ON APPROXIMATELY s-CONVEX FUNCTIONS

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ABSTRACT. A real valued function $f : D \to \mathbb{R}$ defined on an open convex subset D of a normed space X is called *Breckner rationally* (s, d)*-convex* if it satisfies

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y) + d(x, y)$$

for all $x, y \in D$ and $t \in \mathbb{Q} \cap [0, 1]$, where $d : X \times X \to \mathbb{R}$ is a given function and $s \in]0, 1]$ is a fixed parameter.

Our main result is a Bernstein-Doetsch type one. Namely, we prove that, under some natural assumptions, if f is locally bounded from above at a point of D and Breckner rationally (s, d)-convex then it is Breckner (s, d)-convex.

1. INTRODUCTION

It is a well known fact that convexity and its generalization plays important role in different part of mathematics, mainly in optimization theory. In our paper we deal with a common generalization of s-convexity, approximate convexity, and results of Bernstein and Doetsch [BD15].

The concept of s-convexity and rational s-convexity was introduced by Breckner [Bre78]. A real valued function $f : D \to \mathbb{R}$ (where D is a convex, open, nonempty subset of a real (complex) linear space X) is called *Breckner s-convex* (or briefly *s-convex*), if

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y) \tag{1}$$

for every $x, y \in D$ and $\lambda \in [0, 1]$, where $s \in]0, 1]$ is a fixed number. A real valued function $f : D \to \mathbb{R}$ is called *rationally s-convex* if it fulfills (1) for all $\lambda \in \mathbb{Q} \cap [0, 1]$.

This concept is a real generalization of convexity, we get it from (1) with $s = 1$.

In [BO78] and [HM94] it was proved that s-convex functions are nonnegative, when $0 < s < 1$, moreover the set of s-convex functions increases as s decreases (see also [BHJ09] for further information).

We investigate, similarly to [Haz07], the following generalizations of this convexity property. Let $s \in]0, 1]$ be a fixed parameter and let the function $d : X \times X \to \mathbb{R}$ be given. A function $f : D \to \mathbb{R}$ is said to be *Breckner* (s, d)*-convex* (or shortly (s, d)*-convex*) if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y) + d(x, y) \tag{2}$$

for all $x, y \in D$ and $t \in [0, 1]$.

A real valued function $f : D \to \mathbb{R}$ is called *Breckner rationally* (s, d)*-convex (or rationally* (s, d)*-convex)* (in notation $(\mathbb{Q}, s, d)$-convex) if it fulfills (2) for all $t \in \mathbb{Q} \cap [0, 1]$, and it is called *Breckner* (t, s, d)*-convex (or* $(, t, s, d)$*-convex)* if it fulfills (2) for a fixed parameter $t \in]0, 1[$.

Date: October 30, 2009.

1991 *Mathematics Subject Classification.* 26A51, 26B25, 39B62.

Key words and phrases. convexity, approximately convexity, s-convexity, Bernstein–Doetsch theorem, regularity properties of generalized convex functions.

This research has been supported by the Hungarian Scientific Research Fund (OTKA) Grants K-62316 and NK-68040.

In our investigation we assume that the following natural properties of the function $d : X \times X \to \mathbb{R}$ hold, namely,

$$\begin{array}{lrcl} (i) & d(x,y) & \geq & 0, \\ (ii) & d(x,y) & = & d(y,x), \\ (iii) & d(x,y) & \leq & d(x,z)+d(z,y), \\ (iv) & d(x+z,y+z) & = & d(x,y), \\ (v) & d(ux,uy) & \leq & \psi(u)d(x,y) \end{array}$$

for all $x, y, z \in X$ and $u > 0$, where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$. The first three properties declare that d is a semimetric on X, (iv) states the translation invariance of d and (v) is the subhomogeneity of d with respect to ψ. If $(i) - (v)$ hold, then we say that d is a ψ *-subhomogeneous, translation invariant semimetric.*

It is easy to see that all nonnegative constant functions satisfies the properties $(i)-(v)$, with $\psi = 1$ and the function $d(x,y) = \|x-y\|^p$ (where $p \geq 1$) is also ψ-subhomogeneous, translation invariant semimetric functions, with $\psi(u) = u^p$.

In Section 2 we collect some auxiliary result. In Section 3 and 4 we prove some boundedness, continuity and convexity property of approximately Breckner s-convex functions. As the main results of our paper we show that Breckner (t, s, d)-convexity and locally upper boundedness of a function at a point of its domain implies local boundedness and continuity on the whole domain; and rational (s, d)-convexity and local boundedness from above at a point of D implies Breckner (s, d)-convexity.

2. Known results

Throughout this paper let $(X, \|\cdot\|)$ be a normed space and $D \subseteq X$ a nonempty open convex set.

Bernstein and Doetsch in [BD15] proved that the local upper boundedness of a Jensen-convex function yields its continuity and convexity as well. Breckner in [Bre78] proved the following:

Theorem 1. *Let $f : D \to \mathbb{R}$ be a rationally s-convex function. If it is locally bounded from above at a point of D, then it is continuous and s-convex.*

In [BHJ09] was proved the following:

Theorem 2. *Let $H \subseteq [0,1]$ and assume that* 0 *or* 1 *is an accumulation point of H. If $f : D \to \mathbb{R}$ is locally bounded at a point of D, and satisfies* (1) *for all $t \in H$, then it is continuous on D.*

The approximately usual convexity (when $s = 1$) was investigated in several papers. The particular case, the so called $(t, 1, d)$-convexity (where d is a ψ-subhomogeneous, translation invariant semimetric), was studied in [Haz07]:

Theorem 3. *If $f : D \to \mathbb{R}$ satisfies the inequality*

$$f\left(tx+(1-t)y\right) \leq tf(x)+(1-t)f(y)+d(x,y)$$

for all $x, y \in D$, where $t \in]0,1[$ is fixed, and locally bounded from above at a point of D, where the function d is a ψ-subhomogeneous, translation invariant semimetric, then

$$f\left(\lambda x+(1-\lambda)y\right) \leq \lambda f(x)+(1-\lambda)f(y)+\varphi(\lambda)d(x,y) \tag{3}$$

for all $x, y \in D$ and $\lambda \in [0,1]$, where φ the unique and existing fixed point of the operator

$$\left(\mathcal{T}_{\psi,t}\varphi\right)(\lambda) = (1-t)\varphi\left(\frac{\min(\lambda,1-\lambda)}{1-t}\right)+\psi\left(\frac{\min(\lambda,1-\lambda)}{1-t}\right).$$

The particular case of this result, when $d(x,y)=\sum_{i=0}^{k}\varepsilon_i\|x-y\|^{p_i}$, was studied in [HP05]. When $d(x,y)=\varepsilon_0+\varepsilon_1\|x-y\|^p$ is formally a special case of the previous one, however a stronger result was obtained in [Haz05] in this setting. If $d(x,y)=\varepsilon_0+\varepsilon_1\|x-y\|$ and $t=1/2$, then the operator fixed point can be explicitly computed in the terms of the so-called Takagi function (see [HP04]). The case $d(x,y)=\varepsilon_0$ and $t=1/2$ was studied by Nikodem and Ng [NN93], the specialization $d(x,y)=0$ yields the theorem of Bernstein and Doetsch [BD15] (see also [Kuc85] for further references).

3. Regularity properties of approximately Breckner s-convex functions

We deal with boundedness and continuity properties of Breckner (t,s,d)-convex and $(\mathbb{Q},s,d)$-convex functions at first. We recall that a function $f:D\to\mathbb{R}$ is called locally bounded from above on D, if for each point of D, there exists a neighborhood U of this point such that f is bounded from above on U.

Theorem 4. *Assume that the function d is a continuous, ψ-subhomogeneous, translation invariant semimetric. If $f:D\to\mathbb{R}$ is Breckner (t,s,d)-convex and locally bounded from above at a point $w\in D$, then f is locally bounded on D.*

Proof. First we prove that f is locally bounded from above on D. Define the sequence of sets D_n by

$$D_0:=\{w\},\qquad\qquad D_{n+1}:=tD_n+(1-t)D.$$

Then, it follows by induction that

$$D_n=t^nw+(1-t^n)D.$$

Using induction on n, we prove that f is locally upper bounded at each point of D_n. By assumption f is locally upper bounded at $w\in D_0$. Assume that f is locally upper bounded at each point of D_n. For an arbitrary $x\in D_{n+1}$, there exist $x_0\in D_n$ and $y_0\in D$ such that $x=tx_0+(1-t)y_0$. By the inductive assumption, there exists $r>0$ and constant $M_0\geq 0$ such that $f(x')\leq M_0$ and $d(x',x_0)\leq M_0$ for all $x'\in B(x_0,r)$, where $B(x_0,r)$ denotes the open neighborhood centered at x_0 with radius r. Because of the continuity of d, we can choose r such that the previous will be true. Then, by the (t,s,d)-convexity of f, we have

$$\begin{aligned} f\left(tx'+(1-t)y_0\right) &\leq t^sf(x')+(1-t)^sf(y_0)+d(x',y_0)\\ &\leq t^sM_0+(1-t)^sf(y_0)+d(x',x_0)+d(x_0,y_0)\\ &\leq t^sM_0+(1-t)^sf(y_0)+M_0+d(x_0,y_0)=:M.\end{aligned}$$

Therefore, for $y\in U:=tB(x_0,r)+(1-t)y_0=B\left(tx_0+(1-t)y_0,tr\right)=B(x,tr)$, we get that $f(y)\leq M$. Thus f is locally bounded above at $x\in D_{n+1}$, so f is locally bounded above on D_{n+1} .

On the other hand, one can easily see that

$$D=\bigcup_{n=1}^{\infty}D_n.$$

Indeed, for fixed $x\in D$, define the sequence x_n by

$$x_n:=\frac{x-t^nw}{1-t^n}.$$

Then $x_n\to x$ if $n\to\infty$. As the set is open, so there exists an $n_0\in\mathbb{N}$, such that $x_n\in D$ if $n\geq n_0$. Therefore

$$x=t^nw+(1-t^n)x_n\in t^nw+(1-t^n)D=D_n.$$

Thus f is locally bounded from above on D.

We prove now that f is locally bounded from below. Let $q \in D$ be arbitrary. Since f is locally bounded from above at the point q, there exist $\varrho > 0$ and $M > 0$ such that $f(x) \leq M$ and $d(x,q) \leq M$ if $x \in B(q,\varrho)$. (Just like in the first part of the proof, we can find such ϱ, using the continuity of d.) Let $x \in B(q,(1-t)\varrho)$ and $y := \frac{1}{1-t}q - \frac{t}{1-t}x$. Then y is in $B(q,t\varrho) \subset B(q,\varrho)$. Then, by (t,s,d)-convexity of f, and by ψ-subhomogeneity and translation invariance of d, we get

$$\begin{aligned} f(q) &= f(tx+(1-t)y) \leq t^s f(x) + (1-t)^s f(y) + d(x,y) \\ &\leq t^s f(x) + (1-t)^s f(y) + \psi\left(\frac{1}{1-t}\right) d(x,q), \end{aligned}$$

which implies

$$\begin{aligned} f(x) &\geq \frac{1}{t^s} f(q) - \frac{(1-t)^s}{t^s} f(y) - \frac{1}{t^s}\psi\left(\frac{1}{1-t}\right) d(x,q) \\ &\geq \frac{1}{t^s} f(q) - \frac{(1-t)^s}{t^s} M - \frac{1}{t^s}\psi\left(\frac{1}{1-t}\right) M =: M^*. \end{aligned}$$

Therefore f is locally bounded from below at any point of D. □

Corollary 1. *If $f : D \to \mathbb{R}$ is Breckner $(\mathbb{Q},s,d)$-convex and locally bounded from above at a point of D (where d is a continuous, ψ-subhomogeneous, translation invariant semimetric), then f is locally bounded on D.*

The next statement is an immediate consequence of the previous theorem and Steinhaus' and Piccard's theorems (cf. [Ste20], [Pic42]).

Corollary 2. *Let D be an open convex subset of $\mathbb{R}^n$ and $f : D \to \mathbb{R}$ be a (t,s,d)-convex (or $(\mathbb{Q},s,d)$-convex) function. Assume that the function d continuous, ψ-subhomogeneous, translation invariant semimetric and there exist a Lebesgue-measurable set of positive measure (or a Baire-measurable set of second category) $S \subseteq D$ and a Lebesgue-measurable (resp. Baire-measurable) function $g : S \to \mathbb{R}$ such that $f \leq g$ on S. Then f is locally bounded on D.*

The next result states that the local upper boundedness of a rationally (s,d)-convex function at a point of D yields its continuity at this point as well.

Theorem 5. *Assume that the function d is a continuous, ψ-subhomogeneous, translation invariant semimetric and $d(x,x) = 0$. If $f : D \to \mathbb{R}$ is $(\mathbb{Q},s,d)$-convex and locally bounded from above at a a point $w \in D$, then it is continuous at w.*

Proof. Using the local boundedness from above of f at a point $w \in D$, there exist a neighborhood $B(w,r)$ and a constant $K \geq 0$ such that $f(x) \leq K$ for every $x \in B(w,r)$. Let ε be an arbitrary positive constant. Then there exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ is an arbitrarily fixed positive integer, then

$$\left(\frac{1}{n}\right)^s K + \left[\left(1-\frac{1}{n}\right)^s - 1\right] f(w) < \frac{\varepsilon}{4}, \tag{4}$$

$$\left(\frac{1}{n-1}\right)^s K + \left[1 - \frac{1}{(1-\frac{1}{n})^s}\right] f(w) < \frac{\varepsilon}{4} \tag{5}$$

and

$$\frac{2}{(1-\frac{1}{n})^s} < 3. \tag{6}$$

Let $r_1 = \min\{r, \frac{\varepsilon}{4}\}$. By the continuity of d and using the assumption $d(w,w) = 0$, there exists $r_1' < r_1$ such that $d(x,w) < r_1$ if $x \in B(w, r_1')$, and let $\delta < \frac{r_1'}{n}$. We prove that

$$|f(x) - f(w)| < \varepsilon \qquad (x \in B(w,\delta)).$$

For $x \in B(w,\delta)$ there exist $y, z \in B(w, r_1')$ such that

$$\begin{aligned} x &= \frac{1}{n}y + \left(1 - \frac{1}{n}\right)w, \qquad \text{so} \qquad y = nx - (n-1)w \\ w &= \frac{1}{n}z + \left(1 - \frac{1}{n}\right)x, \qquad \text{so} \qquad z = nw - (n-1)x. \end{aligned}$$

Indeed,

$$\|y - w\| = \|nx - nw\| = n\|x - w\| \le n\delta < r_1',$$

and similarly

$$\|z - w\| = \|(n-1)(x - w)\| = (n-1)\|x - w\| \le (n-1)\delta < r_1';$$

that is $y, z \in B(w, r_1')$.

According to $(\mathbb{Q}, s, d)$-convexity of f,

$$\begin{aligned} f(x) &\le \left(\frac{1}{n}\right)^s f(y) + \left(1 - \frac{1}{n}\right)^s f(w) + d(y,w) \qquad (7) \\ &\le \left(\frac{1}{n}\right)^s K + \left(1 - \frac{1}{n}\right)^s f(w) + r_1, \end{aligned}$$

and

$$\begin{aligned} f(w) &\le \left(\frac{1}{n}\right)^s f(z) + \left(1 - \frac{1}{n}\right)^s f(x) + d(z,x) \\ &\le \left(\frac{1}{n}\right)^s K + \left(1 - \frac{1}{n}\right)^s f(x) + d(z,w) + d(w,x) \qquad (8) \\ &\le \left(\frac{1}{n}\right)^s K + \left(1 - \frac{1}{n}\right)^s f(x) + 2r_1. \end{aligned}$$

Using (7) and (4) we get

$$\begin{aligned} f(x) - f(w) &\le \left(\frac{1}{n}\right)^s K + \left[\left(1 - \frac{1}{n}\right)^s - 1\right] f(w) + r_1 \qquad (9) \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon \end{aligned}$$

and using the inequality (8) we get

$$f(x) \ge \frac{f(w) - \left(\frac{1}{n}\right)^s K - 2r_1}{(1 - \frac{1}{n})^s},$$

which together with (5) imply that

$$\begin{aligned} f(x) - f(w) &\ge \left[\frac{1}{(1 - \frac{1}{n})^s} - 1\right] f(w) - \left(\frac{1}{n-1}\right)^s K - \frac{2r_1}{(1 - \frac{1}{n})^s} \\ &> -\left(\frac{\varepsilon}{4} + \frac{2}{(1 - \frac{1}{n})^s}\frac{\varepsilon}{4}\right). \end{aligned}$$

According to (6) we get

$$f(x) - f(w) > -\left(\frac{\varepsilon}{4} + 3\frac{\varepsilon}{4}\right) = -\varepsilon. \tag{10}$$

The inequalities (9) and (10) show that $|f(x) - f(w)| < \varepsilon$, that is f is continuous at w, so the proof is complete. □

Remark 1. *This result is not true for (t, s, d)-convex functions. In the case, when $d(x, y) = 0$ for all $x, y \in D$, in* [BHJ09] *an example was given, which shows that the Breckner $(1/2, s, 0)$-convexity and locally upper boundedness do not imply the continuity of the function. For the reader's convenience we repeat this example: Let*

$$f(x) := \begin{cases} x^s, & \text{if } x \in](2^s - 1)^{1/s}, 1[\setminus \mathbb{Q}; \\ 1, & \text{if } x \in](2^s - 1)^{1/s}, 1[\cap \mathbb{Q}. \end{cases}$$

Then f is Breckner $(1/2, s, 0)$-convex, bounded and nowhere continuous.

Theorem 6. *Assume that the function d is a continuous, translation invariant semimetric and $d(x, x) = 0$. If $f : D \to \mathbb{R}$ is Breckner rationally (s, d)-convex or Breckner (s, d)-convex and locally bounded from above at a point of D, then it is continuous.*

Proof. According to Theorem 4, f is locally bounded at every point of D. So, we can use the previous theorem, which implies the continuity of f at every point of D. □

4. Convexity property of $(\mathbb{Q}, s, d)$-convex functions

The following result offers a generalization of the theorem of Breckner [Bre78] presented in the introduction.

Theorem 7. *Assume that the function d is a continuous, ψ-subhomogeneous, translation invariant semimetric and $d(x, x) = 0$. If $f : D \to \mathbb{R}$ is $(\mathbb{Q}, s, d)$-convex and locally bounded from above at a a point $w \in D$, then it is Breckner (s, d)-convex.*

Proof. We prove that the function f is (t, s, d)-convex for all $t \in [0, 1]$. Let $t \in [0, 1]$ arbitrary. Then there exists a sequence $\{t_n\}_{n\in\mathbb{N}}$ such that $t_n \in \mathbb{Q}$ and $t_n \to t$ (when n tends to ∞). Applying $(\mathbb{Q}, s, d)$-convexity of f, we get

$$f\left(t_n x + (1 - t_n)y\right) \le t_n^s f(x) + (1 - t_n)^s f(y) + d(x, y). \tag{11}$$

The local upper boundedness of f implies the continuity of f (according to Theorem 6). Therefore, taking the limit $n \to \infty$ in (11), we get

$$f\left(tx + (1 - t)y\right) \le t^s f(x) + (1 - t)^s f(y) + d(x, y),$$

which proves the Breckner (s, d)-convexity of f. □

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