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CONSTRUCTION OF EINSTEIN METRICS BY GENERALIZED DEHN FILLING

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ABSTRACT. In this paper, we present a new approach to the construction of Einstein metrics by a generalization of Thurston's Dehn filling. In particular in dimension 3, we will obtain an analytic proof of Thurston's result.

1. Introduction

The goal of this paper is to give an analytic construction of Riemannian metrics g which satisfy the Einstein equation $\mathrm{Ric}_g = -(n-1)g$, by a process similar to Thurston's Dehn filling (see [Thu]). We will first describe the topology of the manifolds on which these metrics live:

Let (N^n, g_h) be a hyperbolic manifold of dimension $n \geq 3$ and finite volume. Denote its ends by $E_1, \ldots, E_p$ and assume that these are diffeomorphic to $T^{n-1} \times (0, \infty)$. We can always choose the E_k so that they are bounded by tori $T_k = \partial E_k$ which are images of horospheres under the universal covering projection and on which $\mathrm{inj} = \mu_n$ where inj is the injectivity radius and μ_n the Margulis constant. (For a more detailed description see subsection 2.1.) Now apply the following surgery procedure: Cut N along the T_k, throw away the cusps E_k and glue in p solid tori $\approx D^2 \times T^{n-2}$ by identifying their boundary with the T_k. The topology of the resulting manifold can be uniquely characterized by the homotopy classes of meridional loops $\sigma_k \subset \partial(D^2 \times T^{n-2})$ inside the T_k (i.e. images of loops $S^1 \times \{pt\} \subset D^2 \times T^{n-2}$ under the gluing identification $\partial(D^2 \times T^{n-2}) \to T_k$). These homotopy classes are simple, i.e. not a nontrivial multiple of another homotopy class. Vice versa, given a homotopy class of a simple closed loop $\sigma_k \subset T_k$ for each k, we can produce a manifold $M_{\overline{\sigma}} = M_{(\sigma_1, \ldots, \sigma_p)}$ by this gluing. In the following we will always assume that the $\sigma_1, \ldots, \sigma_p$ are geodesic representatives (inside $T_1, \ldots, T_p$) of their homotopy classes and set $\ell_k := \ell(\sigma_k)$ and $\ell_{\min\,\text{resp.}\max} := (\min\text{resp.}\max)(\ell_k)$.

The statement of the theorem which we are going to prove, is now:

Theorem 1.1. *There is a constant $L = L(n, V)$ such that whenever $\mathrm{vol}\, N < V$ and $\ell_{\min} > L$, the manifold $M_{\overline{\sigma}}$ carries an Einstein metric $g_{\overline{\sigma}}$.*

Moreover, the metrics on $M_{\overline{\sigma}}$ can be constructed in such a way that as $\ell_{\min} \to \infty$, the $(M_{\overline{\sigma}}, g_{\overline{\sigma}})$ converge to the initial hyperbolic manifold (N, g_h) in the pointed Gromov-Hausdorff sense if the basepoints are chosen away from the cusps.

A slightly weaker statement was also claimed in [And2]. Theorem 1.1 immediately implies the Dehn filling Theorem in dimension 3:

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Corollary 1.2. *Let the dimension* $n = 3$. *There is a constant* $L = L(V)$ *such that whenever* $\operatorname{vol} N < V$ *and* $\ell_{\min} > L$, *the manifold* $M_{\overline{\sigma}}$ *is hyperbolic.*

A slightly weaker version of this theorem was proven by Thurston ([Thu]) using the deformation theory of Kleinian groups. Our methods provide a new and analytic proof of his result.

We will give a short sketch of the proof of Theorem 1.1: First, we endow the solid tori which we will glue into the hyperbolic manifold N with a special Einstein metric called the *black-hole metric.* This metric is asymptotically hyperbolic to its end and and hence each gluing can be arranged arbitrarily smooth for large ℓ_k. Hence, the resulting metric is almost Einstein, i.e. its traceless Ricci tensor is small in some $C^{m,\alpha}$-sense. Eventually, we apply an inverse function theorem like argument to perturb the metric into the desired Einstein metric.

We mention that our proof builds on previous work of Tian ([Tia]) and Anderson ([And2]). Tian established the 3-dimensional case in which N has only one cusp. Later Anderson described a construction for the higher dimensional case and developed new analytical tools of which we will also partly make use here.

We want to point out that the case in which the hyperbolic manifold N has more than one cusp, is substantially more difficult than the case of one cusp. For the following reason: The accuracy of the gluing in the first step (i.e. the construction of the almost Einstein metric) depends exponentially on the minimum $\ell_{\min}$ of the ℓ_k. However, as the ℓ_k get large, the invertibility of the linearized Einstein equation deteriorates linearly in the maximum $\ell_{\max}$ of the ℓ_k. So in the case of one cusp $\ell_{\min} = \ell_{\max}$ and thus the accuracy of the gluing increases more rapidly than the invertibility deteriorates. But if N has more than one cusp and $\ell_{\min}$, $\ell_{\max}$ are not sufficiently controlled towards each other, then this consideration fails. In [And2], Anderson sketches an argument how to get around this issue by looking at certain moduli spaces of solutions of a partly fulfilled Einstein equation. In this paper, we will be able to deal with the problems that arise in this multiple cusp case and we will give a complete proof of Theorem 1.1. In fact, our argument will be more elementary and we find it a more natural way of looking at the problem.

The idea behind our proof is that the reason for the bad invertibility of the linearized Einstein equation lies in certain variations of the metric (so called *trivial Einstein variations*) which correspond to a change of the moduli of the cross-sectional tori of the cusps. It will turn out that with respect to some cleverly chosen norms (see section 4), which treat these trivial Einstein variations separately, the invertibility of the linearized Einstein equation becomes in fact independent of $\ell_{\max}$ (see our Proposition 5.1 as opposed to Proposition 3.2 in [And2]). However, these new norms make it necessary to reprove the inverse function theorem in order to be applicable to our setting (see section 5).

Another difference between our and Anderson's proof is that we have replaced the proof of Lemma 3.4 in [And2] which used to involve the theory of the moduli spaces of conformally compact Einstein metrics, by an elementary argument by which we can even show a slight generalization.

We remark that it still remains an interesting question whether the constant L in Theorem 1.1 can be chosen independent of the volume of N. Hodgson and Kerckhoff (see [HK]) could confirm this in dimension 3 using algebraic techniques.

The paper is organized as follows: In section 2, we will review some basic facts which were also used in [And2]. Section 3 contains a brief recapitulation of the construction of the almost Einstein metric as described in [And2]. In sections 4 to 6 we carry out the main argument. In order to keep these chapters concise, we will defer most of the technical calculations to sections 7 and 8.

I want to thank my advisor Gang Tian for calling my attention on this subject and for his continuous support. I am also indebted to the HIM in Bonn for their hospitality and Hans-Joachim Hein for many inspiring discussions.

2. Preliminaries

2.1. **Hyperbolic manifolds.** We recall the *thick-thin-decomposition* for hyperbolic manifolds

Theorem 2.1. *There is a constant $\mu_n > 0$, the* Margulis constant, *such that the following holds: If N^n is a finite volume hyperbolic manifold then N can be decomposed into a* thin part N_{thin} *and a* thick part N_{thick} *with $N = N_{thin} \dot{\cup} N_{thick}$ such that:*

- inj $\geq \mu_n$ *on N_{thick} and N_{thick} is compact.*
- *N_{thin} is a finite union of connected open sets $E_1, \ldots, E_p$ and $E'_1, \ldots, E'_{p'}$ where the E_k are cusps of the form $(T^{n-1}/\Gamma_k) \times \mathbb{R}$ for finite subgroups $\Gamma_k <$ $\operatorname{Isom} T^{n-1}$ and the E'_k are covered by cylindrical neighborhoods around geodesics in hyperbolic space.*
  Furthermore, we can choose the E_k such that their boundaries are images of horospheres under the universal covering projection and such that inj $=$ *μ_n on the ∂E_k.*

In every dimension, $\operatorname{diam} N_{thick}$ *is bounded from above by a constant which only depends on an upper bound on* $\operatorname{vol} N$ *and in dimension $n \neq 3$, this is even true for the diameter of $N_{thick} \cup E'_1 \cup \ldots \cup E'_{p'} = N \setminus E_1 \cup \ldots \cup E_p$.*

We can compute the volume of the cusps in terms of their boundary surface:

Lemma 2.2. *There is a constant η_n such that for all cusps E_k we have*

$$\operatorname{vol} E_k = \eta_n \operatorname{vol} \partial E_k.$$

So a bound on the volume of N gives us a bound on the volume of the ∂E_k. Since ∂E_k lies in the thick part, we have a bound on the injectivity radius of ∂E_k (which is slightly larger than μ_n since ∂E_k is not totally geodesic). The next lemma shows that in fact we get a bound on the diameter of ∂E_k from an upper volume bound on N. This implies furthermore, that the tori that can occur as cusp cross-sections of a hyperbolic manifold with a given volume bound form a bounded subset in the moduli space of flat tori.

Lemma 2.3. *For every $V < \infty$ and $\iota > 0$ there is a $d(n, V, \iota) < \infty$ such that for any flat torus T^{n-1} we have*

$$\operatorname{vol} T^{n-1} < V \quad \text{and} \quad \operatorname{inj} T^{n-1} > \iota \quad \Longrightarrow \quad \operatorname{diam} T^{n-1} < d.$$

Proof. Let $\gamma : [0, l] \to T^{n-1}$ be a minimizing geodesic. Then the balls $B_\iota(\gamma(\iota))$, $B_\iota(\gamma(3\iota)), \ldots$ are pairwise disjoint and have volume $\omega_{n-1}\iota^{n-1}$. So $l < 2(\frac{V}{\omega_{n-1}\iota^{n-1}} + 1)\iota$. □

2.2. **The Einstein operator.** For any symmetric bilinear form h and any 1-form α on a Riemannian manifold (M, g) we define the divergence and its formal conjugate by

$$\delta_g(h) = -\operatorname{tr}_{12} \nabla h, \qquad (\delta_g^* \alpha)(X, Y) = \frac{1}{2}((\nabla_X \alpha)(Y) + (\nabla_Y \alpha)(X)).$$

Observe that $\delta_g^* \alpha = \frac{1}{2}\mathcal{L}_{\alpha^\sharp} g$. Let h be a bilinear form. We can express the derivative of the Ricci curvature in the direction of H by (for a computation see [Top, sec 2.3])

$$\mathrm{d}\operatorname{Ric}_g(h) = -\frac{1}{2}\triangle_L h - \delta_g^*(\delta_g h + \frac{1}{2}\mathrm{d}\operatorname{tr}_g h).$$

Here $(\triangle_L h)(X,Y) = (\triangle h)(X,Y) + 2R(h)(X,Y) - h(\operatorname{Ric}(X), Y) - h(X, \operatorname{Ric}(Y))$ is the Lichnerowicz Laplacian and $R(h)(X,Y) = \operatorname{tr} h(R(\cdot, X)Y, \cdot)$. Since computing the Ricci tensor is a diffeomorphism invariant operation, we have for any 1-form α

$$\mathrm{d}\operatorname{Ric}_g(\delta_g^* \alpha) = \frac{1}{2}\mathcal{L}_{\alpha^\sharp} \operatorname{Ric}_g. \tag{2.1}$$

Thus $\mathrm{d}\operatorname{Ric}_g$ is not an elliptic operator. In order to make it elliptic, we have to add an extra term: Let $\overline{g}$ be an arbitrary fixed background metric on M. We define $\Psi_{\overline{g}} : \{g \in C^\infty(M; \operatorname{Sym}_2 T^*) \ : \ g > 0\} \to C^\infty(M; \operatorname{Sym}_2 T^*)$ by

$$\Psi_{\overline{g}}(g) := \operatorname{Ric}_g + \delta_g^*(\delta_{\overline{g}} g + \frac{1}{2}\mathrm{d}\operatorname{tr}_{\overline{g}} g).$$

Its derivative at $\overline{g}$ is

$$(\mathrm{d}\Psi_{\overline{g}})_{\overline{g}}(h) = -\frac{1}{2}\triangle_L h$$

hence elliptic. For our purposes we define the Einstein operator $\Phi_{\overline{g}} : \{g \in C^\infty(M; \operatorname{Sym}_2 T^*) \ : \ g > 0\} \to C^\infty(M; \operatorname{Sym}_2 T^*)$ by

$$\Phi_{\overline{g}}(g) = \Psi_{\overline{g}}(g) + (n-1)g.$$

We have

$$\begin{aligned}(\mathrm{d}\Phi_{\overline{g}})_{\overline{g}}(h) &= -\frac{1}{2}\triangle_L h + (n-1)h \\ &= \frac{1}{2}\left(-\triangle h - 2R(h) + \operatorname{Ric} \circ h + h \circ \operatorname{Ric} + 2(n-1)h\right).\end{aligned}$$

Set $L_{\overline{g}} := 2(\mathrm{d}\Phi_{\overline{g}})_{\overline{g}}$ and call elements in the kernel of $L_{\overline{g}}$ *Einstein variations.* Using a Weitzenböck formula, we can express this linear operator as

$$L_{\overline{g}} h = (\delta\delta^* + d^* d)h - R(h) + \frac{1}{2}\operatorname{Ric} \circ h + \frac{1}{2} h \circ \operatorname{Ric} + 2(n-1)h$$

where $d : C^\infty(M; \mathrm{Sym}_2 T^*) \to C^\infty(M; \Lambda_2 T^* \otimes T^*)$ and its formal conjugate $d^* : C^\infty(M; \Lambda_2 T^* \otimes T^*) \to C^\infty(M; \mathrm{Sym}_2 T^*)$ are defined by

$$(dh)(X, Y, Z) = (\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z),$$

$$(d^* t)(X, Y) = -\frac{1}{2}(\mathrm{tr}_{12}(\nabla t)(\cdot, X, Y) + \mathrm{tr}_{12}(\nabla t)(\cdot, Y, X)).$$

If $\overline{g}$ is Einstein with $\mathrm{Ric}_{\overline{g}} = -(n-1)\overline{g}$, we have

$$L_{\overline{g}} h = -\triangle h - 2R(h) = (\delta\delta^* + d^* d)h - R(h) + (n-1)h.$$

Tracing this equation gives us

$$\mathrm{tr}\, L_{\overline{g}} h = \nabla^* \nabla \,\mathrm{tr}\, h + 2(n-1)\,\mathrm{tr}\, h. \tag{2.2}$$

If $\overline{g}$ is hyperbolic of constant sectional curvature -1, we get

$$L_{\overline{g}} h = -\triangle h - 2h + 2(\mathrm{tr}_{\overline{g}}\, h)\overline{g} = (\delta\delta^* + d^* d)h + (\mathrm{tr}\, h)_{\overline{g}}\overline{g} + (n-2)h. \tag{2.3}$$

Lemma 2.4. *If M is closed and $\mathrm{Ric}_g < 0$, then $\Phi_{\overline{g}}(g) = 0$ implies $\mathrm{Ric}_g = -(n-1)g$.*

Proof. We copy the proof from [And2, Lemma 2.1] since we need a variation of the argument later on. Let $\beta_g(h) := \delta_g h + \frac{1}{2} \mathrm{d}\,\mathrm{tr}_g\, h$ be the Bianchi operator. Applying β_g to $\Phi_{\overline{g}}(g) = 0$ yields

$$0 = \beta_g(\Phi_{\overline{g}}(g)) = \beta_g \delta_g^* \beta_{\overline{g}}(g) = \frac{1}{2}\left(\nabla^* \nabla \beta_{\overline{g}}(g) - \mathrm{Ric}_g(\beta_{\overline{g}}(g))\right). \tag{2.4}$$

So $\beta_{\overline{g}}(g) = 0$ and the claim follows. □

Thus, in order to construct Einstein metrics, it suffices to look for zeros of $\Phi_{\overline{g}}$. A similar result to Lemma 2.4 holds in the differential sense:

Lemma 2.5. *If $(M, \overline{g})$ is a complete Einstein manifold and h a symmetric bilinear form such that $|h|(x) \to 0$ for $x \to \infty$, then $L_{\overline{g}} h = 0$ implies*

$$\mathrm{d}\,\mathrm{Ric}_{\overline{g}}(h) = -(n-1)h, \qquad \delta_{\overline{g}} h = 0, \qquad \mathrm{tr}_{\overline{g}}\, h = 0.$$

Proof. The proof is the same as in [And1, Lemma 3.6]. Differentiate (2.4) with respect to g to find

$$0 = \beta_{\overline{g}} L_{\overline{g}} h = \nabla^* \nabla \beta_{\overline{g}} h + (n-1)\beta_{\overline{g}} h. \tag{2.5}$$

So $\beta_{\overline{g}} h = 0$. Moreover, by (2.2) we conclude $\mathrm{tr}\, h = 0$. □

Observe that reversely not every Einstein variation is divergence or trace free.

2.3. **The hyperbolic cusp.** Consider the coordinates $(r, x_2, \ldots, x_n)$ on $\mathbb{R}_+ \times \mathbb{R}^{n-1}$ and the hyperbolic metric

$$g_h = r^{-2} \mathrm{d}r^2 + r^2(\mathrm{d}x_2^2 + \ldots + \mathrm{d}x_n^2).$$

Note that in these coordinates, g_h is not conformally equivalent to the Euclidean metric (as opposed to the coordinates that arise after the transformation $r \to \frac{1}{r}$). Obviously, the metric is invariant under the action of $\mathbb{R}^{n-1}$ by translations on the last factor. We will be interested in Einstein deformations of this metric which are $\mathbb{R}^{n-1}$ invariant. One type of deformation will be very essential: Let u_{ij} be a symmetric $(n-1)\times(n-1)$ matrix indexed by $i,j=2,\dots,n$. Then if $u_{ij} > -\delta_{ij}$

$$r^{-2}\mathrm{d}r^2 + r^2(\mathrm{d}x_2^2 + \ldots + \mathrm{d}x_n^2 + u_{ij}\mathrm{d}x_i\mathrm{d}x_j)$$

is isometric to g_h hence it is also Einstein. We will denote this metric by g_h+u. It can be checked (e.g. using (2.4)) that the equation $\Phi_{g_h}(g_h+u)=0$ is equivalent to $\operatorname{tr} u = 0$. Likewise, dropping the lower bound for u_{ij} and setting $h = r^2 u_{ij}\mathrm{d}x_i\mathrm{d}x_j$, we find that $L_{g_h}h = 0$ iff $\operatorname{tr} u = 0$. We will call variations of this kind *trivial Einstein variations.*

2.4. **The black-hole metric.** We recall the definition of the black-hole metric (M_{BH}, g_{BH}) as in [And2]. Let $m > 0$. Introduce coordinates $(r,\theta,x_3,\dots,x_n)$ on $\mathbb{R}^n = \mathbb{R}^2\times\mathbb{R}^{n-2}$. Here, (r,θ) denote polar coordinates on the first factor such that r is running from $r_+ = (2m)^{1/n-1}$ to ∞ and θ from 0 to $\beta = \frac{4\pi}{(n-1)r_+}$. The black-hole metric is defined as

$$g_{BH} = V^{-1}\mathrm{d}r^2 + V\mathrm{d}\theta^2 + r^2(\mathrm{d}x_3^2 + \ldots + \mathrm{d}x_n^2)$$

where

$$V(r) = r^2 - \frac{2}{r^{n-3}}.$$

By a coordinate transformation one can show that the metric is in fact smooth at the origin. The sectional curvatures of this metric are

$$K_{12} = -1 + \frac{(n-3)(n-2)}{r^{n-1}}, \qquad K_{1i} = K_{2i} = -1 - \frac{n-3}{r^{n-1}}, \qquad i\geq 3,$$
$$K_{ij} = -1 + \frac{2}{r^{n-1}}, \qquad i,j\geq 3$$

Furthermore, g_{BH} is Einstein with $\operatorname{Ric}_{g_{BH}} = -(n-1)g_{BH}$. For $n=3$ this metric is just the standard hyperbolic metric in cylindrical coordinates.

Observe that away from the origin the metric is asymptotic to the standard hyperbolic metric g_h from subsection 2.3. To be precise: The black-hole manifold minus a large cylinder around the core $\mathbb{R}^{n-2}$ is geometrically close to some subset of $\mathbb{H}^n/\langle\gamma\rangle$ where γ is a parabolic transformation. Taking the hyperbolic metric g_h as a background metric, we can verify that $\|\nabla^m(g_{BH}-g_h)\| = O(r^{-(n-1)})$. In dimension 3 this implies the fact that distance cylinders in hyperbolic space metrically approach horospheres.

We analyze the behaviour of this metric under the addition of small trivial Einstein deformations. Let u_{ij} $(i,j=2,\dots,n)$ be a traceless symmetric $(n-1)\times(n-1)$ matrix and set

$$g_{BH} + u = g_{BH} + r^2 u_{ij}\mathrm{d}x_i\mathrm{d}x_j$$

where we set $x_2 = \theta$. Now $g_{BH}+u$ is only smooth away from the origin. By the closeness of g_{BH} to g_h we find for sufficiently small u and say $r > r_+ + 1$

$$\Phi_{g_{BH}}(g_{BH}+u) = |u|O(r^{-n+1}) \qquad \text{and} \qquad L_{g_{BH}}u = |u|O(r^{-n+1}). \tag{2.6}$$

Since we will need it later, we mention the following bound: Let u be small and u' be another traceless symmetric $(n-1)\times(n-1)$ matrix. Then

$$\left|(\mathrm{d}\Phi_{g_{BH}})_{g_{BH}+u}(u') - (\mathrm{d}\Phi_{g_{BH}})_{g_{BH}}(u')\right| = |u||u'|O(r^{-n+1}) \tag{2.7}$$

The same decay holds for all higher covariant derivatives of the left hand side.

It will be useful later to discuss the geometric quotients of the black-hole metric. In dimension $n=3$ the transformations of (M_{BH}, g_{BH}) are the transformations of hyperbolic space. For $n>3$, all transformations leave the topological splitting $M_{BH} = \mathbb{R}^2 \times \mathbb{R}^{n-2}$ and the coordinate r invariant, so they act as a rotation or reflection in the origin on the $\mathbb{R}^2$ factor and as a Euclidean translation on the $\mathbb{R}^{n-2}$-factor.

Let now T^{n-1} be an arbitrary flat torus and $\sigma \subset T^{n-1}$ a simple closed geodesic. Choose R such that $V(R) = (\ell(\sigma)/\beta)^2$. Then $M_{BH}(r=R) \approx S^1 \times \mathbb{R}^{n-2}$ is isometric to a cover $\widetilde{T} \to T^{n-1}$ which allows a lift of the closed curve σ. Consider the group of deck transformations Γ of $\widetilde{T}$. Its action on $M_{BH}(r=R)$ can be uniquely continued to an isometric action on M_{BH} and by the lifting property of σ we know that this continuation is even fixed point free. Hence M_{BH}/Γ is smooth, so the manifold $\hat{E} := (M_{BH}/\Gamma)(r \leq R)$ is diffeomorphic to a solid torus and its boundary $\partial\hat{E} = (M_{BH}/\Gamma)(r=R)$ is isometric to T^{n-1} such that σ corresponds to a meridian. The image of the core $\mathbb{R}^{n-2}$ under the quotient map is a torus $\hat{T}^{n-2} = r^{-1}(r_+)$ which we call the *core torus.* Furthermore, all level sets $r^{-1}(r')$ for $r' > r_+$ are diffeomorphic to T^{n-1} and $(r')^{-1}\operatorname{diam} r^{-1}(r')$ is an increasing function in r' (here diam denotes the intrinsic diameter).

3. The construction process

We will briefly explain how the approximate Einstein metric on the manifold $M_{\overline{\sigma}}$ is constructed. Recall that we are given simple closed geodesics σ_k inside the tori $T_k \subset N$ which bound the cusps E_k, and that $\mathrm{inj} = \mu_n$ on T_k. In dimension 3 it is also important to choose the E'_k such that on their boundary tori T'_k we also have $\mathrm{inj} = \mu_n$.

As mentioned in subsection 2.4, we can find parameters R_k as well as lattices $\Gamma_k < \operatorname{Isom} M_{BH}$ such that the $\hat{E}_k := (M_{BH}/\Gamma_k)(r \leq R_k)$ are topological solid tori with boundary isometric to T_k and such that the σ_k correspond to meridians. Set $R_{\min} := \min R_k$ and observe that $R_{\min} \to \infty$ as $\ell_{\min} \to \infty$.

If we glue together the components $N \setminus \bigcup_{k=1}^p E_k$ and $\hat{E}_k$, we obtain the manifold $M_{\overline{\sigma}}$. We can endow $M_{\overline{\sigma}}$ with an almost Einstein metric $g_{\overline{\sigma}}$ in the following sense (to simplify notation, we will denote this metric by $g_{\overline{\sigma}}$ rather than the final Einstein metric): $g_{\overline{\sigma}}$ equals g_h on the first component and g_{BH} on the $\hat{E}_k$ except on the tubular neighborhoods $\bigcup_{k=1}^p B_1 T_k \cap \hat{E}_k$ of radius 1 around the T_k where an interpolation between g_h and g_{BH} is taking place. Thus $g_{\overline{\sigma}}$ satisfies the Einstein equation on the complement of $\bigcup_{k=1}^p B_1 T_k \cap \hat{E}_k$ whereas on $\bigcup_{k=1}^p B_1 T_k \cap \hat{E}_k$ the quantity $\mathrm{Ric}_{g_{\overline{\sigma}}} + (n-1) g_{\overline{\sigma}}$ and hence $\Phi_{g_{\overline{\sigma}}}(g_{\overline{\sigma}})$ is very small. To be precise: in each $C^{m,\alpha}$-norm we have $\Phi_{g_{\overline{\sigma}}}(g_{\overline{\sigma}}) \to 0$ as $R_{\min} \to \infty$ with the rate $O(R_{\min}^{-n+1})$.

Note that in dimension $n = 3$, the parts E'_k are already isometric to $\hat{E}'_k = (M_{BH}/\Gamma'_k)(r \leq R'_k)$ for certain lattices Γ'_k and numbers R'_k.

For further details of this construction we refer to [And2].

4. Uniform norms on $M_{\overline{\sigma}}$

In the following let $L := L_{g_{\overline{\sigma}}}$ and $0 < \alpha < 1$. We will further fix an upper volume bound on N and call all constants *uniform* which only depend on this bound, but not on N or $\overline{\sigma}$. Observe that the Riemannian manifolds $(M_{\overline{\sigma}}, g_{\overline{\sigma}})$, as constructed in the last section, satisfy the following uniform geometric bounds: The conjugate radius is uniformly bounded from below by some positive constant 2ζ and there are uniform bounds C_m such that $\|\nabla^m R\| < C_m$.

Let h be a symmetric bilinear form on $M_{\overline{\sigma}}$ and $x \in M_{\overline{\sigma}}$. Pull back the bundle $\mathrm{Sym}_2 T^*$ and its section h to the universal cover $\widetilde{B}_\zeta(x)$ of $B_\zeta(x)$. Choose exponential coordinates on $\widetilde{B}_\zeta(x)$ and trivialize $\mathrm{Sym}_2 T^*$ by parallel transport. We can now view h as a vector-valued function on a ball $B_\zeta(0) \subset \mathbb{R}^n$. Define the local Hölder (semi)-norm of h at x by this representation:

$$\|h\|_{m,\alpha;x} := \|h|_{B_\zeta(x)}\|_{m,\alpha}.$$

We note that we have Schauder estimates for these semi-norms:

$$\|h\|_{m,\alpha;x} \leq C \sup_{x' \in B_\zeta(x)} (\|Lh\|_{m-2,\alpha;x'} + \|h\|_{0;x'}) \tag{4.1}$$

such that C is a uniform constant.

Using these semi-norms it is now easy to define the global Hölder norm by

$$\|h\|_{m,\alpha} := \sup_{x \in M_{\overline{\sigma}}} \|h\|_{m,\alpha;x}.$$

We will need another norm that guarantees a certain decay away from the thick part and the core tori. We therefore introduce a weight function W (or rather the inverse of a weight function) on $M_{\overline{\sigma}}$ such that for $n > 3$

$$W = \begin{cases} (\frac{r}{R_k})^{0.1} + r^{-0.1} & \text{on } \hat{E}_k \\ 1 & \text{on } M_{\overline{\sigma}} \setminus \bigcup_{k=1}^p \hat{E}_k \end{cases}$$

In dimension 3 we also choose the weight $(\frac{r}{R'_k})^{0.1} + r^{-0.1}$ on the $\hat{E}'_k$. W is not continuous at the T_k. However, this discontinuity will not be essential since the jump is between 1 and $1 + R_k^{-0.1}$ and $R_k > r_+$. On each $\hat{E}_k$ the weight function W attains its minimum at $r = R_k^{1/2}$. For later use, choose points $c_k \in \hat{E}_k(r = R_k^{1/2})$ (and $c'_k \in E'_k(r = (R'_k)^{1/2})$ in dimension 3). They lie approximately in the centers of the $\hat{E}_k$. Set

$$\|h\|_{m,\alpha;*} := \sup_{x \in M_{\overline{\sigma}}} W^{-1}(x) \|h\|_{m,\alpha;x}.$$

It is immediate that we can derive uniform Schauder estimates for the norms $\|\cdot\|_{m,\alpha}$ and $\|\cdot\|_{m,\alpha;*}$ from (4.1):

$$\|h\|_{m,\alpha} \leq C\left(\|Lh\|_{m-2,\alpha} + \|h\|_0\right)$$
$$\|h\|_{m,\alpha;*} \leq C\left(\|Lh\|_{m-2,\alpha;*} + \|h\|_{0;*}\right)$$

Finally, we have to define a more complicated norm that guarantees decay towards some trivial Einstein variation: Let $\rho_1, \dots, \rho_p$ be cutoff functions on $M_{\overline{\sigma}}$ such that $\rho_k \equiv 1$ on $\hat{E}_k \setminus (B_1 T_k \cup B_2 \hat{T}_k^{n-2})$ and $\rho_k \equiv 0$ on $M_{\overline{\sigma}} \setminus \hat{E}_k$ and $B_1 \hat{T}_k^{n-2}$ where $\hat{T}_k^{n-2}$ is the core torus of $\hat{E}_k$. We may assume that the ρ_k are constructed in such a way that they satisfy some universal C^m bound for each m. Let $u_1, \dots, u_p$ be trivial Einstein variations of the hyperbolic cusp metric which we assume to be defined on the corresponding $\hat{E}_k$. Represent h by

$$h = \bar{h} + \sum_{k=1}^{p} \rho_k u_k. \tag{$*$}$$

and define

$$\|h\|_{m,\alpha;**} := \inf_{\substack{\bar{h}, u_1, \dots, u_p \\ \text{satisfy } (*)}} \left(\|\bar{h}\|_{m,\alpha;*} + \sum_{k=1}^{p} |u_k|\right)$$

where we use an arbitrary uniform norm on the (finite dimensional) space of trivial Einstein deformations. In dimension 3 we have to alter the definition in order to also consider trivial Einstein variations u'_k on the E'_k. Observe that for some uniform C

$$C^{-1}\|h\|_{m,\alpha} \leq \|h\|_{m,\alpha;**} \leq \|h\|_{m,\alpha;*}.$$

Lemma 4.1. *We have the following uniform Schauder estimate for* $\|\cdot\|_{m,\alpha;**}$*:*

$$\|h\|_{m,\alpha;**} \leq C\left(\|Lh\|_{m-2,\alpha;*} + \|h\|_{0;**}\right). \tag{4.2}$$

Note that the second norm is a $*$-norm.

Proof. We carry out the proof for $n > 3$. Choose a decomposition $h = \bar{h} + \sum_{k=1}^{p} \rho_k u_k$. From (2.6) we find that $\|L\rho_k u_k\|_{m-2,\alpha;*} \leq C|u_k|$. Hence

$$\|L\bar{h}\|_{m-2,\alpha;*} \leq \|Lh\|_{m-2,\alpha;*} + C\sum_{k=1}^{p} |u_k|.$$

So by the Schauder estimate for $\|\cdot\|_{m,\alpha;*}$ we find

$$\|\bar{h}\|_{m,\alpha;*} \leq C\left(\|Lh\|_{m-2,\alpha;*} + \sum_{k=1}^{p} |u_k| + \|\bar{h}\|_{0;*}\right)$$

hence the conclusion. □

The following Lemma gives us a tool to estimate the $\|\cdot\|_{m,\alpha;**}$ norm:

Lemma 4.2. *Let h be a symmetric bilinear form on $M_{\overline{\sigma}}$. Choose u_k such that $|(h-u_k)_{c_k}|$ is minimal for each k and set $\bar{h} = h - \sum_{k=1}^p \rho_k u_k$. In dimension 3 also consider u'_k such that $|(h-u'_k)_{c'_k}|$ is minimal. Then there are constants $C_{m,\alpha}$ such that*

$$\|h\|_{m,\alpha;**} \leq \left(\|\bar{h}\|_{m,\alpha;*} + \sum_{k=1}^p |u_k| \right) \leq C_{m,\alpha} \|h\|_{m,\alpha;**}$$

Proof. Assume again $n > 3$. Only the second inequality has to be shown. Let $h = \bar{h}' + \sum_{k=1}^p \rho_k u'_k$ be an arbitrary decomposition of h. First observe that

$$|(u_k)_{c_k}| \leq |h_{c_k}| \leq C\|h\|_{m,\alpha} \leq C\|h\|_{m,\alpha;**}$$

so we have established an upper bound on the u_k.

In order to bound $\bar{h}$, we observe that by the minimal choice of u_k, we have $|(u_k - u'_k)_{c_k}| \leq |(h-u'_k)_{c_k}|$. We use

$$\|\bar{h}\|_{m,\alpha;*} \leq \|\bar{h}'\|_{m,\alpha;*} + \sum_{k=1}^p \|\rho_k(u_k - u'_k)\|_{m,\alpha;*}$$

and bound the last term by $C\sum_{k=1}^p M_k|u_k - u'_k| \leq C\sum_{k=1}^p M_k |(h-u'_k)_{c_k}|$ where

$$M_k = \max_{\hat{E}_k} W^{-1} = W^{-1}(c_k).$$

So if $\rho_k(c_k) = 1$, then $M_k|(h-u'_k)_{c_k}| \leq C\|h - \sum_{l=1}^p \rho_l u'_l\|_{m,\alpha;*} = \|\bar{h}'\|_{m,\alpha;*}$. If not, we have a uniform bound on R_k, hence on M_k and $M_k|(h-u'_k)_{c_k}| \leq C\|\bar{h}'\|_{m,\alpha}$. □

5. Application of the inverse function theorem

We will use the following estimate on L^{-1} which we will prove in the next section:

Proposition 5.1. *There are $R_0 = R_0(n,V), \Lambda = \Lambda(n,V) < \infty$ such that whenever $\operatorname{vol} N < V$ and $R_{\min} > R_0$, then the operator $L = L_{g_{\overline{\sigma}}} : C^{m,\alpha}(M_{\overline{\sigma}}; \operatorname{Sym}_2 T^*) \to C^{m-2,\alpha}(M_{\overline{\sigma}}; \operatorname{Sym}_2 T^*)$ is invertible and*

$$\|h\|_{m,\alpha;**} \leq \Lambda \|L_{g_{\overline{\sigma}}} h\|_{m-2,\alpha;*}$$

for any symmetric bilinear form h on $M_{\overline{\sigma}}$.

Observe that there are different types of norms on both sides of this inequality. Thus in order to construct a perturbation of $g_{\overline{\sigma}}$ which is Einstein we cannot simply use this estimate to strictly apply the inverse function theorem on Banach spaces. However, we will show that the trivial Einstein deformations which make the difference between these two norms, have a weak influence on the nonlinear term of the equation we want to solve.

We will now prove Theorem 1.1 assuming Proposition 5.1.

Proof of Theorem 1.1. We only consider the case $n > 3$. It will be clear how to adapt the proof to the 3 dimensional case.

In the following we set $M = M_{\overline{\sigma}}$, $g = g_{\overline{\sigma}}$, $\Phi = \Phi_{g_{\overline{\sigma}}}$ and $L = L_{g_{\overline{\sigma}}}$.

Assume that $R_{\min} > R_0$. We want to find $h \in C^{m,\alpha}(M; \operatorname{Sym}_2 T^*)$ such that the equation $\Phi(g+h) = 0$ holds. It will then follow from elliptic regularity that h is actually smooth. The equation is equivalent to the fixed point equation

$$h = \Psi(h) = h - L^{-1}\Phi(g+h).$$

In order to solve this equation for large $R_{\min}$, it suffices to show that there is an $\varepsilon = \varepsilon(n, \operatorname{vol} N) > 0$ such that Ψ is $\frac{1}{2}$-Lipschitz with respect to the $\|\cdot\|_{m,\alpha;**}$-norm on $B_\varepsilon = \{h \in C^{m,\alpha}(M; \operatorname{Sym}_2 T^*) \ : \ \|h\|_{m,\alpha;**} < \varepsilon\}$. Then, assuming $R_{\min}$ to be large enough, we can achieve $\|\Psi(0)\|_{m,\alpha;**} = \|L^{-1}\Phi(g)\|_{m,\alpha;**} \leq \Lambda \|\Phi(g)\|_{m-2,\alpha;*} < \frac{1}{2}\varepsilon$ and apply Banach's fixed point theorem.

For $h_0, h_1 \in B_\varepsilon$ and $h_t = (1-t)h_0 + th_1$ we compute

$$\begin{aligned}
\|\Psi(h_0) - \Psi(h_1)\|_{m,\alpha;**} &\leq \left\| \int_0^1 L^{-1} \left(L - \mathrm{d}\Phi_{g+h_t}\right)(h_0 - h_1)\mathrm{d}t \right\|_{m,\alpha;**} \\
&\leq \Lambda \int_0^1 \left\| \left(\mathrm{d}\Phi_g - \mathrm{d}\Phi_{g+h_t}\right)(h_0 - h_1)\right\|_{m-2,\alpha;*} \mathrm{d}t
\end{aligned}$$

(Observe that the subscript of $\mathrm{d}\Phi$ now indicates the point at which the derivative is taken rather than the background metric which we used to define Φ.) Thus, it suffices to show that for any $h \in B_\varepsilon$ and $h' \in C^{m,\alpha}(M; \operatorname{Sym}_2 T^*)$ we have

$$\left\|\mathrm{d}\Phi_g(h') - \mathrm{d}\Phi_{g+h}(h')\right\|_{m-2,\alpha;*} \leq \delta(\varepsilon)\|h'\|_{m,\alpha;**}$$

for some universal $\delta(\epsilon)$ with $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Represent $h = \bar{h} + \sum_k \rho_k u_k$ and $h' = \bar{h}' + \sum_k \rho_k u_k'$ where the u_k, u_k' are trivial Einstein variations. Then

$$\begin{aligned}
&\left\|\mathrm{d}\Phi_g(h') - \mathrm{d}\Phi_{g+h}(h')\right\|_{m-2,\alpha;*} \\
&\quad \leq \left\|\mathrm{d}\Phi_g(\bar{h}') - \mathrm{d}\Phi_{g+h}(\bar{h}')\right\|_{m-2,\alpha;*} + \sum_k \left\|\mathrm{d}\Phi_g(\rho_k u_k') - \mathrm{d}\Phi_{g+h}(\rho_k u_k')\right\|_{m-2,\alpha;*}.
\end{aligned}$$

The first term can immediately be bounded by $C\|h\|_{m,\alpha}\|\bar{h}'\|_{m,\alpha;*} \leq C'\|h\|_{m,\alpha;**}\|\bar{h}'\|_{m,\alpha;*}$. As for the second term we have

$$\begin{aligned}
&\left\|\mathrm{d}\Phi_g(\rho_k u_k') - \mathrm{d}\Phi_{g+h}(\rho_k u_k')\right\|_{m-2,\alpha;*} \\
&\leq \left\|\mathrm{d}\Phi_g(\rho_k u_k') - \mathrm{d}\Phi_{g+\rho_k u_k}(\rho_k u_k')\right\|_{m-2,\alpha;*} + \left\|\mathrm{d}\Phi_{g+\rho_k u_k}(\rho_k u_k') - \mathrm{d}\Phi_{g+\bar{h}+\rho_k u_k}(\rho_k u_k')\right\|_{m-2,\alpha;*}.
\end{aligned}$$

Now, since u_k is a trivial Einstein variation, we can use (2.7) to bound the first term by $C|u_k||u_k'|$. The second term is bounded by $C\|\bar{h}\|_{m,\alpha;*}|u_k'|$. We conclude

$$\begin{aligned}
\left\|\mathrm{d}\Phi_g(h') - \mathrm{d}\Phi_{g+h}(h')\right\|_{m-2,\alpha;*} &\leq C\|h\|_{m,\alpha;**}\|\bar{h}'\|_{m,\alpha;*} + C\sum_k \left(|u_k||u_k'| + \|\bar{h}\|_{m,\alpha;*}|u_k'|\right) \\
&\leq C\|h\|_{m,\alpha;**}\|\bar{h}'\|_{m,\alpha;*} + C\big(\|\bar{h}\|_{m,\alpha;*} + \sum_k |u_k|\big)\sum_l |u_l'|.
\end{aligned}$$

By an appropriate choice of $\bar{h}$ and u_k, the right hand side can be made arbitrarily close to $C\|h\|_{m,\alpha;**}\left(\|\bar{h}'\|_{m,\alpha;*} + \sum_l |u_l'|\right)$ what in turn by a good choice of $\bar{h}'$ and

u_k' can be made arbitrarily close to $C\|h\|_{m,\alpha;**}\|h'\|_{m,\alpha;**} \leq C\varepsilon\|h'\|_{m,\alpha;**}$. This proves the desired bound and hence the theorem. □

6. Estimates for L^{-1}

This section will be occupied with the proof of Proposition 5.1. For the sake of a clear exposition of the main ideas we will defer most of the technical arguments to sections 7 and 8. We first establish a bound on the $\|\cdot\|_{m,\alpha}$-norm:

Lemma 6.1. *There are* $R_0 = R_0(n,V), \Lambda = \Lambda(n,V) < \infty$ *such that whenever* $\operatorname{vol} N < V$ *and* $R_{\min} > R_0$*, then the operator* $L_{g_{\overline{\sigma}}} : C^{m,\alpha}(M_{\overline{\sigma}}; \operatorname{Sym}_2 T^*) \to C^{m-2,\alpha}(M_{\overline{\sigma}}; \operatorname{Sym}_2 T^*)$ *is invertible and*

$$\|h\|_{m,\alpha} \leq \Lambda \|L_{g_{\overline{\sigma}}} h\|_{m-2,\alpha;*}$$

for any symmetric bilinear form h *on* $M_{\overline{\sigma}}$*.*

Proof. The proof of this Lemma is similar to that of [And2, Proposition 3.2]. However, since we have a stronger norm on the right hand side, we can get rid of the extra $\log R_{\max}$-term.

Recall that we have the Schauder estimate

$$\|h\|_{m,\alpha} \leq C(\|L_{g_{\overline{\sigma}}} h\|_{m-2,\alpha} + \|h\|_0)$$

where C is uniform. So it is enough to show that:

There are $R_0 = R_0(n,V), \Lambda' = \Lambda'(n,V) < \infty$ *such that whenever* $\operatorname{vol} N < V$ *and* $R_{\min} > R_0$*, we have*

$$\|h\|_0 \leq \Lambda' \|L_{g_{\overline{\sigma}}} h\|_{m-2,\alpha;*}$$

for all symmetric bilinear forms h *on any* $M_{\overline{\sigma}}$*.*

Assume that this statement was wrong. Then we can find a sequence of hyperbolic manifolds N_i with basepoints $y_i \in N_{thick,i}$ and $\operatorname{vol} N_i$ uniformly bounded from above as well as a sequences of $\overline{\sigma}_i$ such that $R_{\min,i} \to \infty$ and symmetric bilinear forms h_i on $M_i = M_{\overline{\sigma}_i}$ such that for $g_i = g_{\overline{\sigma}_i}$, $L_i = L_{g_i}$ and $f_i := L_i h_i$

$$\|h_i\|_0 = 1, \qquad \text{but} \qquad \|f_i\|_{m-2,\alpha;*} \longrightarrow 0$$

as $i \to \infty$. So there are points $x_i \in M_i$ such that $|h_i|(x_i) > \gamma$ for some universal $\gamma > 0$. The Schauder estimate gives us a uniform $C^{m,\alpha}$-bound for the h_i.

1° In the first step we show that there are sequences $d_i \to \infty$ and $w_i \to 0$ such that $|h_i| < w_i$ on $B_{d_i}(y_i)$.

Consider an arbitrary subsequence of counterexamples. After passing to a subsequence again, the pointed Riemannian manifolds (M_i, y_i) Gromov-Hausdorff converge to a pointed hyperbolic manifold (N_∞, y_∞) of finite volume. Furthermore, the h_i subconverge to a symmetric bilinear form h_∞ on N_∞ such that $L_\infty(h_\infty) = 0$ (here $L_\infty = L_{g_\infty}$).

Denote by N^s_∞ the manifold obtained from N_∞ by truncating its cusps at distance s from the basepoint y_∞. Using Stoke's theorem and (2.3), we find

$$\int_{N^s_\infty} |dh_\infty|^2 + |\delta h_\infty|^2 + (n-2)|h_\infty|^2 + (\operatorname{tr} h_\infty)^2 = \int_{\partial N^s_\infty} Q(h_\infty, \nabla h_\infty)$$

where the right hand side goes to 0 as $s \to \infty$. So $h_\infty \equiv 0$ and we conclude that for any d we have $|h_i| \to 0$ uniformly on $B_d(y_i)$ for a subsequence. Since we started with an arbitrary subsequence, this implies that for any d we have $|h_i| \to 0$ on $B_d(y_i)$ uniformly for the *whole* sequence and hence the claim.

2° Next, we give an estimate for h_i on the $\hat{E}_{k,i}$ (and $E'_{k,i}$ in dimension 3).

Choose coordinates $(r, \theta, x_3, \ldots, x_n)$ on these components (to be precise on their universal covers). Observe that $\hat{E}_{k,i} \setminus B_1(T_{k,i})$ carries the exact black-hole metric. We have $|f_i| < \|f_i\|_{m-2,\alpha;*} W$. Since $\operatorname{dist}(y_i, T_{k,i}) < \operatorname{diam} N_{thick,i}$ is uniformly bounded, we find that $|h_i| < w_i \to 0$ around the boundaries of the $\hat{E}_{k,i}$.

Consider the restriction of h_i and f_i to $\hat{E}_{k,i}$ and take their average under the $S^1 \times \mathbb{R}^{n-2}$-action, i.e. let $T^{n-1}(r') := \hat{E}_{k,i}(r = r')$ be the cross-sectional torus at the coordinate $r = r'$ and set

$$\hat{h}_{st,i}(r) := \frac{1}{\operatorname{vol} T^{n-1}(r)} \int_{T^{n-1}(r)} h_{st,i}.$$

Analogously define $\hat{f}_i$. Obviously, $\hat{h}_i$ and $\hat{f}_i$ are $S^1 \times \mathbb{R}^{n-2}$ invariant and $L_i \hat{h}_i = \hat{f}_i$. Furthermore, still

$$|\hat{f}_i| < \|f_i\|_{m-2,\alpha;*} W = \|f_i\|_{m-2,\alpha;*} \left[\left(\frac{r}{R_{k,i}} \right)^{0.1} + r^{-0.1} \right]$$

and since ∇h_i is uniformly bounded and $\operatorname{diam} T^{n-1}(r) < C \frac{r}{R_{k,i}}$, we conclude

$$|\hat{h}_i - h_i| < C \frac{r}{R_{k,i}} \qquad \text{on} \quad T^{n-1}(r).$$

We can now apply Proposition 8.1 to conclude

$$|h_i| < C \left(w_i + \|f_i\|_{m-2,\alpha;*} + r^{-n+1} + \frac{r}{R_{k,i}} \right). \tag{6.1}$$

3° We can make the following conclusions on x_i: From 1° we already know that $\operatorname{dist}(y_i, x_i) \to \infty$. This implies that x_i eventually lies in some $\hat{E}_{k,i}$ (or $E'_{k,i}$ in dimension 3) and $\frac{r(x_i)}{R_{k,i}} \to 0$. So by (6.1) we conclude that $r(x_i)$ has to stay bounded. This means that the x_i have to stay in bounded distance to some core tori $\hat{T}^{n-2}_{k,i}$ of $\hat{E}_{k,i}$ (or of $E'_{k,i}$).

So there is a sequence d'_i such that the universal covers $(\widetilde{B}_{d'_i}(x_i), x_i)$ Gromov-Hausdorff subconverge to the black-hole metric (M_{BH}, x_∞) and the h_i subconverge to some h_∞ on M_{BH} which satisfies $L_\infty h_\infty = 0$ and $h_\infty(x_\infty) \neq 0$. Moreover, since the the pointed manifolds $(\hat{E}_{k,i}, x_i)$ collapse to a ray, h_∞ is invariant under the $S^1 \times \mathbb{R}^{n-2}$-action. From (6.1) we also conclude that $|h_\infty| < Cr^{-n+1}$.

We can now use Proposition 8.3 to find that $h_\infty \equiv 0$, a contradiction. □

Finally, we can use Lemma 6.1 to refine our result and prove Proposition 5.1:

Proof of Proposition 5.1. Analogously to the proof of Lemma 6.1, we assume that the hypothesis was wrong and that we have sequences $M_i, \overline{\sigma}_i, h_i$ such that $R_{\min,i} \to \infty$ and

$$\|h_i\|_{0;**} = 1, \qquad \text{but} \qquad \|f_i\|_{m-2,\alpha;*} \longrightarrow 0$$

for $f_i = L_i h_i$ (we also used (4.2) here).

By Lemma 6.1 we have $\|h_i\|_0 \to 0$. We now change the h_i by certain trivial Einstein variations of the $\hat{E}_{k,i}$ (or $E'_{k,i}$ in dimension 3): Let $u_{k,i}$ be those trivial Einstein deformations as obtained in Lemma 4.2 and set $\bar{h}_i = h_i - \sum_{k=1}^{p_i} \rho_{k,i} u_{k,i}$. Then $|u_{k,i}| \to 0$ as $i \to \infty$ and by Lemma 4.2

$$\|\bar{h}_i\|_{**;0} \leq \|\bar{h}_i\|_{0;*} \leq C\|\bar{h}_i\|_{0;**}$$

for some uniform constant C. So we conclude that $\frac{1}{2} < \|\bar{h}_i\|_{0;**} < \frac{3}{2}$ for large i and hence we have the uniform estimate $c < \|\bar{h}_i\|_{0;*} < C$. However, we still have $\|\bar{h}_i\|_0 \to 0$. Finally, setting $\bar{f}_i = L_i \bar{h}_i$, we get $\|\bar{f}_i\|_{m-2,\alpha;*} \to 0$.

By the lower bound on $\|\bar{h}_i\|_{0;*}$, we can find points $x_i \in M_i$ such that

$$W^{-1}(x_i)|\bar{h}_i|(x_i) > \gamma > 0.$$

Since $\|\bar{h}_i\|_0 \to 0$, we conclude $W(x_i) \to 0$. So the x_i eventually lie in certain $\hat{E}_{k_i,i}$ (or $E'_{k,i}$), $R_{k_i,i} \to \infty$ and the distance of the x_i to both $T_{k_i,i}$ as well as $\hat{T}^{n-2}_{k_i,i}$ goes to infinity. So there is a sequence d'_i such that the universal covers $(\widetilde{B}_{d'_i}(x_i), x_i)$ converge to hyperbolic space $(\mathbb{H}^n, x_\infty)$ on which we can choose coordinates $(r_\infty, x_2, \ldots, x_n)$ with $r_\infty(x_\infty) = 1$ and $\frac{r}{r_i} \to r_\infty$ where $r_i := r(x_i)$. In order to analyze the limiting behaviour of $\bar{h}_i$, we have to distinguish three cases:

1° For a subsequence we have $r_i R_{k_i,i}^{-1/2} \to \infty$.

Then we have the (local) convergence

$$\left(\frac{R_{k_i,i}}{r_i}\right)^{0.1} W = \left(\frac{r}{r_i}\right)^{0.1} + \left(\frac{R_{k_i,i}}{r_i^2} \cdot \frac{r_i}{r}\right)^{0.1} \longrightarrow r_\infty^{0.1}.$$

So $(\frac{R_{k_i,i}}{r_i})^{0.1}\bar{h}_i$ is locally bounded and $(\frac{R_{k_i,i}}{r_i})^{0.1}\bar{f}_i \to 0$ locally. Hence the $\bar{h}_i$ subconverge to some nonzero $\bar{h}_\infty$ on $\mathbb{H}^n$ which satisfies $|\bar{h}_\infty| < Cr_\infty^{0.1}$ and $L_\infty \bar{h}_\infty = 0$. Since the sequence $(B_{d'_i}(x_i), x_i)$ collapses to a line, $\bar{h}_\infty$ must be invariant under the group $\mathbb{R}^{n-1}$ acting on the last coordinates. We can now use Proposition 7.1 to obtain a contradiction.

2° For a subsequence we have $r_i R_{k_i,i}^{-1/2} \to 0$.

This time we have the convergence

$$r_i^{0.1}W = \left(\frac{r_i^2}{R_{k_i,i}} \cdot \frac{r}{r_i}\right)^{0.1} + \left(\frac{r}{r_i}\right)^{-0.1} \longrightarrow r_\infty^{-0.1}.$$

Now we can use the same arguments as in 1° to construct $\bar{h}_\infty$ on $\mathbb{H}^n$ which obeys the bound $|\bar{h}_\infty| < Cr_\infty^{-0.1}$. This also contradicts Proposition 7.1.

3° For a subsequence we have $r_i R_{k_i,i}^{-1/2} \to q$ where $0 < q < \infty$.

This means that the points x_i stay within bounded distance to the $c_{k_i,i}$. Let $c_\infty \in \mathbb{H}^n$ be one of their limit points. We have the convergence

$$r_i^{0.1}W = \left(\frac{r_i^2}{R_{k_i,i}} \cdot \frac{r}{r_i}\right)^{0.1} + \left(\frac{r}{r_i}\right)^{-0.1} \longrightarrow q^{0.2}r_\infty^{0.1} + r_\infty^{-0.1}.$$

Hence the same reasoning as in 1° yields a nonzero $\bar{h}_\infty$ which satisfies $|\bar{h}_\infty| < C(r_\infty^{0.1} + r_\infty^{-0.1})$. So by Proposition 7.1, $\bar{h}_\infty$ must be trivial.

However, by the construction of the $\bar{h}_i$ we get that $|(\bar{h}_\infty)_{c_\infty}| \leq |(\bar{h}_\infty - u)_{c_\infty}|$ for any trivial Einstein variation u, contradicting the fact that $\bar{h}_\infty$ is nonzero. ☐

7. Einstein variations of the hyperbolic cusp metric

Consider the hyperbolic metric

$$g_h = r^{-2}\mathrm{d}r^2 + r^2(\mathrm{d}x_2^2 + \ldots + \mathrm{d}x_n^2)$$

on $\mathbb{R}^+ \times \mathbb{R}^{n-1}$ and the parabolic isometric action of $\mathbb{R}^{n-1}$ by translations on the second factor.

Set $L := L_{g_h}$. We will prove the following result:

Proposition 7.1. *Let h be a symmetric bilinear form on $\mathbb{H}^n$ that is invariant under the $\mathbb{R}^{n-1}$-action. Assume furthermore that $|h| < r^{0.1} + r^{-0.1}$.*
Then $Lh = 0$ implies that h is trivial.

Thus, if even $|h| < r^{\pm 0.1}$, then $h \equiv 0$.

Proof. We assume $|h| < r^{0.1} + r^{-0.1}$. Express $h = h_{ij}\mathrm{d}x_i\mathrm{d}x_j$ where we set $x_1 = r$. Then the h_{ij} only depend on r and the bound on $|h|$ implies to

$$|h_{11}|(r) < r^{-2.1} + r^{-1.9}, \quad |h_{1i}|(r) < r^{-0.1} + r^{0.1} \quad \text{and} \quad |h_{ij}|(r) < r^{1.9} + r^{2.1}$$

for $i, j > 1$.

The equation $Lh = 0$ writes out as (see (2.2) and (2.3))

$$-\triangle h - 2h + 2(\operatorname{tr} h)g_h = 0$$

where the trace part simplifies to

$$-\triangle \operatorname{tr} h + 2(n-1)\operatorname{tr} h = 0.$$

So we can compute that for $q(r) = \operatorname{tr} h(r)$

$$-r^2q'' - nrq' + 2(n-1)q = 0 \tag{7.1}$$

whose solutions are of the form $A_1r^{\gamma_1} + A_2r^{\gamma_2}$ with $\gamma_{1/2} = \frac{1}{2}(-n+1 \pm \sqrt{n^2+6n-7})$. Hence by the bound on $|h|$ we get $\operatorname{tr} h \equiv 0$ and combining this with (2.3) gives

$\triangle h + 2h = 0$. In terms of the h_{ij} this equation is equivalent to the following system of differential equations $(i, j > 1)$

$$r^2 h''_{11} + (n+4) r h'_{11} + 6 h_{11} + \frac{2}{r^4} \sum_{k=2}^n h_{kk} = 0 \tag{I}$$

$$r^2 h''_{ij} + (n-4) r h'_{ij} - 2(n-3) h_{ij} + 2 r^4 h_{11} \delta_{ij} = 0 \tag{II}$$

$$r^2 h''_{1i} + n r h'_{1i} - n h_{1i} = 0 \tag{III}$$

By the trace-freeness, the first equation reduces to

$$r^2 h''_{11} + (n+4) r h'_{11} + 4 h_{11} = 0, \tag{I'}$$

so $h_{11}(r) = A_1 r^{\gamma_1} + A_2 r^{\gamma_2}$ where $\gamma_{1/2} = \frac{1}{2}(-n-3 \pm \sqrt{n^2+6n-7})$. By the bound on h_{11} this implies $h_{11} \equiv 0$.

Plugging this into (II) yields $h_{ij}(r) = A_1 r^2 + A_2 r^{-n+3}$ and thus $h_{ij} = A_1 r^2$.

Finally, (III) implies $h_{1i}(r) = A_1 r + A_2 r^{-n}$, hence $h_{1i} \equiv 0$. □

8. Variations of the black hole metric

Consider the black-hole metric

$$g = g_{BH} = V^{-1} \mathrm{d}r^2 + V \mathrm{d}\theta^2 + r^2(\mathrm{d}x_3^2 + \ldots + \mathrm{d}x_n^2)$$

on $M_{BH} \approx \mathbb{R}^2 \times \mathbb{R}^{n-2}$. Set $L = L_g$. Recall that g is asymptotic to the hyperbolic metric

$$g_h = r^{-2} \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \mathrm{d}x_2^2 + \ldots + \mathrm{d}x_n^2)$$

for $r \to \infty$ in the sense that $|\nabla^m (g - g_h)| = O(r^{-n+1})$. This is why we can estimate

$$|L_{g_{BH}} h - L_{g_h} h| \lesssim O(r^{-n+1})|h| + O(r^{-n+1})|\nabla h| + O(r^{-n+1})|\nabla^2 h|$$

for $r \to \infty$.

In the following we will analyze Einstein variations of g_{BH} or variations which are almost Einstein. We will hereby always assume that these variations are invariant under the $S^1 \times \mathbb{R}^{n-2}$ action. When we compare g_{BH} with g_h, this action becomes the parabolic $\mathbb{R}^{n-1}$ action.

We remark that Olivier Biquard has independently found elementary proofs of some of the following results ([Biq]).

Proposition 8.1. *Let $R > r_+$ and assume that on $M_{BH}(r \leq R)$ we have $Lh = f$ for $S^1 \times \mathbb{R}^{n-2}$ invariant h and f satisfying $|h|(r) < 1$ and*

$$|f|(r) < \alpha \left[\left(\frac{r}{R} \right)^{0.1} + r^{-0.1} \right]$$

for all $r \leq R$ and some $\alpha < 1$. Then

$$|h|(r) < C \left(|h|(R) + \alpha + r^{-n+1} \right)$$

for some universal constant C (which is independent of R).

We will need a technical Lemma. Note that from now on whenever we use the notation $O(\varphi(r))$ for a function $\varphi(r)$, we indicate an error term whose absolute value is *always* (not only for $r \to \infty$) bounded above by $C\varphi(r)$ where C is a universal constant.

Lemma 8.2. *Let $a, b \in \mathbb{R}$ and $0 \le B_1, B_2 \le \infty$. Consider a solution $f : (B_1, B_2) \to \mathbb{R}$ of the ODE*

$$r^2 f''(r) + arf'(r) + bf(r) = \varphi(r)$$

for some $\varphi : (B_1, B_2) \to \mathbb{R}$. Assume that a, b are chosen in such a way that the corresponding homogeneous ODE (for $\varphi \equiv 0$) has the general solution $f(r) = A_1 r^{\gamma_1} + A_2 r^{\gamma_2}$ with $\gamma_1, \gamma_2 \in \mathbb{R}$ and $\gamma_1 < \gamma_2$.

Now, suppose $\varphi(r) = \sum_{k=1}^{p} O(r^{\delta_k})$ where we assume that $\delta_k \neq \gamma_1, \gamma_2$ for each k. Then $f(r) = A_1 r^{\gamma_1} + A_2 r^{\gamma_2} + \sum_{k=1}^{p} O(r^{\delta_k})$.

Here the coefficients in $O(r^{\delta_k})$ only depend on a, b, δ and the coefficients in the error terms of φ.

Proof. To simplify the argument, we will assume that $p = 1$ and $\delta_1 = \delta$.

Observe that γ_1, γ_2 satisfy the quadratic equation $\gamma^2 + (a-1)\gamma + b = 0$. Set $g(r) = r^{-\gamma_1} f(r)$. Then with $c = 1 - \gamma_2 + \gamma_1$ we compute

$$r^2 g''(r) + crg'(r) = r^{-\gamma_1}\varphi = O(r^{\delta-\gamma_1}).$$

So

$$\left(r^{1-\gamma_2+\gamma_1} g'\right)'(r) = O(r^{-1+\delta-\gamma_2}),$$

which integrates to

$$r^{1-\gamma_2+\gamma_1} g'(r) = A_2 + O(r^{\delta-\gamma_2})$$

and thus

$$g'(r) = A_2 r^{-1+\gamma_2-\gamma_1} + O(r^{-1+\delta-\gamma_1}).$$

A final integration yields

$$r^{-\gamma_1} f(r) = g(r) = A_2 r^{\gamma_2-\gamma_1} + A_1 + O(r^{\delta-\gamma_1})$$

hence the desired result. □

Proof of Proposition 8.1. We assume from now on that $|h|(r) < 1$. Using the Schauder estimates we find that this implies $|\nabla^l h| < C_l$, so

$$|L_g h - L_{g_h} h| = O(r^{-n+1}) \tag{8.1}$$

for $r > r_+ + 1$. In coordinates, the bound on h translates to

$$|h_{11}|(r) \lesssim r^{-2}, \quad , |h_{1i}|(r) \lesssim 1, \quad |h_{ij}|(r) \lesssim r^2$$

where $i, j > 1$ and the first inequality only holds for $r > r_+ + 1$.

We will use the equations from the last section to derive a better estimate on h. Set $H = |h|(R)$.

In the following four paragraphs we will always assume $r > r_+ + 1$.

1° Consider the $1i$ entry of h for $i > 1$. By equation (III) of the last section and (8.1) it satisfies

$$r^2 h_{1i}'' + nrh_{1i}' - nh_{1i} = f_{1i} + O(r^{-n+1}).$$

where $f_{1i} = O(\alpha(\frac{r}{R})^{0.1}) + O(\alpha r^{-0.1})$. Lemma 8.2 gives us

$$h_{1i}(r) = A_1 r + A_2 r^{-n} + O(\alpha(\tfrac{r}{R})^{0.1}) + O(\alpha r^{-0.1}) + O(r^{-n+1}).$$

Since $|h_{1i}|(r)$ and the error terms above are bounded for say $r \in (r_+ + 1, r_+ + 2)$, we can deduce a bound on $|A_1|$ and $|A_2|$ (although this bound is rather bad). Now setting $r = R$ in the equation above, we conclude that even $|A_1| < CH\frac{1}{R} + O(R^{-n-1}) + O(\alpha R^{-1}) + O(R^{-n})$ and hence

$$|h_{1i}|(r) < C\left(H\frac{r}{R} + \alpha + r^{-n+1}\right) \leq C\left(H + \alpha + r^{-n+1}\right).$$

2° Next, we apply the same principle to $q = \operatorname{tr} h$. Using (7.1) we get

$$r^2 q'' + nrq' - 2(n-1)q = O(\alpha(\tfrac{r}{R})^{0.1}) + O(\alpha r^{-0.1}) + O(r^{-n+1}).$$

So

$$q(r) = A_1 r^{\gamma_1} + A_2 r^{\gamma_2} + O(\alpha(\tfrac{r}{R})^{0.1}) + O(\alpha r^{-0.1}) + O(r^{-n+1})$$

where $\gamma_{1/2} = \frac{1}{2}(-n + 1 \pm \sqrt{n^2 + 6n - 7})$. Observe that $\gamma_1 > 0.1$ and $\gamma_2 < -n + 1$. By the same argument as before, we conclude that $|A_2| < C$ and $|A_1| < CH\frac{1}{R^{\gamma_1}} + O(R^{\gamma_2 - \gamma_1}) + O(\alpha R^{-\gamma_1}) + O(R^{-n+1-\gamma_1})$. Thus

$$|\operatorname{tr} h|(r) < C\left[H\left(\frac{r}{R}\right)^{\gamma_1} + \alpha\left(\frac{r}{R}\right)^{0.1} + \alpha r^{-0.1} + r^{-n+1}\right] \leq C\left(H + \alpha + r^{-n+1}\right).$$

3° With the help of this estimate, we can now bound h_{11}. By equation (I) resp. (I')

$$\begin{aligned} r^2 h_{11}'' &+ (n+4)rh_{11}' + 4h_{11} \\ &= r^{-2}\left(O(H(\tfrac{r}{R})^{\gamma_1}) + O(\alpha(\tfrac{r}{R})^{0.1}) + O(\alpha r^{-0.1}) + O(r^{-n+1})\right). \end{aligned}$$

At this point we have to be a bit careful since the exponent of one fundamental solution coincides with the exponent of the first error term. A precise calculation would give us a log-term, however we solve this issue a bit sloppy by decreasing the exponent γ_1 slightly (recall that $r \leq R$). Lemma 8.2 implies then

$$\begin{aligned} r^2 h_{11}(r) &= A_1 r^{\gamma_1} + A_2 r^{\gamma_2} \\ &\quad + O(H(\tfrac{r}{R})^{\gamma_1 - 0.1}) + O(\alpha(\tfrac{r}{R})^{0.1}) + O(\alpha r^{-0.1}) + O(r^{-n+1}). \end{aligned}$$

By the same arguments as in the last two paragraphs we find

$$r^2|h_{11}|(r) < C\left[H\left(\frac{r}{R}\right)^{\gamma_1-0.1} + \alpha\left(\frac{r}{R}\right)^{0.1} + \alpha r^{-0.1} + r^{-n+1}\right] \\ \leq C\left(H + \alpha + r^{-n+1}\right).$$

4° Finally, we bound h_{ij} for $i, j > 1$. Observe that similar estimates to the ones in paragraph 1°, 2° and 3° could also have been obtained if we hadn't assumed exponential decay for f (such as Anderson did in [And2]). However, the exponential decay will be essential in the following estimate since it saves us from an extra log-term. By (II)

$$r^2 h''_{ij} + (n-4)r h'_{ij} - 2(n-3)h_{ij} \\ = r^2\left[O(H(\tfrac{r}{R})^{\gamma_1-0.1}) + O(\alpha(\tfrac{r}{R})^{0.1}) + O(\alpha r^{-0.1}) + O(r^{-n+1})\right].$$

Thus

$$r^{-2}h_{ij}(r) = A_1 + A_2 r^{-n+1} \\ + O(H(\tfrac{r}{R})^{\gamma_1-0.1}) + O(\alpha(\tfrac{r}{R})^{0.1}) + O(\alpha r^{-0.1}) + O(r^{-n+1}).$$

We find that $|A_2| < C$ and $|A_1| < CH + O(H+\alpha) + O(R^{-n+1})$, so

$$r^{-2}|h_{ij}|(r) < C\left(H + \alpha + r^{-n+1}\right)$$ □

We will now prove the second result of this section.

Proposition 8.3. *Let h be an $S^1 \times \mathbb{R}^{n-2}$ invariant Einstein variation of g_{BH} and assume $|h|(r) \to 0$ for $r \to \infty$. Then $h \equiv 0$.*

We note that with a little more work, it is even possible to deduce that any $S^1 \times \mathbb{R}^{n-2}$ invariant Einstein variation h that suffices a bound $|h|(r) < r^\delta$ with $\delta < \frac{1}{2}(-n+1+\sqrt{n^2+6n-7})$, is of the form

$$h = -\operatorname{tr} u \frac{n-1}{V r^{n-1}} \mathrm{d}r^2 - \operatorname{tr} u \frac{VV'}{2r} \mathrm{d}\theta^2 + 2(\operatorname{tr} u) r^{-n+3}(\mathrm{d}x_3^2 + \ldots + \mathrm{d}x_n^2) + u_{ij} r^2 \mathrm{d}x_i \mathrm{d}x_j$$

for some symmetric $(n-2) \times (n-2)$ matrix u_{ij} indexed by $i, j = 3, \ldots, n$.

Assume from now on that $|h|(r) \to 0$ as $r \to \infty$ and that $Lh = 0$. Using Proposition 8.1, we find that we even have $|h|(r) < r^{-n+1}$ after multiplying h by a suitable constant. (Note that we could also have assumed this decay rate in the assumption of Proposition 8.3 since we have already obtained this rate in section 6.) By Schauder's estimates we can deduce the same decay for all covariant derivatives of h.

Lemma 8.4. *We have $\operatorname{tr} h \equiv 0$, $\delta h \equiv 0$ and hence $\mathrm{d}\,\mathrm{Ric}_g(h) + (n-1)h = 0$.*

Proof. This follows from the maximum principle applied to (2.2) resp. (2.5) and the fact that $\operatorname{tr} h$ and $\beta(h)$ are decaying. □

Lemma 8.5. *We have $h_{1i} = h_{i1} \equiv 0$ for all $i \geq 2$.*

Proof. Writing out the equation $\delta_g h = 0$ in terms of the h_{ij} gives

$$0 = -(\delta_g h)_1 = V h'_{11} + (\frac{3}{2}V' + (n-2)\frac{V}{r})h_{11} - \frac{V'}{2V^2}h_{22} - \sum_{i\geq 3}\frac{1}{r^3}h_{ii}$$

$$0 = -(\delta_g h)_2 = V h'_{12} + (V' + (n-2)\frac{V}{r})h_{12}$$

$$0 = -(\delta_g h)_i = V h'_{1i} + (V' + (n-2)\frac{V}{r})h_{1i} \qquad (i \geq 3)$$

The solutions of the last two ODEs behave like $\frac{1}{r-r_+}$ for $r \to r_+$, so h_{12} and h_{1i} must be constantly zero. □

Now we will alter h by an infinitesimal diffeomorphism $\delta_g^* \xi$ for some 1-form ξ to eliminate its 11 entry. Observe that by (2.1) for every 1-form ξ we have

$$\mathrm{d}\,\mathrm{Ric}_g(\delta_g^*\xi) + (n-1)\delta_g^*\xi = 0$$

since g is Einstein. So for any 1-form ξ the bilinear form $h + \delta_g^*\xi$ will still be an infinitesimal Einstein variation. However, we might lose the divergence or trace freeness.

Lemma 8.6. *There is an $S^1 \times \mathbb{R}^{n-2}$ invariant 1-form $\xi = \xi(r)$ such that for $k = h + \delta_g^*\xi$ we have $k_{1i} = k_{i1} = 0$ for $i = 1, \ldots, n$. Moreover, $|k|(r) < C$.*

Proof. Assume $\xi = \xi_1 dr$. Then

$$(\delta^*\xi)_{11} = \xi_1' + \frac{V'}{2V}\xi_1,$$

$$(\delta^*\xi)_{22} = \frac{1}{2}VV'\xi_1$$

$$(\delta^*\xi)_{ii} = rV\xi_1 \qquad \text{for } i \geq 3$$

The remaining components are zero.

We now solve the ODE $(\delta^*\xi)_{11} = -h_{11}$. We will need a particular solution. Observe that the ODE can be written as $(V^{1/2}\xi_1)' = -V^{1/2}h_{11} = -\frac{1}{V^{1/2}}f$, where $|f| < Cr^{-n+1}$. Choose the following solution

$$\xi_1(r) := -\frac{1}{V^{1/2}}\int_{r_+}^{r}\frac{f}{V^{1/2}}.$$

It is easy to see that the integral exists and that $V^{1/2}|\xi_1| \leq C(r-r_+)^{1/2}$. This implies that ξ is smooth. By the decay of f we find furthermore that $V^{1/2}|\xi_1| < C$. Thus $V^{-1}|(\delta^*\xi)_{22}| < C$ and $r^{-2}|(\delta^*\xi)_{ii}| < C$ for $i \geq 3$. This proves the second statement. □

We will now show that k has a very simple form.

For convenience we introduce a new coordinate $s = s(r)$ (the distance to the origin) with the property that $s(r_+) = 0$ and $g_{BH} = \mathrm{d}s^2 + V(r(s))\mathrm{d}\theta^2 +$

$r^2(s)(\mathrm{d}x_3^2 + \ldots + \mathrm{d}x_n^2)$. From now on we will only work in the coordinate system $(s, \theta, x_3, \ldots, x_n)$. Consider a metric $\hat{g}$ of the form

$$\hat{g}(s) = \begin{pmatrix} 1 & 0 \\ 0 & M(s) \end{pmatrix}.$$

The condition of being Einstein with $\mathrm{Ric}_{\hat{g}} = -(n-1)\hat{g}$ is equivalent to the following system of ODEs (see e.g. [Lin]):

$$\left(\sqrt{\det M} M' M^{-1}\right)' - 2(n-1)\sqrt{\det M} E_{n-1} = 0 \tag{I}$$

$$\chi_{n-2}(M' M^{-1}) - 2(n-2)(n-1) = 0 \tag{II}$$

where E_{n-1} denotes the unit matrix of rank $n-1$ and χ_{n-2} the $(n-2)$-th coefficient of the characteristic polynomial, i.e. the elementary symmetric polynomial of degree 2 in the eigenvalues. The prime denotes differentiation by s.

Given $M(1)$ and $M'(1)$, equation (I) has a unique solution $M(s)$. Furthermore, it can be checked that if $M(1)$ and $M'(1)$ satisfy (II) *at* $s = 1$, then this property is preserved by (I).

Now denote by $M = M^0$ the matrix corresponding to $g = g_{BH}$ and the given Einstein-variation $k_{2 \leq i,j \leq n}$ (in the new coordinate system) by $\dot{M}$. Obviously, $\dot{M}$ is an infinitesimal variation of the ODEs (I), (II). We can integrate this infinitesimal variation to a variation M^t of solutions of (I), (II) by a "variation of the initial conditions" argument: First vary $M^t(1)$ and $(M^t)'(1)$ such that $\frac{d}{dt}|_{t=0} M^t(1) = \dot{M}(1)$, $\frac{d}{dt}|_{t=0} (M')^t(1) = \dot{M}'(1)$ and such that $M^t(1)$ and $(M^t)'(1)$ satisfy equation (II) for all t. Next, for every t integrate (I) to get M^t. Equation (II) is automatically preserved. The variation corresponds to a variation of Einstein metrics, however it is not guaranteed that the metrics are smooth at the origin.

A short analysis of (I) yields

Lemma 8.7. $\sqrt{\det M^0} = A \sinh(n-1)s$ *for some* $A > 0$ *and* $\frac{d}{dt}|_{t=0} \sqrt{\det M^t} = \dot{A}\sqrt{\det M^t}$. *This implies* $\mathrm{tr}_g\, k = \mathrm{tr}\, \dot{M}(M^0)^{-1} \equiv 2\dot{A}$.

Proof. Set $u^t = \sqrt{\det M^t}$. Then $(u^t)' = \frac{1}{2}\sqrt{\det M^t}\, \mathrm{tr}\left((M^t)'(M^t)^{-1}\right)$ and

$$(u^0)^{\cdot} = \frac{1}{2}\sqrt{\det M^0}\, \mathrm{tr}\left(\dot{M}(M^0)^{-1}\right) = \frac{1}{2} u^0\, \mathrm{tr}\, k.$$

So taking the trace of (I) yields

$$(u^t)'' - (n-1)^2 u^t = 0.$$

Since $u^0(0) = 0$, we get $u^0 = A\sinh(n-1)s$ and by differentiating the equation above with respect to t gives $\dot{u}^0 = \dot{A}\sinh(n-1)s + \dot{B}\cosh(n-1)s$. So $\frac{1}{2}\mathrm{tr}\, k = \dot{A}A^{-1} + \dot{B}A^{-1}\cosh(n-1)s/\sinh(n-1)s$. Since $\mathrm{tr}\, k$ is bounded, we conclude $\dot{B} = 0$ and therefore $\mathrm{tr}\, k \equiv \mathrm{const}$. □

Now observe that by the symmetries $x_i \to -x_i$ ($i \geq 3$), also the matrix

$$\dot{M}^\perp := \begin{pmatrix} \dot{M}_{22} & -\dot{M}_{23} & \cdots & -\dot{M}_{2n} \\ -\dot{M}_{32} & \dot{M}_{33} & \cdots & \dot{M}_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ -\dot{M}_{n2} & \dot{M}_{n3} & \cdots & \dot{M}_{nn} \end{pmatrix}$$

corresponds to an Einstein variation. Hence, we conclude that $\dot{M}^* := \frac{1}{2}(\dot{M} + \dot{M}^\perp)$ corresponds to an Einstein variation, too. Observe that $\dot{M}^*_{23} = \ldots = \dot{M}^*_{2n} = 0$.

Lemma 8.8. *$\dot{M}^* = QM^*$ where Q is a symmetric matrix with $Q_{2i} = Q_{i2} = 0$ for $i = 2, \ldots, n$.*

Proof. Assume for the moment that the variation M^t is already of the type $\dot{M}^0 = \dot{M}^*$. A variation of (I) gives

$$\big(\sinh((n-1)s)(M'M^{-1})^\cdot\big)' = 0.$$

Hence

$$(M'M^{-1})^\cdot = \frac{1}{\sinh(n-1)s} P.$$

For some constant matrix P. Since M, M' and $\dot{M}(= \dot{M}^*), \dot{M}' = ((\dot{M}^*)')$ are in block form (blocks of size 1 and $n-2$), we conclude that P is also of block form. Moreover, since the lower block of $(M'M^{-1})^\cdot = \dot{M}'M^{-1} - M'M^{-1}\dot{M}M^{-1}$ stays bounded for $s \to 0$, we find that the lower block of $(\sinh(n-1)s)^{-1}P$ must also stay bounded. Hence, the lower block of P must be zero. Moreover, since

$$\big(M'M^{-1}\big)^\cdot = \dot{M}'M^{-1} - M'M^{-1}\dot{M}M^{-1} = M(M^{-1}\dot{M})'M^{-1}$$

and $\operatorname{tr} M^{-1}\dot{M} = \operatorname{tr} \dot{M}M^{-1} \equiv 2\dot{A}$, we find $\operatorname{tr}(M'M^{-1})^\cdot = 0$ and hence $\operatorname{tr} P = 0$. So $P = 0$ and we conclude $\dot{M}M^{-1} = Q$ for some constant matrix Q.

Obviously, Q is of block form and $\operatorname{tr} Q = 2\dot{A}$. By smoothness at the origin, it is easy to conclude that the upper 1×1 block must be zero. □

So we conclude that the variation $\dot{M}^* = QM$ corresponds to a variation of the form $r^2 \sum_{i,j=3}^n u_{ij} \mathrm{d}x_i \mathrm{d}x_j$ where u_{ij} is a $(n-2) \times (n-2)$ matrix. We will come back to this form later.

Assume now that the given variation is of the type $\dot{M} = \frac{1}{2}(\dot{M} - \dot{M}^\perp)$. So

$$\dot{M} = \begin{pmatrix} 0 & \dot{M}_{23} & \cdots & \dot{M}_{2n} \\ \dot{M}_{32} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \dot{M}_{n2} & 0 & \cdots & 0 \end{pmatrix}.$$

The space of variations of ODE (I) which are of this form, is $2(n-2)$ dimensional. We will write down an element of this vector space which depends on $2(n-2)$ parameters $a_3, \ldots, a_n, b_3, \ldots, b_n$, so the given variation must be of this form: Write the black-hole solution as $M = \operatorname{diag}(p, q, \ldots, q)$. Consider the variation $\dot{S}$

with the only nonzero entries $\dot{S}_{23} = \dot{S}_{32} = a_3 p + b_3 q, \ldots, \dot{S}_{2n} = \dot{S}_{n2} = a_n p + b_n q$. We will show that this variation is a solution of (I). Let $N^t = E_{n-1} + \dot{N}t$ where

$$\dot{N} = \begin{pmatrix} 0 & b_3 & \cdots & b_n \\ a_3 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_n & 0 & \cdots & 0 \end{pmatrix}.$$

Then $\hat{M}^t := N^t M (N^t)^T$ is a solution of (I) and $\dot{S} = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} \hat{M}^t$.

Now extend M from $[0, \infty)$ to $\mathbb{R}$. We can do this by assuming that $M_{ij}(s) = \sigma_{ij} M_{ij}(-s)$ where $\sigma_{ij} = \sigma_i' \sigma_j'$ and $\sigma_i' = -1$ for $i = 2$ and 1 for $i \geq 3$. By smoothness at the origin, M is smooth and satisfies ODE (I) everywhere. $\dot{M}$ can be extended to $\mathbb{R}$ in the same way. Observe that p and q are now even functions, whereas $\dot{M}_{2i} = \dot{M}_{i2}$ must be odd implying, $a_3 = \ldots = a_n = b_3 = \ldots = b_n = 0$ and hence $\dot{M}_{2i} = \dot{M}_{i2} \equiv 0$ (the argument seems to be unfamiliar, but it is actually only a quick way of tracking the different vanishing orders at the origin).

We can now summarize the discussion above: Returning to the old coordinates $(r, \theta, x_3, \ldots, x_n)$, we have proven so far that h takes the following form

$$h = -\delta_g^* \xi + r^2 \sum_{i,j=3}^{n} u_{ij} \mathrm{d}x_i \mathrm{d}x_j.$$

So $h_{22} = -(\delta_g^* \xi)_{22}$. By the equations from the proof of Lemma 8.6, we conclude from the decay of h that $V^{1/2} \xi_1(r) < C r^{-n+1}$ hence $r^{-2} |(\delta_g^* \xi)_{ii}| < C r^{-n+1}$. But then we must have $u_{ij} = 0$ hence $h = -\delta_g^* \xi$. The trace freeness of h implies by the formulas in Lemma 8.6

$$V \xi_1' + \left(V' + (n-2) \frac{V}{r} \right) \xi_1 = 0.$$

This implies

$$\xi_1(r) = \frac{C}{V r^{n-2}}$$

for some C. This solution behaves like $\frac{C}{2(n-1)} \frac{1}{r - r_+}$ as $r \to r_+$. So by smoothness of ξ we must have $C = 0$. This concludes the proof of Proposition 8.3.

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