



# Decomposition of higher-order Wright-convex functions <sup>☆</sup>

Gyula Maksa, Zsolt Páles\*

Institute of Mathematics, University of Debrecen, H-4010 Debrecen, Pf. 12, Hungary

## ARTICLE INFO

### Article history:

Received 8 January 2009  
 Available online 27 May 2009  
 Submitted by M. Laczko

### Keywords:

Convexity  
 Higher-order convexity  
 Wright convexity  
 Decomposition

## ABSTRACT

In this paper we consider higher-order Wright-convex functions and prove that they are representable as the sum of a continuous higher-order convex function and a polynomial function.

© 2009 Elsevier Inc. All rights reserved.

## 1. Introduction

Throughout this paper  $\mathbb{N}$ ,  $\mathbb{Z}$ ,  $\mathbb{Q}$ ,  $\mathbb{R}$ , and  $I$  will denote the sets of all positive integers, integers, rational numbers, real numbers, and a non-void open subinterval of  $\mathbb{R}$ , respectively. By the standard definition (cf. [13,18]), a real-valued function  $f : I \rightarrow \mathbb{R}$  is called *convex* if

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y) \quad (x, y \in I, t \in [0, 1]). \tag{1}$$

If the above inequality holds for all  $x, y \in I$  with  $t = 1/2$ , then  $f$  is usually said to be *Jensen-convex*. In 1954, E.M. Wright [20] introduced a new convexity property for real functions: A function  $f : I \rightarrow \mathbb{R}$  is called *Wright-convex* (cf. [18]) if

$$f(tx + (1 - t)y) + f((1 - t)x + ty) \leq f(x) + f(y) \quad (x, y \in I, t \in [0, 1]). \tag{2}$$

One can easily see that convex functions are Wright-convex. On the other hand, if  $f : \mathbb{R} \rightarrow \mathbb{R}$  is *additive*, that is,

$$f(x + y) = f(x) + f(y) \quad (x, y \in \mathbb{R}),$$

then  $f$  is also Wright-convex. The following result of Ng [10] is the much more surprising statement that any Wright-convex function can be decomposed as the sum of such functions.

**Ng's Theorem.** (See [10, Corollary 5].) *Let  $f : I \rightarrow \mathbb{R}$  be a function. Then  $f$  is Wright-convex if and only if there exist a convex function  $C : I \rightarrow \mathbb{R}$  and an additive function  $A : \mathbb{R} \rightarrow \mathbb{R}$  such that*

$$f(x) = C(x) + A(x) \quad (x \in I).$$

In the theory of convex functions, decomposition theorems play an important role. For instance, in the context of approximately convex functions and midconvex functions, various forms of decomposition theorems have been established, cf.

<sup>☆</sup> This research has been supported by the Hungarian Scientific Research Fund (OTKA) Grants K-62316, NK-68040.

\* Corresponding author.

E-mail addresses: maksa@math.klte.hu (G. Maksa), pales@math.klte.hu (Z. Páles).

Hyers and Ulam [6], Green [5], Ger [3], Ng [11], Nikodem [14,15], Ng and Nikodem [12], Kominek [7], Mrowiec [9], Páles [16], Dilworth et al. [1].

To define higher-order convexity concepts, we need to recall the notions of the *translation and difference operators*. For a fixed real number  $h$ , these operators  $\tau_h$  and  $\Delta_h$ , acting on a real function  $f: I \rightarrow \mathbb{R}$ , are defined by

$$\tau_h f(x) := f(x+h) \quad (x \in I-h) \quad \text{and} \quad \Delta_h f(x) := f(x+h) - f(x) \quad (x \in I \cap (I-h)),$$

respectively. Obviously, if  $|h|$  is small enough, then  $I \cap (I-h)$  is a non-void open interval again and the product of these operators can also be defined in the usual way (see e.g. Kuczma [8, p. 365]).

Taking  $x, y \in I$ ,  $x < y$ , with the notation  $h := (y-x)/2$ , the Jensen-convexity of a function  $f$  can be rephrased as

$$\Delta_h^2 f(x) \geq 0 \quad (h > 0, x \in I \cap (I-2h)).$$

Similarly, with the notations  $h_1 := t(y-x)$ ,  $h_2 := (1-t)(y-x)$ , the Wright-convexity inequality (2) can be written as

$$\Delta_{h_1} \Delta_{h_2} f(x) \geq 0 \quad (h_1, h_2 > 0, x \in I \cap (I-(h_1+h_2))).$$

Observe that this also yields the increasingness of the function  $\Delta_h f$  on  $I \cap (I-h)$  for each  $0 < h$ .

The aim of this paper is to give a generalization of the above concepts and results to the setting of higher-order convexity. As it is extensively discussed in [8], for  $n \in \mathbb{N}$ , a function  $f: I \rightarrow \mathbb{R}$  is called *Jensen-convex of order  $n$*  if

$$\Delta_h^{n+1} f(x) \geq 0 \quad (h > 0, x \in I \cap (I-(n+1)h)). \quad (3)$$

The notion of  *$n$ th-order convexity* was defined in terms of divided differences by Popoviciu [17] (cf. also [18]). However, under the assumption of continuity, Jensen-convexity of order  $n$  and  $n$ th-order convexity are equivalent properties. It seems to be natural to introduce the notion of the Wright-convexity of order  $n$ , as well. In the paper [4] the following definition has been given: For  $n \in \mathbb{N}$ , a function  $f: I \rightarrow \mathbb{R}$  is called *Wright-convex of order  $n$*  (or shortly  *$n$ -Wright-convex*) if

$$\Delta_{h_1} \cdots \Delta_{h_{n+1}} f(x) \geq 0 \quad (h_1, \dots, h_{n+1} > 0, x \in I \cap (I-(h_1+\cdots+h_{n+1}))). \quad (4)$$

In the investigation of inequalities (3) and (4), those functions that satisfy these inequalities with equality play a crucial role. For  $n \in \mathbb{N}$ , a function  $P: \mathbb{R} \rightarrow \mathbb{R}$  is called a *polynomial function of degree at most  $n$*  if it satisfies the Fréchet functional equation, i.e., if

$$\Delta_h^{n+1} P(x) = 0 \quad (h, x \in \mathbb{R}).$$

It is well known (see [8,19]) that  $P: \mathbb{R} \rightarrow \mathbb{R}$  is a polynomial function of degree at most  $n$  if, and only if, it has the representation

$$P(x) = a_0 + a_1(x) + \cdots + a_n(x) \quad (x \in \mathbb{R}),$$

where  $a_0 \in \mathbb{R}$  and  $a_k$  is the *diagonalization* of some  $k$ -additive and symmetric function  $A_k: \mathbb{R}^k \rightarrow \mathbb{R}$ , that is,  $a_k(x) = A_k(x, \dots, x)$  ( $x \in \mathbb{R}$ ,  $k = 1, \dots, n$ ). Polynomials are exactly the continuous polynomial functions, however, in terms of Hamel bases, one can construct non-continuous polynomial functions [8].

In this note we give a generalization of Ng's Theorem by proving that any Wright-convex function of order  $n$  can be represented as the sum of a continuous  $n$ -convex function and a polynomial function of degree at most  $n$ .

## 2. The main result

It is well known that convex functions defined on an open interval are continuous. Thus Ng's Theorem implies that, if the second difference functions (with positive increments) of a function defined on an open interval are non-negative then their first difference functions (with positive increments) are continuous. Applying this observation to the  $(n-1)$ st-order differences of  $f$ , we get the following easy consequence of Ng's Theorem.

**Lemma 1.** *Let  $f: I \rightarrow \mathbb{R}$  and  $n \in \mathbb{N}$ . If  $f$  is  $n$ -Wright-convex then, for all  $h_1, \dots, h_n \in ]0, +\infty[$ , the function  $\Delta_{h_1} \cdots \Delta_{h_n} f$  is continuous on  $I \cap (I-(h_1+\cdots+h_n))$ .*

The next statement establishes the equivalence of convexity and Wright-convexity under a weak regularity assumption (cf. [8]).

**Lemma 2.** *Let  $f: I \rightarrow \mathbb{R}$  be bounded on a subinterval  $I$  of positive length and  $n \in \mathbb{N}$ . Then  $f$  is  $n$ -Wright-convex if and only if it is convex of order  $n$ . Furthermore, in this case,  $f$  is continuous (moreover, it is  $(n-1)$ -times continuously differentiable) on  $I$ .*

The following definition will be useful in formulating some technical details.

**Definition.** Let  $f: I \rightarrow \mathbb{R}$ ,  $n \in \mathbb{N}$  be fixed and  $I_0 \subset I$  be a subinterval of positive length. We say that  $f$  is *decomposable* on  $I_0$  if there exist a locally Riemann integrable function  $f_0: I_0 \rightarrow \mathbb{R}$  and a polynomial function  $P_0$  of degree at most  $n$  such that  $P_0(\mathbb{Q}) = \{0\}$  and  $f = f_0 + P_0$  on  $I_0$ .

The next result of Gajda [2, Corollary 1] derives a decomposability of 1-periodic functions from the local Riemann integrability of their  $n$ th-order differences.

**Lemma 3.** Let  $n \in \mathbb{N}$  and  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a 1-periodic function such that  $\Delta_h^n f$  is locally Riemann integrable on  $\mathbb{R}$  for each  $h \in \mathbb{R}$ . Then there exist a locally Riemann integrable  $f_0 : \mathbb{R} \rightarrow \mathbb{R}$  and a polynomial function  $P_0 : \mathbb{R} \rightarrow \mathbb{R}$  of degree at most  $n$  such that  $f = f_0 + P_0$ .

**Remark.** In fact, Corollary 1 of paper [2] asserts a decomposition also with continuity instead of local Riemann integrability. However, in the proof of our main result, we could not utilize this type of decomposition.

To prove our main result we need the following

**Lemma 4.** Let  $f : I \rightarrow \mathbb{R}$  and  $n \in \mathbb{N}$ .

- (a) If  $f$  is decomposable on the subintervals  $I_1, I_2$  of  $I$  and the interval  $I_1 \cap I_2$  has positive length then  $f$  is also decomposable on  $I_1 \cup I_2$ .
- (b) If  $(I_k)$  is an increasing sequence of subintervals of positive length of  $I$  on which  $f$  is decomposable then  $f$  is also decomposable on  $\bigcup_{k=1}^{\infty} I_k$ .
- (c) If, for all  $a, b \in I, a < b$ ,  $f$  is decomposable on  $[a, \frac{a+b}{2}]$  then  $f$  is also decomposable on  $I$ .

**Proof.** (a) Since  $f = f_i + P_i$  on  $I_i$  with locally Riemann integrable  $f_i : I_i \rightarrow \mathbb{R}$  and polynomial functions  $P_i$  ( $i = 1, 2$ ), we have that  $P_1 - P_2$  is locally Riemann integrable on  $I_1 \cap I_2$  and vanishes at its rational points. Therefore  $P_1 = P_2$  on  $\mathbb{R}$  and  $f_1 = f_2$  on  $I_1 \cap I_2$ . Thus the function  $f_0 = f_1 \cup f_2$  is well defined, it is locally Riemann integrable and  $f = f_0 + P_1$  on  $I_1 \cup I_2$ , that is,  $f$  is decomposable on  $I_1 \cup I_2$ .

(b) Since, for all  $k \in \mathbb{N}$ ,  $f = f_k + P_k$  on  $I_k$  with locally Riemann integrable  $f_k : I_k \rightarrow \mathbb{R}$  and polynomial functions  $P_k$  of degree at most  $n$  with  $P_k(\mathbb{Q}) = \{0\}$ , we have that  $P_k - P_{k+1}$  is locally Riemann integrable on  $I_k$  and vanishes on  $I_k \cap \mathbb{Q}$ . Therefore  $P_k = P_1$  on  $\mathbb{R}$  for all  $k \in \mathbb{N}$ . Thus the function  $f_0 = \bigcup_{k=1}^{\infty} f_k$  is well defined, it is locally Riemann integrable and  $f = f_0 + P_1$  on  $\bigcup_{k=1}^{\infty} I_k$ , that is,  $f$  is decomposable on  $\bigcup_{k=1}^{\infty} I_k$ .

(c) Let  $b_k = 2^{-k}(a + (2^k - 1)b)$  ( $k \in \mathbb{N}$ ). Then the sequence  $b_k$  is strictly increasing and  $b_k \rightarrow b$  as  $k \rightarrow \infty$ . Therefore the sequence  $([a, b_k])$  is increasing and  $\bigcup_{k=1}^{\infty} [a, b_k] = [a, b[$ . By induction on  $k$ , we prove that  $f$  is decomposable on each  $[a, b_k]$ . Obviously, this is true for  $k = 1$ . Suppose that  $f$  is decomposable on  $[a, b_k]$  and let the real number  $\varepsilon > 0$  be so that  $b + \varepsilon \in I$  and  $\varepsilon < b_k - a$ . Since  $f$  is decomposable on  $[b_k - \varepsilon, \frac{b_k - \varepsilon + b + \varepsilon}{2}] = [b_k - \varepsilon, b_{k+1}]$  and also on  $[a, b_k]$ , by the part (a), we obtain that  $f$  is decomposable on  $[a, b_{k+1}]$ . Thus it follows from the part (b) that  $f$  is decomposable on  $[a, b[$ . Let now the positive real number  $\delta$  be so that  $b + \delta \in I$  and  $\delta < b - a$ . Since  $f$  is decomposable on  $[b - \delta, \frac{b - \delta + b + \delta}{2}] = [b - \delta, b]$ , the part (a) implies that  $f$  is decomposable on  $[a, b]$ . Finally, let  $(I_k)$  be an increasing sequence of compact subintervals of  $I$  such that  $I = \bigcup_{k=1}^{\infty} I_k$ . Since  $f$  is decomposable on any  $I_k$ , the part (b) implies the part (c).  $\square$

Our main result is the following

**Theorem.** Let  $n \in \mathbb{N}$  and  $f : I \rightarrow \mathbb{R}$ . Then  $f$  is an  $n$ -Wright-convex function if and only if  $f$  is of the form

$$f(x) = C(x) + P(x) \quad (x \in I), \tag{5}$$

where  $C : I \rightarrow \mathbb{R}$  is continuous  $n$ -convex function and  $P : \mathbb{R} \rightarrow \mathbb{R}$  is a polynomial function of degree at most  $n$  with  $P(\mathbb{Q}) = \{0\}$ . Furthermore, under the assumption  $P(\mathbb{Q}) = \{0\}$ , the decomposition (5) is unique.

**Proof.** Lemma 1 implies that  $\Delta_h^{n-1} \Delta_s f$  is continuous, consequently locally Riemann integrable for all  $0 < h, 0 < s$  on  $I \cap (I - (n - 1)h - s)$ . Therefore, by Lemma 4, it is enough to prove that, for all  $a, b \in I, a < b$ ,  $f$  is decomposable on the interval  $[a, \frac{a+b}{2}]$ . Indeed, in this case we shall have (5) with a locally Riemann integrable  $C : I \rightarrow \mathbb{R}$  and a polynomial function  $P$  of degree at most  $n$ . On the other hand, Eq. (5) implies, for all  $h > 0$ , that  $\Delta_h^{n+1} f = \Delta_h^{n+1} C$  on  $I \cap (I - (n + 1)h)$ , which shows that  $C$  is an  $n$ -Jensen-convex locally Riemann integrable function. Thus, by [8, p. 383],  $C$  is continuous.

Let  $[a, b] \subset I$  be a compact subinterval of positive length

$$\varphi(x) := \frac{1}{2}(b - a)x + a \quad (x \in \mathbb{R}), \tag{6}$$

$J := \varphi^{-1}(I)$ , and

$$f_1(x) = f(\varphi(x)) - x \left[ f\left(\frac{a+b}{2}\right) - f(a) \right] \quad (x \in J). \tag{7}$$

Then  $[0, 2] \subset J$ ,  $f_1(0) = f_1(1)$  and, for all  $0 < h, 0 < s, x \in J \cap (J - (n - 1)h - s)$ , by (7) and (6) and a well-known identity (see e.g. [8, p. 368]), for  $1 < n \in \mathbb{N}$ , we get that

$$\begin{aligned}
 \Delta_h^{n-1} \Delta_s f_1(x) &= \sum_{k=0}^{n-1} (-1)^{n-1-k} \binom{n-1}{k} \Delta_s f_1(x+kh) \\
 &= \sum_{k=0}^{n-1} (-1)^{n-1-k} \binom{n-1}{k} (f_1(x+kh+s) - f_1(x+kh)) \\
 &= \sum_{k=0}^{n-1} (-1)^{n-1-k} \binom{n-1}{k} (f(\varphi(x+kh+s)) - f(\varphi(x+kh))) \\
 &= \sum_{k=0}^{n-1} (-1)^{n-1-k} \binom{n-1}{k} \left( f\left(\varphi(x) + k\frac{1}{2}(b-a)h + \frac{1}{2}(b-a)s\right) - f\left(\varphi(x) + k\frac{1}{2}(b-a)h\right) \right) \\
 &= \sum_{k=0}^{n-1} (-1)^{n-1-k} \binom{n-1}{k} \Delta_{\frac{1}{2}(b-a)s} f\left(\varphi(x) + k\frac{1}{2}(b-a)h\right) \\
 &= (\Delta_{\frac{1}{2}(b-a)h}^{n-1} \Delta_{\frac{1}{2}(b-a)s} f)(\varphi(x)).
 \end{aligned}$$

For the case  $n = 1$ , one can obtain that

$$\begin{aligned}
 \Delta_h^{n-1} \Delta_s f_1(x) &= \Delta_s f_1(x) = \Delta_{\frac{1}{2}(b-a)s} f(\varphi(x)) - \left[ f\left(\frac{a+b}{2}\right) - f(a) \right] s \\
 &= (\Delta_{\frac{1}{2}(b-a)h}^{n-1} \Delta_{\frac{1}{2}(b-a)s} f)(\varphi(x)) - \left[ f\left(\frac{a+b}{2}\right) - f(a) \right] s.
 \end{aligned}$$

These identities show that  $\Delta_h^{n-1} \Delta_s f_1$  is continuous on  $J \cap (J - (n-1)h - s)$  for all  $0 < h, 0 < s$ .

Define the function  $g$  on  $\mathbb{R}$  as the 1-periodic extension of the restriction of  $f_1$  to the interval  $[0, 1]$  and the function  $f_2$  on  $J$  by

$$f_2(x) := f_1(x) - g(x). \tag{8}$$

We prove that, for each  $h > 0$ ,  $\Delta_h^{n-1} f_2$  is locally Riemann integrable on  $[0, +\infty[ \cap J \cap (J - (n-1)h)$ . Since

$$\begin{aligned}
 f_2(x) &= f_1(x) - g(x) = f_1(\{x\} + [x]) - g(\{x\} + [x]) \\
 &= f_1(\{x\} + [x]) - g(\{x\}) = f_1(\{x\} + [x]) - f_1(\{x\}) = \Delta_{[x]} f_1(\{x\})
 \end{aligned}$$

for all  $x \in J$ , therefore

$$\Delta_h^{n-1} f_2(x) = (\Delta_h^{n-1} \Delta_{[x]} f_1)(\{x\})$$

for all  $x \in J \cap (J - (n-1)h)$ . (Here and in the sequel,  $\{x\}$  and  $[x]$  denote the fractional and integer parts of  $x$ , respectively.) Thus, for each  $h > 0$ ,  $\Delta_h^{n-1} f_2$  is right-continuous on  $[0, +\infty[ \cap J \cap (J - (n-1)h)$ , left-continuous on  $([0, +\infty[ \setminus \mathbb{Z}) \cap J \cap (J - (n-1)h)$  and its left-sided finite limit exists at the positive integer points of  $[0, +\infty[ \cap J \cap (J - (n-1)h)$ . Therefore, for each  $h > 0$ ,  $\Delta_h^{n-1} f_2$  is locally Riemann integrable on  $[0, +\infty[ \cap J \cap (J - (n-1)h)$ . Since, for each  $0 < h \leq \frac{1}{n}$ , Eq. (8) implies that

$$\Delta_h^n f_2(x) = \Delta_h^n f_1(x) - \Delta_h^n g(x) \quad (x \in [0, 1]),$$

hence we have that  $\Delta_h^n g$  is locally Riemann integrable on  $\mathbb{R}$  for each  $0 < h \leq \frac{1}{n}$ . On the other hand, since the difference and the translation operators are linear and they commute, we get that

$$\Delta_h g = (\tau_h - \tau_0)g = (\tau_{\frac{h}{N}}^N - \tau_0^N)g = \sum_{j=0}^{N-1} \tau_{\frac{h}{N}}^j (\tau_{\frac{h}{N}} - \tau_0)g = \sum_{j=0}^{N-1} \tau_{\frac{h}{N}}^j \Delta_{\frac{h}{N}} g$$

hence

$$\Delta_h^n g = \sum_{j_1=0, \dots, j_n=0}^{N-1} \tau_{\frac{h}{N}}^{j_1+\dots+j_n} \Delta_{\frac{h}{N}}^n g$$

holds for all  $1 < N \in \mathbb{N}$ . Therefore  $\Delta_h^n g$  is locally Riemann integrable on  $\mathbb{R}$  for all  $0 < h$ . Finally the identity  $\Delta_h^n g = (-1)^n \tau_h^n \Delta_{-h}^n g$  shows that  $\Delta_h^n g$  is locally Riemann integrable on  $\mathbb{R}$  for all  $h \in \mathbb{R}$ . Applying Lemma 3, we get that  $g$  is the sum of a locally Riemann integrable function and a polynomial function of degree at most  $n$  on  $\mathbb{R}$ . Therefore, by (7) and (6),  $f$  also has this property on the interval  $[a, \frac{a+b}{2}]$ , that is,

$$f(x) = f_0(x) + P_0(x) \quad \left( x \in \left[ a, \frac{a+b}{2} \right] \right), \tag{9}$$

where  $f_0 : [a, \frac{a+b}{2}] \rightarrow \mathbb{R}$  is locally Riemann integrable and  $P_0 : \mathbb{R} \rightarrow \mathbb{R}$  is a polynomial function of degree at most  $n$ . As we have mentioned,  $P_0$  has the representation

$$P_0(x) = a_0 + a_1(x) + \cdots + a_n(x) \quad (x \in \mathbb{R})$$

with some  $a_0 \in \mathbb{R}$  and the diagonalization  $a_k$  of some  $k$ -additive symmetric function ( $k = 1, \dots, n$ ). Thus, with the definitions

$$C(x) := f_0(x) + a_0 + \sum_{k=1}^n a_k(1)x^k \quad \left(x \in \left[a, \frac{a+b}{2}\right]\right)$$

and

$$P(x) := P_0(x) - a_0 - \sum_{k=1}^n a_k(1)x^k \quad (x \in \mathbb{R}),$$

we have that  $P(\mathbb{Q}) = \{0\}$  and Eq. (9) implies that  $f = C + P$  on  $[a, \frac{a+b}{2}]$ , that is,  $f$  is decomposable on the interval  $[a, \frac{a+b}{2}]$ .

The uniqueness of the representation (5) on  $I$  is obvious because of the property  $P(\mathbb{Q}) = \{0\}$ . Indeed, if  $f = C_1 + P_1 = C_2 + P_2$  on  $I$  and  $P_1(\mathbb{Q}) = P_2(\mathbb{Q}) = \{0\}$ , then  $P_2 - P_1 = C_2 - C_1$  is a continuous polynomial function which vanishes at rational points of  $I$ . Hence  $P_2 - P_1 = 0$  on  $\mathbb{R}$ .  $\square$

### Acknowledgment

We thank the referee for calling our attention to some additional important references.

### References

- [1] S.J. Dilworth, R. Howard, J.W. Roberts, A general theory of almost convex functions, *Trans. Amer. Math. Soc.* 358 (8) (2006) 3413–3445.
- [2] Z. Gajda, Difference properties of higher orders for continuity and Riemann integrability, *Colloq. Math.* 53 (2) (1987) 275–288.
- [3] R. Ger, Almost approximately convex functions, *Math. Slovaca* 38 (1) (1988) 61–78.
- [4] A. Gilányi, Zs. Páles, On convex and Wright-convex functions of higher order, *Math. Inequal. Appl.* 11 (2) (2008) 271–282.
- [5] J.W. Green, Approximately convex functions, *Duke Math. J.* 19 (1952) 499–504.
- [6] D.H. Hyers, S.M. Ulam, Approximately convex functions, *Proc. Amer. Math. Soc.* 3 (1952) 821–828.
- [7] Z. Kominek, On a problem of K. Nikodem, *Arch. Math. (Basel)* 50 (3) (1988) 287–288.
- [8] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities*, Pr. Nauk. Uniw. Śl. Katow., vol. 489, Państwowe Wydawnictwo Naukowe – Uniwersytet Śląski, Warszawa, Kraków, Katowice, 1985.
- [9] J. Mrowiec, On the stability of Wright-convex functions, *Aequationes Math.* 65 (1–2) (2003) 158–164.
- [10] C.T. Ng, Functions generating Schur-convex sums, in: W. Walter (Ed.), *General Inequalities*, 5, Oberwolfach, 1986, in: *Internat. Ser. Numer. Math.*, vol. 80, Birkhäuser, Basel, Boston, 1987, pp. 433–438.
- [11] C.T. Ng, On midconvex functions with midconcave bounds, *Proc. Amer. Math. Soc.* 102 (3) (1988) 538–540.
- [12] C.T. Ng, K. Nikodem, On approximately convex functions, *Proc. Amer. Math. Soc.* 118 (1) (1993) 103–108.
- [13] C.P. Niculescu, L.-E. Persson, *Convex Functions and Their Applications*, CMS Books Math./Ouvrages Math. SMC, vol. 23, Springer-Verlag, New York, 2006, a contemporary approach.
- [14] K. Nikodem, Midpoint convex functions majorized by midpoint concave functions, *Aequationes Math.* 32 (1) (1987) 45–51.
- [15] K. Nikodem, On some class of midconvex functions, *Ann. Polon. Math.* 50 (2) (1989) 145–151.
- [16] Zs. Páles, On approximately convex functions, *Proc. Amer. Math. Soc.* 131 (1) (2003) 243–252.
- [17] T. Popoviciu, *Les Fonctions Convexes*, Hermann et Cie, Paris, 1944.
- [18] A.W. Roberts, D.E. Varberg, *Convex Functions*, Pure Appl. Math., vol. 57, Academic Press, New York, London, 1973.
- [19] L. Székelyhidi, *Convolution Type Functional Equations on Topological Abelian Groups*, World Scientific Publishing Co. Inc., Teaneck, NJ, 1991.
- [20] E.M. Wright, An inequality for convex functions, *Amer. Math. Monthly* 61 (1954) 620–622.