type 757 for  $i = 1, 2, 3$ . We apply again Lemma 5.8 and see that  $rad(z_i, \{8\text{-vert. in } K\}) \leq \arccos(-\frac{1}{2\sqrt{2}})$  for  $i = 1, 2, 3$ . Hence,  $x_1$  is an  $8T'$ -vertex. A contradiction to Lemma 5.10.  $\square$

If we combine the Lemmata 5.12 and 5.15 we obtain the following result, which is the main step towards the proof of Theorem 5.24.

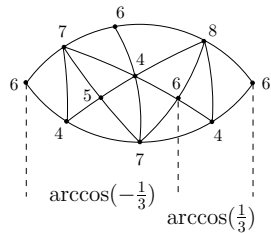
**Corollary 5.17.** *All 8-vertices in  $K$  have antipodes in  $K$ .*

Now we proceed to prove that the other vertices in  $K$  must also have antipodes in  $K$ . We use the information about types of segments between vertices in the Coxeter complex of type  $E_8$  listed in Section 3.4.

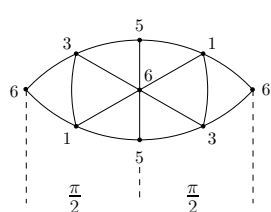
**Lemma 5.18.** *All 2-vertices in  $K$  have antipodes in  $K$ .*

*Proof.* First note that a  $2A$ -vertex  $x \in K$  cannot be adjacent to an 8-vertex in  $K$ . Otherwise let  $y \in K$  be an antipode of the 8-vertex adjacent to  $x$ . The segment  $xy$  is of type 2828. This is not possible due to Lemma 4.1 and 5.17.

Suppose there is a  $2A$ -vertex  $x \in K$ . There exists  $x' \in G \cdot x$  with  $d(x, x') > \frac{\pi}{2}$ . From the observation above it follows, that  $d(x, x') \neq \arccos(-\frac{1}{4})$ .  $d(x, x')$  cannot be  $\arccos(-\frac{3}{4})$  either, because in this case the midpoint of the segment  $xx'$  is an 8-vertex. It follows that  $d(x, x') = \frac{2\pi}{3}$  and the segment  $xx'$  is of type 26262. Let  $y := m(x, x')$ , it is also a  $2A$ -vertex. Therefore we can find  $y' \in G \cdot y$  with  $d(y, y') = \frac{2\pi}{3}$ . Suppose w.l.o.g. that  $\angle_y(x, y') \geq \frac{\pi}{2}$ . Then  $d(x, y') > \frac{\pi}{2}$ , thus  $d(x, y') = \frac{2\pi}{3}$ . This implies by triangle comparison, that  $\angle_y(x, y') \leq \arccos(-\frac{1}{3})$ .



If  $\angle_y(x, y') = \arccos(-\frac{1}{3})$ , then either this angle is of type 686, which is not possible because  $K$  contains no 8-vertex adjacent to  $y$ ; or the simplicial convex hull of the segment  $\overrightarrow{yxy'}$  contains a 7-vertex adjacent to  $\overrightarrow{yx}$ . The segment connecting  $\overrightarrow{yx}$  and  $\overrightarrow{yx'}$  through this 7-vertex is of type 67686. This cannot happen either. Hence,  $\angle_y(x, y') = \frac{\pi}{2}$ . It follows that  $\angle_y(x', y') \geq \frac{\pi}{2}$  and we conclude analogously that  $\angle_y(x', y') = \frac{\pi}{2}$ .



Let  $\gamma \subset \Sigma_y K$  be the geodesic connecting  $\overrightarrow{yx}$  and  $\overrightarrow{yx'}$  through  $\overrightarrow{yy'}$ . The simplicial convex hull of  $\gamma$  is either 3-dimensional, in which case the direction  $\overrightarrow{yxy'}$  spans a simplex of type 578 and in particular,  $\Sigma_y K$  contains 8-vertices, but this is not possible; or it is 2-dimensional and it contains a pair of 1-vertices adjacent to  $\overrightarrow{yy'}$ . Let  $z := m(y, y')$  and let  $w$  be the 6-vertex  $m(y, z)$ . The segment joining  $\overrightarrow{wy}$  and  $\overrightarrow{wz}$  through the 1-vertex adjacent to  $\overrightarrow{wy}$  is of type 2152. It follows that  $\overrightarrow{zy}$  is adjacent to a 5-vertex. The geodesic connecting  $\overrightarrow{zy}$



and  $\overrightarrow{zy'}$  through this 5-vertex is of type 65856, but  $z$  is a 2A-vertex and  $\Sigma_z K$  cannot contain 8-vertices.  $\square$

**Lemma 5.19.** *All 7-vertices in  $K$  have antipodes in  $K$ .*

*Proof.* Considering the singular circles in  $E_8$ , we observe again that a 7A-vertex cannot be adjacent to 2- or 8-vertices in  $K$ . Suppose  $K$  contains 7A-vertices, then there exist 7A-vertices  $x_1, x_2 \in K$  at distance  $> \frac{\pi}{2}$ . There are two types of segments  $x_1x_2$  of length  $> \frac{\pi}{2}$  and so that the simplices containing  $\overrightarrow{x_1x_2}$  in their interiors have no 2- or 8-vertices. They are of type 76867 and 7342437. These segments have a vertex of type 2 or 8 in their interiors, which yields a contradiction.  $\square$

**Lemma 5.20.** *All 1-vertices in  $K$  have antipodes in  $K$ .*

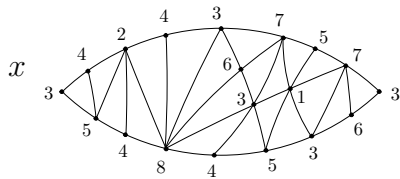
*Proof.* Suppose  $x$  is an 1A-vertex in  $K$ . Then  $x$  cannot be adjacent to 2-, 7- or 8-vertices in  $K$ . Let  $x' \in G \cdot x$  be another 1A-vertex at distance  $> \frac{\pi}{2}$  to  $x$ . It follows that the simplex spanned by the direction  $\overrightarrow{xx'}$  has no 2-, 7- or 8-vertices. There are four possible types of segments  $xx'$ . If  $d(x, x') = \arccos(-\frac{3}{8})$ , then the simplicial convex hull of  $xx'$  contains an 8-vertex adjacent to  $x'$ . If  $d(x, x') = \frac{2\pi}{3}$  or  $\arccos(-\frac{7}{8})$ , then the midpoint of  $xx'$  is an 8-vertex. If  $d(x, x') = \arccos(-\frac{5}{8})$ , then the midpoint of  $xx'$  is a 7-vertex. This is not possible by Lemma 4.1. Hence, there are no 1A-vertices in  $K$ .  $\square$

**Lemma 5.21.** *All 6-vertices in  $K$  have antipodes in  $K$ .*

*Proof.* Let  $x$  be a 6A-vertex. By the previous lemmata and according to the list of singular 1-spheres in the Coxeter complex of type  $E_8$ ,  $x$  cannot be adjacent to vertices of type 1, 2, 7 or 8. There exists another 6A-vertex  $x' \in K$  at distance  $> \frac{\pi}{2}$  to  $x$ . It follows that the direction  $\overrightarrow{xx'}$  span a simplex with no 1, 2, 7 or 8-vertices. Hence  $d(x, x') \in \{\arccos(-\frac{1}{4}), \frac{2\pi}{3}, \arccos(-\frac{3}{4})\}$ . In the first case the midpoint of  $xx'$  is an 8-vertex and in the third case, it is a 7-vertex. In the second case the simplicial convex hull of  $xx'$  contains an 8-vertex adjacent to  $x'$ . A contradiction.  $\square$

**Lemma 5.22.** *All 3-vertices in  $K$  have antipodes in  $K$ .*

*Proof.* Observe, that a 3A-vertex is not adjacent to a vertex of type 1, 2, 6, 7 or 8.



If  $K$  contains 3A-vertices, then it contains at least two distinct 3A-vertices  $x, x'$ . Then  $\overrightarrow{xx'}$  is contained in an edge of type 45. Consider the bigon in the Coxeter complex of type  $E_8$ , which is the convex hull of a simplex of type 345 and the antipode of the 3-vertex of this simplex. We see that there are only three possibilities for the type of the segment  $xx'$ . In one of them, the midpoint of  $xx'$  is a 2-vertex; and in another possibility, it is an 8-vertex. The simplicial convex hull of  $xx'$  for the last possibility contains an 8-vertex adjacent to  $x'$ . We obtain again a contradiction to Lemma 4.1.  $\square$

**Lemma 5.23.** *All 4- and 5-vertices in  $K$  have antipodes in  $K$ .*

*Proof.* A vertex in  $K$  of type 4 or 5 without antipodes in  $K$  cannot have vertices of type 1, 2, 3, 6, 7 or 8 in  $K$  adjacent to it. It follows that, if  $K$  contains 4A- or 5A-vertices, then it has dimension  $\leq 1$ . A contradiction.  $\square$

We have shown in the previous lemmata that all vertices of a counterexample  $K$  have antipodes in  $K$ , contradicting Lemma 4.2. This proves our main result:

**Theorem 5.24.** *The Center Conjecture 1.1 holds for spherical buildings of type  $E_8$ .*

**Remark 5.25.** Our proof actually shows that  $K$  is a subbuilding or the action of the group  $Aut_B(K) \curvearrowright K$  fixes a point, where  $Aut_B(K) \supseteq Stab_{Aut(B)}(K)$  denotes the possibly larger group of isometries of  $K$  preserving the polyhedral structure of  $K$  induced by the polyhedral structure of  $B$  and such that the permutation in the labelling of the vertices of  $K$  is induced by a symmetry of the Dynkin diagram of  $B$ . Notice that the automorphisms in  $Aut_B(K)$  are not necessarily extendable to automorphisms of  $B$ , as the following example shows. Let  $\sigma \subset B$  be a panel and let  $K_\sigma$  be the union of the Weyl chambers in  $B$  containing  $\sigma$ . It is a convex subcomplex of  $B$  and  $Aut_B(K_\sigma)$  is the group of permutations of the set of Weyl chambers containing  $\sigma$ . This group is very large if e.g. the set of Weyl chambers containing  $\sigma$  is uncountable.

## 5.2 The $E_7$ -case

**Theorem 5.26.** *The Center Conjecture 1.1 holds for spherical buildings of type  $E_7$ .*

*Proof.* It can be deduced from the  $E_8$ -case as follows: Let  $K \subset B$  be a convex subcomplex of a spherical building of type  $E_7$ . Suppose that  $K$  is not a subbuilding. Let  $\tilde{B}$  be the suspension of  $B$ , i.e. the spherical join of  $B$  and a 0-sphere  $\{p_1, p_2\}$ . There is a natural embedding  $B \hookrightarrow \tilde{B}$ , so we can consider  $B$  as a subset of  $\tilde{B}$ . Notice that the map  $v \mapsto \overline{p_i v}$  for  $v \in B \subset \tilde{B}$  is an isometry  $B \cong B_{p_i} := \Sigma_{p_i} \tilde{B}$ . Let  $\tilde{K} \subset \tilde{B}$  be the suspension of  $K$  and let  $K_{p_i} := \Sigma_{p_i} \tilde{K} \cong K$ .

Recall that the link of an 8-vertex in the Coxeter complex of type  $E_8$  is a Coxeter complex of type  $E_7$ . Hence a chart  $(S^6, W_{E_7}) \hookrightarrow B_{p_1}$  of the building  $B_{p_1}$  induces a chart  $(S^7, W_{E_8}) \hookrightarrow \tilde{B}$ , giving  $\tilde{B}$  a structure of spherical building of type  $E_8$ , where  $p_1$  and  $p_2$  are 8-vertices.

Observe that there is a natural isomorphism  $Aut(B_{p_1}) \cong Stab_{Aut(\tilde{B})}(p_1)$ . The embedding  $Aut(B_{p_1}) \hookrightarrow Aut(\tilde{B})$  restricts to an embedding  $G := Stab_{Aut(B_{p_1})}(K_{p_1}) \hookrightarrow \tilde{G} := Stab_{Aut(\tilde{B})}(\tilde{K})$ .

There is an isometry  $\phi_0$  of  $\tilde{B}$  that exchanges the points  $p_1 \leftrightarrow p_2$  and preserves the geodesics connecting  $p_1$  and  $p_2$ . The restriction of  $\phi_0$  to an apartment of  $\tilde{B}$  is the reflection on the wall orthogonal to  $p_1, p_2$ . Hence  $\phi_0$  is an automorphism of  $\tilde{B}$  and  $\phi_0 \in \tilde{G}$ .

We apply now the Center Conjecture for buildings of type  $E_8$  (Theorem 5.24) to the building  $\tilde{B}$  and the convex subcomplex  $\tilde{K}$ . Since  $K$  is not a subbuilding, then  $\tilde{K}$  is not a subbuilding either. It follows that  $\tilde{G}$  fixes a point  $x \in \tilde{K}$  and since  $\phi_0 \in \tilde{G}$ , this fixed point cannot be  $p_1$  or  $p_2$ . The image of  $G$  in  $\tilde{G}$  fixes  $x$ ,  $p_1$  and  $p_2$ , hence it fixes pointwise the geodesic  $\gamma \subset \tilde{K}$  through  $x$  connecting  $p_1$  and  $p_2$ . Therefore, the action  $G \curvearrowright K_{p_1} \cong K$  has a fixed point  $\overline{p_1 x} \in K_{p_1}$ .  $\square$

**Remark 5.27.** Notice that the embedding  $G \hookrightarrow \tilde{G}$  in the proof of Theorem 5.26 extends to an embedding  $Aut_{B_{p_1}}(K_{p_1}) \hookrightarrow Aut_{\tilde{B}}(\tilde{K})$ . Then by Remark 5.25, the proof actually shows that

$K$  is a subbuilding or the action of the group  $Aut_B(K) \curvearrowright K$  fixes a point.

## A Vector space realizations of Coxeter complexes

In this appendix we present a vector space realization of the irreducible spherical Coxeter complexes. The information on the root systems can be found in [GB71, Ch. 5]. The orders of the irreducible Weyl groups can be found in [GB71, p. 80].

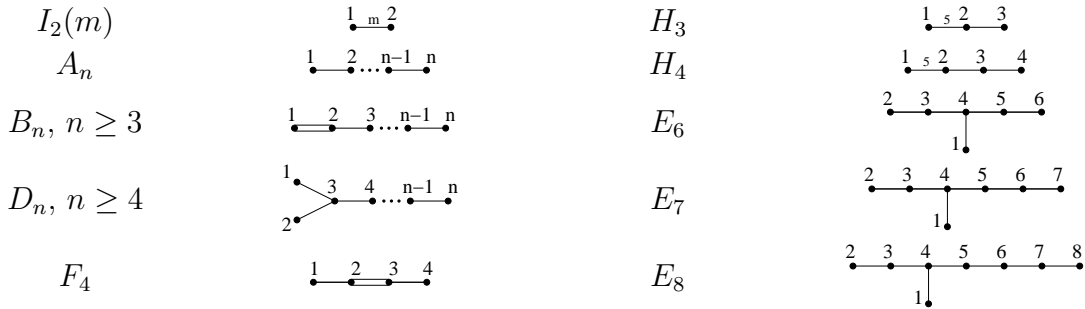
We consider the spherical Coxeter complex  $(S^{n-1}, W)$  embedded in  $\mathbb{R}^n$  as the unit sphere. Let  $\{e_i\}_{i=1}^n$  denote the canonical base of  $\mathbb{R}^n$ .

The *root system* of a Coxeter complex  $(S, W)$  is the set of (unit) vectors orthogonal to the hyperplanes inducing the reflections in  $W$ . The elements of the root system are called root vectors.

A subset  $F$  of the root system is called a base if there is a vector  $v \in \mathbb{R}^n$  such that  $\langle r, v \rangle \neq 0$  for all root vectors  $r$ , and  $F$  is minimal with respect to the property that any root vector  $r$ , such that  $\langle r, v \rangle > 0$ , can be written as a linear combination of elements in  $F$  with nonnegative coefficients. The *fundamental root vectors* are the elements of a given base of the root system. The *fundamental Weyl chamber* of  $(S, W)$  is  $\Delta := \bar{\Delta} \cap S$ , where  $\bar{\Delta}$  is the intersection of the half spaces  $\langle r_i, \cdot \rangle \geq 0$ , where  $r_1, \dots, r_n$  are the fundamental root vectors.  $\bar{\Delta}$  is a fundamental domain for the action of  $W$  in  $\mathbb{R}^n$ .

Let  $v_i$  be the vertex of  $\Delta$  opposite to the face determined by  $\langle r_i, \cdot \rangle = 0$ . We say that a vertex of  $(S, W)$  is of type  $i$ , if it lies on the orbit  $W \cdot v_i$ .

We use the following labelling of the Dynkin diagrams of the irreducible spherical Coxeter complexes:



Recall, that the link  $\Sigma_v S$  of a vertex  $x \in S$  is a spherical Coxeter complex with Weyl group  $Stab_W(v)$  and with Dynkin diagram obtained from the Dynkin diagram of  $(S, W)$  by deleting the vertex with label corresponding to the type of  $v$ .

The antipodal involution  $v \mapsto -v$  is type preserving for the spherical Coxeter complexes of type  $I_2(m)$  ( $m$  even),  $B_n$  ( $n \geq 3$ ),  $D_{2n}$  ( $n \geq 2$ ),  $H_3$ ,  $H_4$ ,  $F_4$ ,  $E_7$  and  $E_8$ . It exchanges the types  $1 \leftrightarrow 2$  in  $I_2(m)$  for  $m$  odd; the types  $i \leftrightarrow (n + 1 - i)$ , for  $i = 1, \dots, \lfloor \frac{n}{2} \rfloor$  in  $A_n$ ; the types  $1 \leftrightarrow 2$ , in  $D_{2n+1}$ ,  $n \geq 2$  and the types  $2 \leftrightarrow 6$  and  $3 \leftrightarrow 5$  in  $E_6$ .

Suppose  $xy$  is an edge of  $S$  of type  $ij$ . By deleting the vertex with label  $j$  from the Dynkin diagram of  $(S, W)$ , we obtain the Dynkin diagram of  $(\Sigma_y S, Stab_W(y))$ . We can easily read off this Dynkin diagram which type the antipode of  $\overrightarrow{yx}$  in  $\Sigma_y S$  has. Say it has type  $k$ , then the

edge  $xy$  extends to a segment of type  $ijk$ . Repeating this procedure and taking into account the lengths of the different types of segments (which can be deduced from the description of the fundamental Weyl chamber), we can determine the different singular 1-spheres in  $S$ . A similar consideration can be used to determine the 2-dimensional singular bigons bounded by singular segments and with it the 2-dimensional singular spheres.

To determine the different types of segments modulo the action of the Weyl group connecting a vertex of type  $i$  with a vertex of type  $j$ , it suffices to compute the vertices of type  $j$  in the spherical bigon  $\beta_i := CH(\Delta, \widehat{v}_i) \subset S$ , where  $\widehat{v}_i$  is the vertex antipodal to  $v_i$ .

The bigon  $\beta_i$  can be described by the set of inequalities  $\{\langle r_l, \cdot \rangle \geq 0\}_{l \neq i}$ .

More generally, suppose we want to determine the different types of segments connecting a vertex  $x$  of type  $i$  and a vertex  $y$  of type  $j$ , such that the vertices of the simplex in  $\Sigma_x S$  spanned by the direction  $\overrightarrow{xy}$  are not of type  $i_1, \dots, i_k \neq i$ . Then, it suffices to compute the vertices of type  $j$  in the spherical bigon  $\beta_i(i_1, \dots, i_k) := CH(\Delta(i_1, \dots, i_k), \widehat{v}_i)$ . Here,  $\Delta(i_1, \dots, i_k)$  denotes the face of the fundamental Weyl chamber  $\Delta$ , which does not contain the vertices  $v_{i_1}, \dots, v_{i_k}$ .

The bigon  $\beta_i(i_1, \dots, i_k)$  can be described by the set of (in)equalities

$$\{\langle r_l, \cdot \rangle \geq 0\}_{l \neq i, i_1, \dots, i_k}, \quad \{\langle r_l, \cdot \rangle = 0\}_{l = i_1, \dots, i_k}.$$

Given a table listing the  $j$ -vertices in the bigon  $\beta_i$ , this list can be verified as follows. First, we have to check that the vertices listed indeed are of type  $j$  and are contained in  $\beta_i$ . Next we notice that  $\beta_i$  is a fundamental domain for the action  $Stab_W(v_i) \curvearrowright S$ . For a  $j$ -vertex  $x$  in the list, let  $\sigma_x$  be the face of  $\Delta$  spanned by the initial part of the segment  $v_i x$ . Then the orbit  $Stab_W(v_i) \cdot x$  has cardinality  $|Stab_w(v_i)|/|Stab_W(\sigma_x)|$ . Since the stabilizers are again Weyl groups of spherical Coxeter complexes, their orders can be found in the table in [GB71, p. 80]. It remains to verify that the union of the orbits  $Stab_W(v_i) \cdot x$  exhausts all the  $j$ -vertices in  $S$ , that is, we have to check that  $\sum_{x \text{ in the list}} \frac{|Stab_W(v_i)|}{|Stab_W(\sigma_x)|} = \frac{|W|}{|Stab_W(v_i)|}$ .

## A.1 $D_n$

Let  $n \geq 4$ . The Weyl group  $W_{D_n}$  of type  $D_n$  is the finite group of isometries of  $\mathbb{R}^n$  generated by the reflections at the hyperplanes orthogonal to the *fundamental root vectors*:

$$r_1 = e_1 + e_2, \quad r_i = e_i - e_{i-1} \text{ for } 2 \leq i \leq n$$

The *fundamental Weyl chamber*  $\Delta$  can be described by the inequalities:

$$\begin{array}{cccc} (1) & (2) & (3) & (n) \\ -x_2 \leq x_1 \leq x_2 \leq \dots \leq x_n. \end{array}$$

Next we exhibit an element representing the vertices of the fundamental Weyl chamber  $\Delta$ , i.e. elements of  $\mathbb{R}^+ \cdot v_i$ :

1-vertex:	$v_1$	$(1, 1, 1, \dots, 1)$
2-vertex:	$v_2$	$(-1, 1, 1, \dots, 1)$
3-vertex:	$v_3$	$(0, 0, 1, \dots, 1)$
$\vdots$	$\vdots$	$\vdots$
$(n-1)$ -vertex:	$v_{n-1}$	$(0, \dots, 0, 1, 1)$
$n$ -vertex:	$v_n$	$(0, \dots, 0, 0, 1)$

The Weyl group  $W_{D_n}$  acts on  $\mathbb{R}^n$  by permutations of the coordinates and change of signs in an even number of places.

## A.2 $E_6$

The Weyl group  $W_{E_6}$  of type  $E_6$  is the finite group of isometries of  $\mathbb{R}^6 \cong \{(x_1, \dots, x_8) \in \mathbb{R}^8 \mid x_6 = x_7 = x_8\}$  generated by the reflections at the hyperplanes orthogonal to the *fundamental root vectors*:

$$r_1 = \frac{1}{2}(1, 1, 1, -1, -1, -1, -1, -1), \quad r_i = e_i - e_{i-1} \text{ for } 2 \leq i \leq 5; .$$

$$\text{and } r_6 = \frac{1}{2}(1, 1, 1, 1, -1, 1, 1, 1).$$

The *fundamental Weyl chamber*  $\Delta$  can be described by the inequalities:

$$x_4 + x_5 + \dots + x_8 \stackrel{(1)}{\leq} x_1 + x_2 + x_3; \quad x_1 \stackrel{(2)}{\leq} x_2 \stackrel{(3)}{\leq} \dots \stackrel{(5)}{\leq} x_5; \quad x_5 \stackrel{(6)}{\leq} x_1 + \dots + x_4 + x_6 + x_7 + x_8.$$

Next we exhibit an element representing the vertices of the fundamental Weyl chamber  $\Delta$ , i.e. elements of  $\mathbb{R}^+ \cdot v_i$ :

1-vertex:	$v_1$	$(1, 1, 1, 1, 1, -1, -1, -1)$
2-vertex:	$v_2$	$(-3, 3, 3, 3, 3, -1, -1, -1)$
3-vertex:	$v_3$	$(0, 0, 3, 3, 3, -1, -1, -1)$
4-vertex:	$v_4$	$(1, 1, 1, 3, 3, -1, -1, -1)$
5-vertex:	$v_5$	$(3, 3, 3, 3, 9, -1, -1, -1)$
6-vertex:	$v_6$	$(3, 3, 3, 3, 3, 1, 1, 1)$

We list now the orbits of the 2-vertices of  $\Delta$  under the action of the Weyl group (modulo the following elements of the Weyl group: permutations of the first five coordinates and change of sign in an even number of places in the first five coordinates). We give representing vectors for the vertices. The 6-vertices are just the antipodes of the 2-vertices.

$$\begin{aligned} \text{2-vertices} \quad & (-3, 3, 3, 3, 3, -1, -1, -1), \quad (0, 0, 0, 0, 3, 1, 1, 1), \\ & (0, 0, 0, 0, 0, -1, -1, -1). \end{aligned}$$

This list can be verified by checking that the vertices listed indeed lie on the orbit  $W_{E_6} \cdot v_2$  and there are as many as  $|W_{E_6}|/|W_{D_5}| = 3^3$ .

We describe in the following table the 2- and 6-vertices  $x$  in  $\beta_2$ . Let  $\sigma$  be the face of  $\Sigma_{v_2} \Delta$  containing  $\overrightarrow{v_2 x}$  in its interior.

	$x$	$d(x, v_2)$	Type of $\sigma$
2-vertices $x \neq v_2$	$(3, -3, 3, 3, 3, -1, -1, -1)$	$\arccos(\frac{1}{4})$	3
	$(3, 0, 0, 0, 0, 1, 1, 1)$	$\frac{2\pi}{3}$	6
6-vertices $x \neq \widehat{v}_2$	$(3, 3, 3, 3, 3, 1, 1, 1)$	$\frac{\pi}{3}$	6
	$(3, 0, 0, 0, 0, -1, -1, -1)$	$\arccos(-\frac{1}{4})$	1

### A.3 $E_7$

The Weyl group  $W_{E_7}$  of type  $E_7$  is the finite group of isometries of  $\mathbb{R}^7 \cong \{(x_1, \dots, x_8) \in \mathbb{R}^8 \mid x_7 = x_8\}$  generated by the reflections at the hyperplanes orthogonal to the *fundamental root vectors*:

$$r_1 = \frac{1}{2}(1, 1, 1, -1, -1, -1, -1, -1), \quad r_i = e_i - e_{i-1} \text{ for } 2 \leq i \leq 6$$

$$\text{and } r_7 = \frac{1}{2}(1, 1, 1, 1, 1, -1, 1, 1).$$

The *fundamental Weyl chamber*  $\Delta$  can be described by the inequalities:

$$x_4 + x_5 + \dots + x_8 \stackrel{(1)}{\leq} x_1 + x_2 + x_3; \quad x_1 \stackrel{(2)}{\leq} x_2 \stackrel{(3)}{\leq} \dots \stackrel{(6)}{\leq} x_6; \quad x_6 \stackrel{(7)}{\leq} x_1 + \dots + x_5 + x_7 + x_8.$$

Next we exhibit an element representing the vertices of the fundamental Weyl chamber  $\Delta$ , i.e. elements of  $\mathbb{R}^+ \cdot v_i$ :

$$\begin{aligned} \text{1-vertex: } v_1 & \quad (1, 1, 1, 1, 1, 1, -2, -2) \\ \text{2-vertex: } v_2 & \quad (-1, 1, 1, 1, 1, 1, -1, -1) \\ \text{3-vertex: } v_3 & \quad (0, 0, 1, 1, 1, 1, -1, -1) \\ \text{4-vertex: } v_4 & \quad (1, 1, 1, 3, 3, 3, -3, -3) \\ \text{5-vertex: } v_5 & \quad (1, 1, 1, 1, 3, 3, -2, -2) \\ \text{6-vertex: } v_6 & \quad (1, 1, 1, 1, 1, 3, -1, -1) \\ \text{7-vertex: } v_7 & \quad (1, 1, 1, 1, 1, 1, 0, 0) \end{aligned}$$

We list now the orbits of the 2- and 7-vertices of  $\Delta$  under the action of the Weyl group (modulo the following elements of the Weyl group: permutations of the first six coordinates, change of sign in an even number of places in the first six coordinates and simultaneous change of sign of the last two coordinates). We give representing vectors for the vertices.

$$\begin{aligned} \text{2-vertices} & \quad (-1, 1, 1, 1, 1, 1, -1, -1), \quad (0, 0, 0, 0, 0, 0, 2, 2), \\ & \quad (0, 0, 0, 0, 2, 2, 0, 0). \end{aligned}$$

$$\text{7-vertices} \quad (1, 1, 1, 1, 1, 1, 0, 0), \quad (0, 0, 0, 0, 0, 2, 1, 1).$$

This list can be verified by checking that the vertices listed indeed lie on the orbits  $W_{E_7} \cdot v_i$  and there are as many as  $|W_{E_7}|/|\text{Stab}_{W_{E_7}}(v_i)|$ .

We describe in the following table the 2-vertices  $x$  in  $\beta_2$ . Let  $\sigma$  be the face of  $\Sigma_{v_2} \Delta$  containing  $\overrightarrow{v_2 x}$  in its interior.

	$x$	$d(x, v_2)$	Type of $\sigma$
2-vertices $x \neq v_2, \widehat{v}_2$	( 1, -1, 1, 1, 1, 1, -1, -1)	$\frac{\pi}{3}$	3
	( 2, 0, 0, 0, 0, 2, 0, 0)	$\frac{\pi}{2}$	6
	( 2, -2, 0, 0, 0, 0, 0, 0)	$\frac{2\pi}{3}$	3

We describe in the following table the 2- and 7-vertices  $x$  in  $\beta_7$ . Let  $\sigma$  be the face of  $\Sigma_{v_7}\Delta$  containing  $\overrightarrow{v_7x}$  in its interior.

	$x$	$d(x, v_7)$	Type of $\sigma$
2-vertices $x$	(-1, 1, 1, 1, 1, 1, -1, -1)	$\arccos(\frac{1}{\sqrt{3}})$	2
	( 0, 0, 0, 0, 0, 0, -2, -2)	$\frac{\pi}{2}$	1
	(-1, -1, -1, -1, -1, 1, -1, -1)	$\arccos(-\frac{1}{\sqrt{3}})$	6
7-vertices $x \neq v_7, \widehat{v}_7$	( 0, 0, 0, 0, 0, 2, 1, 1)	$\arccos(\frac{1}{3})$	6
	(-2, 0, 0, 0, 0, 0, 1, 1)	$\arccos(-\frac{1}{3})$	2

#### A.4 $E_8$

The Weyl group  $W_{E_8}$  of type  $E_8$  is the finite group of isometries of  $\mathbb{R}^8$  generated by the reflections at the hyperplanes orthogonal to the *fundamental root vectors*:

$$r_1 = \frac{1}{2}(1, 1, 1, -1, -1, -1, -1, -1) \text{ and } r_i = e_i - e_{i-1} \text{ for } 2 \leq i \leq 8.$$

The *fundamental Weyl chamber*  $\Delta$  can be described by the inequalities:

$$x_4 + x_5 + \dots + x_8 \stackrel{(1)}{\leq} x_1 + x_2 + x_3; \quad x_1 \stackrel{(2)}{\leq} x_2 \stackrel{(3)}{\leq} x_3 \stackrel{(4)}{\leq} \dots \stackrel{(8)}{\leq} x_8.$$

Next we exhibit an element representing the vertices of the fundamental Weyl chamber  $\Delta$ , i.e. elements of  $\mathbb{R}^+ \cdot v_i$ :

1-vertex: $v_1$	(-1, -1, -1, -1, -1, -1, -1, -1)
2-vertex: $v_2$	(-3, -1, -1, -1, -1, -1, -1, -1)
3-vertex: $v_3$	(-2, -2, -1, -1, -1, -1, -1, -1)
4-vertex: $v_4$	(-5, -5, -5, -3, -3, -3, -3, -3)
5-vertex: $v_5$	(-2, -2, -2, -2, -1, -1, -1, -1)
6-vertex: $v_6$	(-3, -3, -3, -3, -3, -1, -1, -1)
7-vertex: $v_7$	(-1, -1, -1, -1, -1, -1, 0, 0)
8-vertex: $v_8$	(-1, -1, -1, -1, -1, -1, -1, 1)

We list now (modulo the following elements of the Weyl group: permutations of the coordinates and change of sign in an even number of places) the orbits of the vertices of  $\Delta$  of type 1, 2, 6, 7, 8 under the action of the Weyl group. We give representing vectors for the vertices.

1-vertices	$(-1, -1, -1, -1, -1, -1, -1, -1),$ $\frac{1}{2}(3, 3, 3, 1, 1, 1, 1, 1),$	$\frac{1}{2}(-5, 1, 1, 1, 1, 1, 1, 1),$ $(0, 0, 0, 1, 1, 1, 1, 2).$
2-vertices	$(-3, -1, -1, -1, -1, -1, -1, -1),$ $(4, 0, 0, 0, 0, 0, 0, 0).$	$(2, 2, 2, 2, 0, 0, 0, 0),$
6-vertices	$(-3, -3, -3, -3, -3, -1, -1, -1),$ $(4, 4, 4, 0, 0, 0, 0, 0),$ $(4, 4, 2, 2, 2, 2, 0, 0).$	$(6, 2, 2, 2, 0, 0, 0, 0),$ $(-5, 3, 3, 1, 1, 1, 1, 1),$
7-vertices	$(-1, -1, -1, -1, -1, -1, 0, 0),$ $\frac{1}{2}(-3, 3, 1, 1, 1, 1, 1, 1).$	$(2, 1, 1, 0, 0, 0, 0, 0),$
8-vertices	$(-1, -1, -1, -1, -1, -1, -1, 1),$	$(2, 2, 0, 0, 0, 0, 0, 0).$

This list can be verified by checking that the vertices listed indeed lie on the orbits  $W_{E_8} \cdot v_i$  and there are as many as  $|W_{E_8}|/|Stab_{W_{E_8}}(v_i)|$ .

We describe in the following table the 2- and 8-vertices  $x$  in  $\beta_2$ . Let  $\sigma$  be the face of  $\Sigma_{v_2}\Delta$  containing  $\overrightarrow{v_2x}$  in its interior.

	$x$	$d(x, v_2)$	Type of $\sigma$
2-vertices $x \neq v_2, \widehat{v}_2$	$(1, -3, -1, -1, -1, -1, -1, 1)$	$\arccos(\frac{3}{4})$	3
	$(0, -2, -2, -2, -2, 0, 0, 0)$	$\frac{\pi}{3}$	6
	$(1, -3, -1, -1, -1, -1, -1, 1)$	$\arccos(\frac{1}{4})$	38
	$(1, -1, -1, -1, -1, -1, -1, 3)$	$\frac{\pi}{2}$	8
	$(2, -2, -2, -2, 0, 0, 0, 0)$	$\frac{\pi}{2}$	5
	$(3, -1, -1, -1, -1, -1, -1, 1)$	$\arccos(-\frac{1}{4})$	18
	$(3, -1, -1, -1, -1, 1, 1, 1)$	$\frac{2\pi}{3}$	6
	$(4, 0, 0, 0, 0, 0, 0, 0)$	$\arccos(-\frac{3}{4})$	1
8-vertices $x$	$(-1, -1, -1, -1, -1, -1, -1, 1)$	$\frac{\pi}{4}$	8
	$(1, -1, -1, -1, -1, -1, -1, -1)$	$\arccos(\frac{1}{2\sqrt{2}})$	1
	$(1, -1, -1, -1, -1, -1, 1, 1)$	$\frac{\pi}{2}$	7
	$(2, -2, 0, 0, 0, 0, 0, 0)$	$\arccos(-\frac{1}{2\sqrt{2}})$	3
	$(2, 0, 0, 0, 0, 0, 0, 2)$	$\arccos(\frac{3\pi}{4})$	8

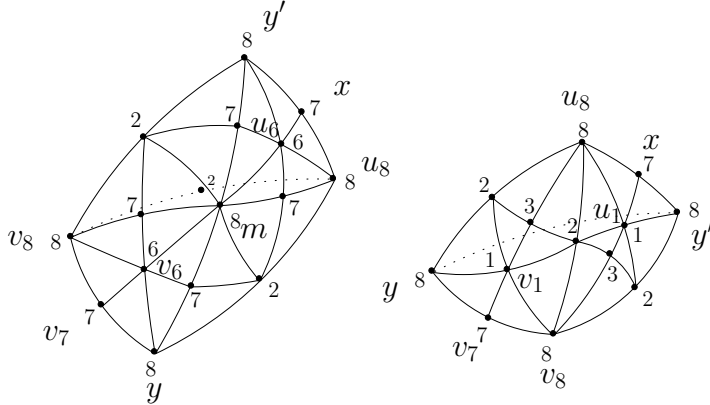
We describe in the following table the 7-vertices  $x$  in  $\beta_7$ , such that  $d(x, v_7) = \arccos(-\frac{1}{3})$  or  $\arccos(-\frac{1}{6})$ , and the 8-vertices  $x$  in  $\beta_7$ , such that  $d(x, v_7) > \frac{\pi}{2}$ . Let  $\sigma$  be the face of  $\Sigma_{v_7}\Delta$  containing  $\overrightarrow{v_7x}$  in its interior.



	$x$	$d(x, v_7)$	Type of $\sigma$
7-vertices $x$	$(0, 0, 0, 0, 0, 2, -1, -1)$	$\arccos(-\frac{1}{3})$	6
	$(0, 0, 0, 0, 1, 1, -2, 0)$	$\arccos(-\frac{1}{3})$	58
	$\frac{1}{2}(-1, 1, 1, 1, 1, 1, -3, -3)$	$\arccos(-\frac{1}{3})$	12
	$(0, 0, 0, 0, 0, 1, -2, 1)$	$\arccos(-\frac{1}{6})$	68
	$\frac{1}{2}(-3, 1, 1, 1, 1, 1, -3, -1)$	$\arccos(-\frac{1}{6})$	28
	$(0, 0, 0, 0, 0, 1, -2, -1)$	$\arccos(-\frac{1}{6})$	168
8-vertices $x$	$(1, 1, 1, 1, 1, 1, -1, 1)$	$\arccos(-\frac{\sqrt{3}}{2})$	8
	$(-1, 1, 1, 1, 1, 1, -1, -1)$	$\arccos(-\frac{1}{\sqrt{3}})$	2
	$(0, 0, 0, 0, 0, 2, -2, 0)$	$\arccos(-\frac{1}{2\sqrt{3}})$	68

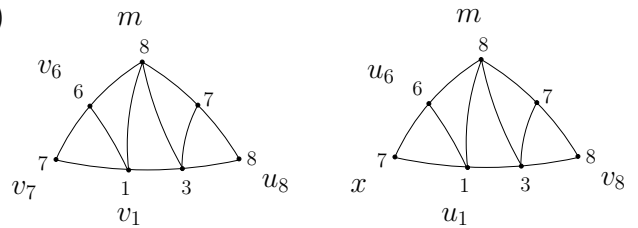
In order to make it easier to verify the table above, we present the complete table in Appendix B.

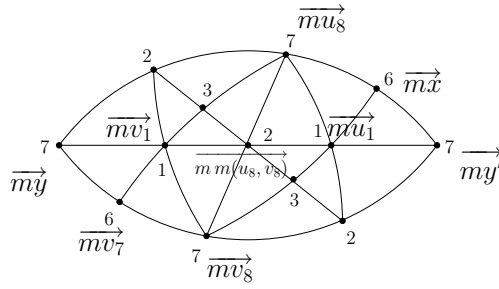
We want to describe the simplicial convex hull  $C$  of the segment  $v_7x$  for the 7-vertex  $x = (0, 0, 0, 0, 0, 1, -2, -1)$ , for this we present first a larger 3-dimensional spherical polyhedron, namely the tetrahedron  $C' := CH(v_8, y, u_8, y')$ , where  $y = (-1, -1, -1, -1, -1, -1, 1, -1)$ ,  $u_8 = (0, 0, 0, 0, 0, 0, -2, -2)$  and  $y' = (0, 0, 0, 0, 0, 2, -2, 0)$ . Notice that  $v_7 = m(v_8, y)$  and  $x = m(y', u_8)$ .  $C'$  is a subcomplex with four 2-dimensional faces: the triangles  $CH(v_8, y, y')$ ,  $CH(z, y, y')$ ,  $CH(y, u_8, v_8)$  and  $CH(y', u_8, v_8)$ . The figures illustrate the tetrahedron  $C'$  from the front and from behind.



$$\begin{aligned}
x &= (0, 0, 0, 0, 0, 1, -2, -1) \\
y &= (-1, -1, -1, -1, -1, -1, 1, -1) \\
m &= (-1, -1, -1, -1, -1, 1, -1, -1) \\
y' &= (0, 0, 0, 0, 0, 2, -2, 0) \\
u_1 &= (-1, -1, -1, -1, -1, 1, -5, -1) \\
u_6 &= (-1, -1, -1, -1, -1, 3, -5, -3) \\
u_8 &= (0, 0, 0, 0, 0, 0, -2, -2) \\
m(u_8, v_8) &= \\
&(-1, -1, -1, -1, -1, -1, -3, -1)
\end{aligned}$$

The triangles  $CH(v_8, m, x)$  and  $CH(v_7, m, u_8)$  are 2-dimensional subcomplexes. If we cut  $C'$  along these triangles, we obtain a convex subcomplex  $C'' := CH(v_7, v_8, x, u_8, m)$ . It has six 2-dimensional faces: the triangles  $CH(m, v_7, u_8)$ ,  $CH(m, x, v_8)$ ,  $CH(m, v_7, v_8)$ ,  $CH(m, x, u_8)$ ,  $CH(v_7, v_8, u_8)$  and  $CH(x, u_8, v_8)$ . Recall that the direction  $\overrightarrow{v_7x}$  spans the 168-face in  $\Sigma_{v_7}\Delta$ , this implies that  $v_1, v_6$  and  $v_8$  are contained in the simplicial convex hull  $C$  of  $v_7x$ . We can also see that the direction  $\overrightarrow{xv_7}$  spans the 168-face with vertices  $\overrightarrow{xu_1}$ ,  $\overrightarrow{xu_6}$  and  $\overrightarrow{xu_8}$ . In particular,  $u_8 \in C$ . Considering the triangle  $CH(v_7, m, u_8)$  we deduce that also  $m \in C$ . It follows that  $C = C''$ . The next figure shows the link  $\Sigma_m C'$ .





We describe in the following table the 8-vertices  $x$  in  $\beta_8$ . Let  $\sigma$  be the face of  $\Sigma_{v_8}\Delta$  containing  $\overrightarrow{v_8x}$  in its interior.

	$x$	$d(x, v_8)$	Type of $\sigma$
8-vertices $x \neq v_8, \widehat{v}_8$	$(-1, -1, -1, -1, -1, -1, 1, -1)$	$\frac{\pi}{3}$	7
	$(-2, 0, 0, 0, 0, 0, 0, -2)$	$\frac{\pi}{2}$	2
	$(0, 0, 0, 0, 0, 0, 2, -2)$	$\frac{2\pi}{3}$	7

We describe in the following table the 7-vertices  $x$  in  $\beta_7(2, 8)$  with  $d(x, v_7) > \frac{\pi}{2}$ . Let  $\sigma$  be the face of  $\Sigma_{v_7}\Delta(2, 8)$  containing  $\overrightarrow{v_7x}$  in its interior.

	$x$	$d(x, v_7)$	Type of $\sigma$
7-vertices	$(0, 0, 1, 1, 1, 1, -1, -1)$	$\arccos(-\frac{2}{3})$	3
$x \neq v_7, \widehat{v}_7$	$(0, 0, 0, 0, 0, 2, -1, -1)$	$\arccos(-\frac{1}{3})$	6

In order to make it easier to verify the table above, we present the complete table in Appendix B.

We describe in the following table the 1-vertices  $x$  in  $\beta_1(2, 7, 8)$  with  $d(x, v_1) > \frac{\pi}{2}$ . Let  $\sigma$  be the face of  $\Sigma_{v_1}\Delta(2, 7, 8)$  containing  $\overrightarrow{v_1x}$  in its interior.

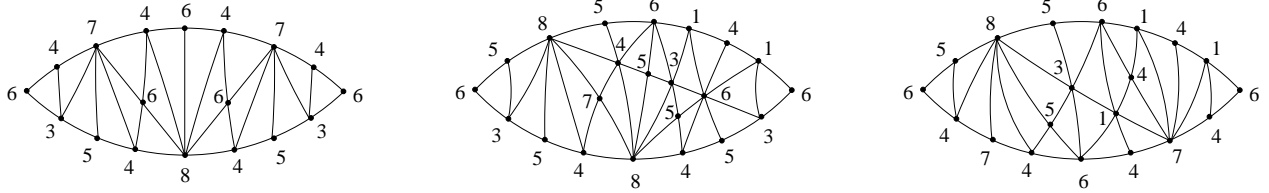
	$x$	$d(x, v_1)$	Type of $\sigma$
1-vertices $x \neq v_1, \widehat{v}_1$	$\frac{1}{2}(-1, -1, -1, -1, 1, 3, 3, 3)$	$\arccos(-\frac{3}{8})$	56
	$(-1, -1, 1, 1, 1, 1, 1, 1)$	$\frac{2\pi}{3}$	3
	$\frac{1}{2}(-1, -1, 1, 1, 1, 3, 3, 3)$	$\arccos(-\frac{5}{8})$	36
	$\frac{1}{2}(1, 1, 1, 1, 1, 3, 3, 3)$	$\arccos(-\frac{7}{8})$	6

In order to make it easier to verify the table above, we present the complete table in Appendix B.

We describe in the following table the 6-vertices  $x$  in  $\beta_6(1, 2, 7, 8)$  with  $d(x, v_6) > \frac{\pi}{2}$ . Let  $\sigma$  be the face of  $\Sigma_{v_6}\Delta(1, 2, 7, 8)$  containing  $\overrightarrow{v_6x}$  in its interior.

	$x$	$d(x, v_6)$	Type of $\sigma$
6-vertices $x \neq v_6, \widehat{v}_6$	$(0, 0, 0, 0, 6, -2, -2, -2)$	$\arccos(-\frac{1}{4})$	5
	$(0, 0, 2, 4, 4, -2, -2, -2)$	$\frac{2\pi}{3}$	34
	$(1, 1, 3, 3, 5, -1, -1, -1)$	$\arccos(-\frac{3}{4})$	35

Let us verify this last table. By considering the following 2-dimensional bigons, we can see that if there are 6-vertices missing in the table above, they must lie in the interior of  $\beta_6(1, 2, 7, 8)$ .



A 6-vertex  $x$  in the interior of  $\beta_6(1, 2, 7, 8)$  should satisfy

$$x_4 + x_5 + \cdots + x_8 \stackrel{(1)}{=} x_1 + x_2 + x_3; \quad x_1 \stackrel{(2)}{=} x_2 \stackrel{(3)}{<} x_3 \stackrel{(4)}{<} x_4 \stackrel{(5)}{<} x_5; \quad x_5 > x_6 \stackrel{(7)}{=} x_7 \stackrel{(8)}{=} x_8.$$

In particular, we have four different values  $x_2 < x_3 < x_4 < x_5$ . Hence,  $x$  cannot be a permutation of  $(\pm 4, \pm 4, \pm 4, 0, 0, 0, 0, 0)$ .

If  $x$  is obtained from  $(-3, -3, -3, -3, -3, -1, -1, -1)$  by permutations of the coordinates and change of sign in an even number of places, then  $x_1 = x_2 = -3$ ,  $x_3 = -1$ ,  $x_4 = 1$  and  $x_5 = 3$ . By the equalities (1), (2), (7) and (8), it follows that  $x_6 = -\frac{11}{3}$ , which is not possible.

If  $x$  is obtained from  $(\pm 6, \pm 2, \pm 2, \pm 2, 0, 0, 0, 0)$  by permutations of the coordinates, then  $(x_2, x_3, x_4, x_5) = (-6, -2, 0, 2)$ , but  $x_1 = x_2 = -6$  is not possible; or  $(x_2, x_3, x_4, x_5) = (-2, 0, 2, 6)$ . In this case the equalities (1), (2), (7) and (8) imply  $x_6 = -4$ , which is not possible.

If  $x$  is obtained from  $(-5, 3, 3, 1, 1, 1, 1, 1)$  by permutations of the coordinates and change of sign in an even number of places, then  $x_2 \in \{-5, -3, -1\}$ .  $x_1 = x_2 = -5$  is not possible.  $x_2 = -3$  implies  $(x_1, x_2, x_3, x_4, x_5) = (-3, -3, -1, 1, 5)$  and equalities (1), (2), (7) and (8) imply  $x_6 = -\frac{13}{3}$ . This is again impossible.  $x_2 = -1$  implies  $(x_1, x_2, x_3, x_4, x_5) = (-1, -1, 1, 3, 5)$  and equalities (1), (2), (7) and (8) imply  $x_6 = x_7 = x_8 = -3$ , which cannot happen.

If  $x$  is obtained from  $(\pm 4, \pm 4, \pm 2, \pm 2, \pm 2, \pm 2, 0, 0)$  by permutations of the coordinates, then  $(x_1, x_2, x_3, x_4, x_5) = (-4, -4, -2, 0, 2)$  or  $(-2, -2, 0, 2, 4)$ . In both cases the equalities  $x_6 = x_7 = x_8$  cannot be satisfied.

So we have verified that  $\beta_6(1, 2, 7, 8)$  contains no 6-vertices in its interior and therefore our table is complete.

## B More information about $E_8$

In this section, we complete some tables given in Appendix A.4. Although this information is not directly used in the proof of our main result, we present it here in order to make it easier to verify the tables in Appendix A.4.

The next table lists the 7-vertices  $x$  in  $\beta_7$  with  $d(x, v_7) \geq \frac{\pi}{2}$ . The vertices marked with \* are the ones at distance  $= \frac{\pi}{2}$  to  $v_7$ . Let  $\sigma$  be the face of  $\Sigma_{v_7}\Delta$  containing  $\overrightarrow{v_7x}$  in its interior. Let  $\sigma_x$  be the face of  $\Delta$  spanned by the initial part of the segment  $v_7x$ .

$x$	Type of $\sigma$	$ Stab_{W_{E_8}}(v_7) \cdot x  = \frac{ Stab_{W_{E_8}}(v_7) }{ Stab_{W_{E_8}}(\sigma_x) }$
( 1, 1, 1, 1, 1, 1, 0, 0)		$ W_{E_6}  W_{A_1} /( W_{E_6}  W_{A_1} ) = 1$
( 0, 0, 1, 1, 1, 1, -1, -1)	3	$ W_{E_6}  W_{A_1} /( W_{A_1}  W_{A_4}  W_{A_1} ) = 216$
* $\frac{1}{2}$ (-1, -1, -1, 1, 1, 1, -3, -3)	4	$ W_{E_6}  W_{A_1} /( W_{A_2}  W_{A_1}  W_{A_2}  W_{A_1} ) = 720$
( 0, 0, 0, 0, 0, 2, -1, -1)	6	$ W_{E_6}  W_{A_1} /( W_{D_5}  W_{A_1} ) = 27$
$\frac{1}{2}$ ( 1, 1, 1, 1, 1, 1, -3, 3)	8	$ W_{E_6}  W_{A_1} / W_{E_6}  = 2$
$\frac{1}{2}$ (-1, 1, 1, 1, 1, 1, -3, -3)	12	$ W_{E_6}  W_{A_1} /( W_{A_4}  W_{A_1} ) = 432$
( 0, 1, 1, 1, 1, 1, -1, 0)	28	$ W_{E_6}  W_{A_1} / W_{D_5}  = 54$
$\frac{1}{2}$ (-3, 1, 1, 1, 1, 1, -3, -1)	28	$ W_{E_6}  W_{A_1} / W_{D_5}  = 54$
( 0, 0, 0, 0, 1, 1, -2, 0)	58	$ W_{E_6}  W_{A_1} /( W_{A_4}  W_{A_1} ) = 432$
( 0, 0, 0, 0, 0, 1, -2, 1)	68	$ W_{E_6}  W_{A_1} / W_{D_5}  = 54$
$\frac{1}{2}$ ( 1, 1, 1, 1, 1, 3, -3, 1)	68	$ W_{E_6}  W_{A_1} / W_{D_5}  = 54$
( 0, 0, 0, 0, 0, 1, -2, -1)	168	$ W_{E_6}  W_{A_1} / W_{A_4}  = 864$
* (-1, 0, 0, 0, 0, 1, -2, 0)	268	$ W_{E_6}  W_{A_1} / W_{D_4}  = 540$
$\frac{1}{2}$ (-1, 1, 1, 1, 1, 3, -3, -1)	268	$ W_{E_6}  W_{A_1} / W_{D_4}  = 540$

Notice that since the antipode  $\widehat{v}_7$  of  $v_7$  is also a 7-vertex, then the number of 7-vertices in  $S$  at distance  $\leq \frac{\pi}{2}$  to  $v_7$  is the same as the number of 7-vertices in  $S$  at distance  $\geq \frac{\pi}{2}$  to  $v_7$ . It follows that the number of 7-vertices in  $S$  is two times the number of 7-vertices in  $S$  at distance  $\leq \frac{\pi}{2}$  to  $v_7$  minus the number of 7-vertices at distance  $= \frac{\pi}{2}$  to  $v_7$ . With this observation and the one at the end of the introductory section of Appendix A, we can verify the correctness of the list above:  $2(1+216+720+27+2+432+54+54+432+54+54+864+540+540) - 720 - 540 = 6720 = \frac{|W_{E_8}|}{|Stab_{W_{E_8}}(v_7)|} = \#\{7\text{- vertices in } S\}$ .

The next table lists the 8-vertices  $x$  in  $\beta_7$  with  $d(x, v_7) \geq \frac{\pi}{2}$ . The vertices marked with \* are the ones at distance  $= \frac{\pi}{2}$  to  $v_7$ . Let  $\sigma$  be the face of  $\Sigma_{v_7}\Delta$  containing  $\overrightarrow{v_7x}$  in its interior.

$x$	Type of $\sigma$
* ( 0, 0, 0, 0, 0, 0, -2, -2)	1
(-1, 1, 1, 1, 1, 1, -1, -1)	2
( 1, 1, 1, 1, 1, 1, -1, 1)	8
* ( 0, 0, 0, 0, 0, 0, -2, 2)	8
( 0, 0, 0, 0, 0, 2, -2, 0)	68

The next table lists the 1-vertices  $x$  in  $\beta_1$  with  $d(x, v_1) \geq \frac{\pi}{2}$ . The vertices marked with \* are the ones at distance  $= \frac{\pi}{2}$  to  $v_1$ . Let  $\sigma$  be the face of  $\Sigma_{v_1}\Delta$  containing  $\overrightarrow{v_1x}$  in its interior.

$x$	Type of $\sigma$	$ Stab_{W_{E_8}}(v_1) \cdot x  = \frac{ Stab_{W_{E_8}}(v_1) }{ Stab_{W_{E_8}}(\sigma_x) }$
( 1, 1, 1, 1, 1, 1, 1, 1)		$ W_{A_7} / W_{A_7}  = 1$
$\frac{1}{2}(-5, 1, 1, 1, 1, 1, 1, 1)$	2	$ W_{A_7} / W_{A_6}  = 8$
(-1, -1, 1, 1, 1, 1, 1, 1)	3	$ W_{A_7} /( W_{A_1}  W_{A_5} ) = 28$
* (-1, -1, -1, -1, 1, 1, 1, 1)	5	$ W_{A_7} /( W_{A_3}  W_{A_3} ) = 70$
$\frac{1}{2}( 1, 1, 1, 1, 1, 3, 3, 3)$	6	$ W_{A_7} /( W_{A_2}  W_{A_4} ) = 56$
(-2, 0, 0, 0, 1, 1, 1, 1)	25	$ W_{A_7} /( W_{A_2}  W_{A_3} ) = 280$
$\frac{1}{2}(-1, 1, 1, 1, 1, 1, 1, 5)$	28	$ W_{A_7} / W_{A_5}  = 56$
$\frac{1}{2}(-1, -1, 1, 1, 1, 3, 3, 3)$	36	$ W_{A_7} /( W_{A_1}  W_{A_2}  W_{A_2} ) = 560$
$\frac{1}{2}(-3, -3, 1, 1, 1, 1, 1, 3)$	38	$ W_{A_7} /( W_{A_1}  W_{A_4} ) = 168$
( 0, 0, 0, 1, 1, 1, 1, 2)	48	$ W_{A_7} /( W_{A_2}  W_{A_3} ) = 280$
$\frac{1}{2}(-1, -1, -1, 1, 1, 1, 1, 5)$	48	$ W_{A_7} /( W_{A_2}  W_{A_3} ) = 280$
$\frac{1}{2}(-1, -1, -1, -1, 1, 3, 3, 3)$	56	$ W_{A_7} /( W_{A_2}  W_{A_3} ) = 280$
$\frac{1}{2}(-1, -1, -1, -1, -1, 1, 1, 5)$	68	$ W_{A_7} /( W_{A_4}  W_{A_1} ) = 168$
* (-2, -1, 0, 0, 0, 1, 1, 1)	236	$ W_{A_7} /( W_{A_2}  W_{A_2} ) = 1120$
$\frac{1}{2}(-3, -1, 1, 1, 1, 1, 3, 3)$	237	$ W_{A_7} /( W_{A_3}  W_{A_1} ) = 840$
$\frac{1}{2}(-3, -1, -1, -1, 1, 1, 3, 3)$	257	$ W_{A_7} /( W_{A_2}  W_{A_1}  W_{A_1} ) = 1680$
(-1, 0, 0, 0, 1, 1, 1, 2)	258	$ W_{A_7} /( W_{A_2}  W_{A_2} ) = 1120$
(-1, -1, 0, 0, 0, 1, 1, 2)	368	$ W_{A_7} /( W_{A_2}  W_{A_1}  W_{A_1} ) = 1680$
* (-1, -1, -1, 0, 0, 0, 1, 2)	478	$ W_{A_7} /( W_{A_2}  W_{A_2} ) = 1120$

We can verify this table as we did with the table above:  $2(1 + 8 + 28 + 70 + 56 + 280 + 56 + 560 + 168 + 280 + 280 + 280 + 168 + 1120 + 840 + 1680 + 1120 + 1680 + 1120) - 70 - 1120 - 1120 = 17280 = \frac{|W_{E_8}|}{|W_{A_7}|} = \#\{1\text{- vertices in } S\}$ .

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