On the Integral Cohomology of Bianchi groups

Mehmet Haluk Şengün

Bianchi groups are groups of the form $\mathrm{PSL}_2(\mathcal{O})$ where $\mathcal{O}$ is the ring of integers of an imaginary quadratic field. They form an important class of arithmetic Kleinian groups. In fact, it is well known that any finite covolume arithmetic Kleinian group, after conjugation, is commensurable with a Bianchi group.

In this paper, I will be interested in the cohomology of Bianchi groups with certain $\mathcal{O}$-module coefficients. These cohomology groups are fundamental to the study of Bianchi modular forms, that is, modular forms (for $\mathbf{GL}_2$) over an imaginary quadratic field. Unlike their analogs over totally real fields, i.e. Hilbert modular forms, the arithmetic of Bianchi modular forms is little understood. One of the features that obstruct the application of standard methods is the torsion in the cohomology of Bianchi groups.

The first computations of torsion were performed in 1981 by Elstrodt, Grunewald and Mennicke [EGM] where they computed the abelianizations $\Gamma_0(\mathfrak{p})^{ab}$ ($\simeq H_1(\Gamma_0(\mathfrak{p}), \mathbb{Z})$) of congruence subgroups $\Gamma_0(\mathfrak{p})$ for prime ideals $\mathfrak{p}$ of $\mathbb{Z}[i]$ of residue degree 1 and norm ≤ 400. In the same paper, numerical evidence suggesting a connection, à la Serre [S2], between some of the 2-torsion classes and certain S_3-extension of $\mathbb{Q}(i)$ was exhibited. Later in 1994, Figueiredo [Fi] provided several more examples which suggested the same kind of connection between certain 3-torsion classes and certain $\mathrm{SL}_2(\mathbb{F}_3)$-extensions of the fields $Q(\sqrt{-d})$ with $d = 1, 2, 3$. See also [Se2]. So far, there is no general arithmetic interpretation for torsion classes although it is speculated that there should be one.

In 1983, Grunewald and Schwermer [GS] determined the conjugacy classes of small index subgroups of several Bianchi groups. In particular, they computed the abelianizations of a vast amount of small index (≤ 12) subgroups. Depending on the data they collected and the data in [EGM], they speculated that for any finite index subgroup Γ of a Bianchi group G, if a prime p appears as an exponent for an element of Γ^{ab}, then p should be smaller than half of the index of Γ in G.

In 1988, Taylor in his unpublished thesis [T] proved the existence of p^k-torsion classes in $H_1(\Gamma_1(Np^r), E_{k,\ell}(\mathcal{O}))$ (notation will be explained later) with $k \neq \ell$ under special circumstances. In her unpublished 2000 thesis [P], Priplata numerically investigated the torsion for some Bianchi groups, mostly for coefficient modules $E_{k,\ell}$ where $k \neq \ell$. It is worth remarking that the (co)homology of Bianchi groups is related to cuspidal Bianchi modular forms only in the parallel weight case, that is, in the case $k = \ell$.

In this paper I report on my extensive systematic computations of the integral (co)homology of Bianchi groups. More specifically, I work with the groups $\mathrm{PSL}(\mathcal{O}_d)$ and $\mathrm{PGL}_2(\mathcal{O}_d)$ with $-d = 1, 2, 3, 7, 11$ and also their congruence subgroups. Motivated by the possibility of congruences between cuspidal Bianchi modular forms and torsion classes, and also the arithmetical connections mentioned above, I limit myself to the (co)homology with parallel weights. The data I collected suggest that the torsion in H^1 is always “small” while the torsion of H^2 can be sporadically large. The data also show the speculations of Grunewald and Schwermer mentioned above are false. In another direction, the data support a recent conjecture of Long, Maclahlan and Reid [LMR] on existence of certain families of rational homology spheres. Moreover, motivated by the recent work of Müller [Mü] and Bergeron-Venkatesh [BV], I collect data that is relevant to the asymptotic behaviour of the size of the torsion in the homology. Along with the computational work, I establish some basic properties of the integral cohomology, such as the smallness of the torsion in the first cohomology and in the second cohomology of the boundary and discuss the liftibility issue of Hecke eigenvalue systems.

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Dedication After the first draft of this work was completed, Fritz Grunewald tragically passed away. Over the years, I benefited greatly from his invaluable guidance and generous support. I dedicate this paper to his memory with admiration, gratitude and love.

1. the modules

Given a commutative a ring R, let $E_k(R) \simeq R[x,y]_k$ where the latter is the space of homogeneous degree k polynomials in two variables over R. Note that $\{x^{k-i}y^i : 0 \leq i \leq k\}$ is an R-basis of $E_k(R)$.

For a polynomial $P(X,Y)$ in $E_k(R)$ and a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{M}_2(R)$, the we have the right action

$$\big(P \cdot \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\big)\big(X,Y\big) = P\big(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)\big) = P\big(aX+bY, cX+dY\big).$$

Let $\mathcal{O}$ be the ring of integers of an imaginary quadratic field. Consider the $\mathrm{M}_2(\mathcal{O})$-module

$$E_{k,\ell}(\mathcal{O}) := E_k(\mathcal{O}) \otimes_{\mathcal{O}} \overline{E_\ell(\mathcal{O})}.$$

Here the overline on the second factor is to indicate that action on the second factor is twisted with complex conjugation. Note that we should insist that $k+\ell$ is even so that $-Id$ acts trivially and thus $\mathrm{PSL}(2,\mathcal{O})$ acts on it as well.

It is useful to remark that $E_{k,\ell}(\mathcal{O}) \simeq \mathrm{Sym}^k(\mathcal{O}^2) \otimes_{\mathcal{O}} \overline{\mathrm{Sym}}^\ell(\mathcal{O}^2)$ as $\mathrm{M}_2(\mathcal{O})$-modules where $\mathrm{Sym}^i(\mathcal{O}^2)$ is the ith symmetric power of the standard representation of $\mathrm{M}_2(\mathcal{O})$ on $\mathcal{O}^2$. Here the overline on the second factor means that the action is twisted with complex conjugation.

Let π be a prime element of $\mathcal{O}$ over a rational prime p. Put κ_π for its residue field. We put

$$E_{k,l}(\kappa_\pi) := E_{k,\ell}(\mathcal{O}) \otimes_{\mathcal{O}} \kappa_\pi.$$

If p splits in $\mathcal{O}$, then

$$E_{k,l}(\kappa_\pi) \simeq E_k(\kappa_\pi) \otimes E_\ell(\kappa_{\bar{\pi}}).$$

Thus $\mathrm{PSL}_2(\mathcal{O})$ acts on it by reduction mod π on the first factor and by reduction mod $\bar{\pi}$ on the second. If p is inert in $\mathcal{O}$, then

$$E_{k,l}(\kappa_\pi) \simeq E_k(\kappa_\pi) \otimes E_\ell(\kappa_\pi)^\sigma.$$

Here $\mathrm{PSL}_2(\mathcal{O})$ acts by reduction mod π. The action on the second factor is twisted by the nontrivial automorphism σ of κ_π.

Finally, when p is ramified in $\mathcal{O}$, we have

$$E_{k,l}(\kappa_\pi) \simeq E_k(\kappa_\pi) \otimes E_\ell(\kappa_\pi).$$

Here the action of $\mathrm{PSL}_2(\mathcal{O})$ is via reduction mod π and is the same on both factors.

A result of Brauer and Nesbitt [BN] tells us that in the inert and split cases, the $\mathrm{PSL}_2(\mathcal{O})$-modules $E_{k,\ell}(\kappa_\pi)$ are irreducible only when $0 \leq k, \ell \leq p-1$. In the ramified case, $E_{k,l}(\kappa_\pi)$ is never an irreducible $\mathrm{PSL}_2(\mathcal{O})$-module unless $k = 0 \leq \ell \leq p-1$ or $\ell = 0 \leq k \leq p-1$. For more on the structure of these modules, we refer to reader to [ST].

The following will be used later.

Proposition 1.1. *Let $\mathcal{O}$ be the ring of integers of an imaginary quadratic field. Let $k \geq \ell$ and put $R = \mathcal{O}[\frac{1}{k!}]$. Then there is a perfect pairing*

$$E_{k,\ell}(R) \times E_{k,\ell}(R) \to R.$$

It is well known (see e.g. [Wi] Lemma 2.4) that there is a perfect pairing on $\mathrm{Sym}^n(R^2)$ coming from the determinant pairing on R^2 whenever $n!$ is invertible in the ring R. The proposition follows by taking the product of the two pairings associated to the two factors of $E_{k,\ell}$. For an explicit description of this pairing, see Section 2.4. of [Be]. As a corollary, we see that the modules $E_{k,\ell}(R)$ are self-dual.

2. the cohomology

In this section I will investigate the integral cohomology of Bianchi groups. My treatment is heavily influenced by work of Haberland [H], Hida [Hi], Wang [Wa] and Wiese [Wi].

Let K be an imaginary quadratic field. Let $\mathcal{O}$ be its ring of integers. Let G be the associated Bianchi group. Let Γ be a finite index subgroup of G.

In this paper, we will focus on the $\mathcal{O}$-modules

$$H^i(\Gamma, E_{k,l}(\mathcal{O})), \quad i = 1, 2.$$

It is well known that these are finitely generated $\mathcal{O}$-modules.

Definition 2.1. Let $\pi \in \mathcal{O}$ be a prime element over the rational prime p. Assume that $H^i(\Gamma, E_{k,\ell}(\mathcal{O}))$ has π-torsion, i.e. it contains a non-zero class c such that $\pi \cdot c = 0$. We say that π is a *large torsion* if $k, \ell < p$. Otherwise, we say that π is a *small torsion*.

Proposition 2.2. *Let π be prime element of $\mathcal{O}$ over the rational prime p. Put κ_π for its residue field. Let Γ be a torsion-free finite index subgroup of the Bianchi group G.*

1. *If Γ surjects onto $\mathrm{PSL}_2(\kappa_\pi)$ then $H^1(\Gamma, E_{k,\ell}(\mathcal{O}))$ has no large π-torsion.*
2. *If π is ramified in $\mathcal{O}$, then $H^1(\Gamma, E_{k,\ell}(\mathcal{O}))$ has π-torsion unless $k = 0 \leq \ell \leq p-1$ or $\ell = 0 \leq k \leq p-1$.*
3. *The obstruction to the lifting of a class in $H^1(\Gamma, E_{k,\ell}(\kappa_\pi))$ to $H^1(\Gamma, E_{k,\ell}(\mathcal{O}))$ is the π-torsion in $H^2(\Gamma, E_{k,\ell}(\mathcal{O}))$.*
4. *$H^2(\Gamma, E_{k,\ell}(\mathcal{O})) \otimes \kappa_\pi \simeq H^2(\Gamma, E_{k,\ell}(\kappa_\pi))$ for every $k, \ell \geq 0$.*

Proof. In the following, let us put $E = E_{k,\ell}$. Consider the following short exact sequence

$$0 \to E(\mathcal{O}) \xrightarrow{\cdot\pi} E(\mathcal{O}) \to E(\kappa_\pi) \to 0.$$

where $\cdot\pi$ is the multiplication by π map.

The associated long exact sequence gives the following short exact sequence

$$0 \to H^i(\Gamma, E(\mathcal{O})) \otimes \kappa_\pi \to H^i(\Gamma, E(\kappa_\pi)) \to H^{i+1}(\Gamma, E(\mathcal{O}))[\pi] \to 0$$

for $i \geq 0$. Here $H^j(\Gamma, E(\mathcal{O}))[\pi]$ denotes the kernel of the map induced by $\cdot\pi$.
Putting $i = 0$, we get

$$E(\kappa_\pi)^\Gamma \simeq H^1(\Gamma, E(\mathcal{O}))[\pi]$$

Now (a) and (b) follow via the irreducibility discussions of the previous section. For $i = 1$ we get

$$0 \to H^1(\Gamma, E(\mathcal{O})) \otimes \kappa_\pi \to H^1(\Gamma, E(\kappa_\pi)) \to H^2(\Gamma, E(\mathcal{O}))[\pi] \to 0$$

which explains the claim (c). It is known that the virtual cohomological dimension of a Bianchi group is 2. Setting $i = 2$, we get

$$H^2(\Gamma, E(\mathcal{O})) \otimes \kappa_\pi \simeq H^2(\Gamma, E(\kappa_\pi))$$

finishing the proof. □

Remark 2.3. The cohomology spaces come equipped with a commuting family of Hecke operators acting on them, see [ST]. Using a lifting theorem of Ash and Stevens ([AS], Prop.1.2.2), we can lift Hecke eigenvalue systems living on $H^2(\Gamma, E(\kappa_\pi))$ to $H^2(\Gamma, E(\mathcal{O}_\pi))$ where $\mathcal{O}_\pi$ is the completion of $\mathcal{O}$ at π. The possible p-torsion in $H^2(\Gamma, E(\mathcal{O}))$ obstructs us from lifting eigenvalue systems living on $H^1(\Gamma, E(\kappa_\pi))$ using the lifting theorem.

2.1. cuspidal cohomology

There is a subspace of the cohomology that is of special interest due to the fact that it can be identified with cuspidal Bianchi modular forms.

Let $K = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field of class number h_K with ring of integers $\mathcal{O} = \mathcal{O}_d$. Let $\mathbb{P}$ denote the projective line over K and $G = \mathrm{PSL}_2(\mathcal{O})$. The group $\mathrm{PSL}_2(K)$ acts naturally on K^2 and thus on $\mathbb{P}$. It is well-known that the cardinality $|\mathbb{P}/G|$ of the set $\mathbb{P}/G$ of G-orbits of $\mathbb{P}$ is equal to h_K. Hence $|\mathbb{P}/\Gamma|$ is finite for any finite index subgroup Γ of G. We will call the elements of $\mathbb{P}/\Gamma$ the *cusps* of Γ.

For every $D \in \mathbb{P}$, let B_D be the Borel subgroup of G defined by the (setwise) stabilizer of D in G. Then the pointwise stabilizer of D in G is the unipotent radical U_D of the Borel subgroup B_D. Let Γ be a torsion-free finite index subgroup of G and D_c be a representative for a cusp c of Γ. Define

$$\Gamma_c := B_{D_c} \cap \Gamma.$$

If Γ_c is torsion-free, this is automatic if Γ is itself torsion-free or $-d \neq 1, 3$, then $\Gamma_c = U_{D_c} \cap \Gamma$ and Γ_c is free abelian of rank two (see [S1] p.507). The group

$$U(\Gamma) := \bigoplus_{c \in \mathbb{P}/\Gamma} \Gamma_c$$

is independent of the choice of representatives taken for the cusps of Γ.

Let E be a Γ-module. Consider the long exact sequence of relative group cohomology for the pair $(\Gamma, U(\Gamma))$

$$\ldots \to H_c^{i-1}(\Gamma, E) \to H^i(\Gamma, E) \to H^i(U(\Gamma), E) \to \ldots$$

where $H_c^n(\Gamma, E) := H^n(\Gamma; U(\Gamma), E)$ and the third arrow is given by the restriction maps.

Definition 2.4. The *cuspidal cohomology* $H^i_{cusp}(\Gamma, E)$ is defined as the image of the cohomology with compact support in $H^i(\Gamma, E)$, or equivalently as the kernel of the restriction map $H^i(\Gamma, E) \to H^i(U(\Gamma), E)$.

Proposition 2.5. *Let Γ be a finite index subgroup of the Bianchi group $\mathrm{PSL}_2(\mathcal{O}_d)$. Assume either that Γ is torsion-free or that $-d \neq 1, 3$. Then $H^2(U(\Gamma), E_{k,\ell}(\mathcal{O}))$ has no large torsion.*

Proof. It is enough to prove the claim for a single cusp c of Γ, that is, for $H^2(\Gamma_c, E)$. So fix a cusp c and Γ_c. Let $E = E_{k,\ell}$ and $t = \max\{k, \ell\}$. Put $R = \mathcal{O}[\frac{1}{t!}]$. Composition of the cup product and the perfect pairing of Proposition 1.1 gives us a pairing

$$\begin{array}{ccc} H^0(\Gamma_c, E(R)) \times H^2(\Gamma_c, E(R)) & \xrightarrow{\cup} & H^2(\Gamma_c, E(R) \otimes_R E(R)) \\ & & \big\downarrow {\scriptstyle (\cdot,\cdot)} \\ & & H^2(\Gamma_c, R) \simeq R. \end{array}$$

That $H^2(\Gamma_c, R) \simeq R$ can be shown as follows. Recall that Γ_c is free abelian with two generators, say a, u. It is known, see [M] p.188, that the tensor product of the two resolutions

$$0 \longrightarrow R[\langle a\rangle] \xrightarrow{1-a} R[\langle a\rangle] \xrightarrow{\varepsilon} R \longrightarrow 0,$$

$$0 \longrightarrow R[\langle u\rangle] \xrightarrow{1-u} R[\langle u\rangle] \xrightarrow{\varepsilon} R \longrightarrow 0,$$

where ε is the usual augmentation map, gives a resolution of Γ_c. One sees from this resolution that the second cohomology of Γ_c with any R-module (right) M can be described as

$$H^2(\Gamma_c, M) \simeq M/\left(M(1-a) + M(1-u)\right).$$

In the case of trivial module R, it follows immediately that $H^2(\Gamma_c, R) \simeq R$.

The above pairing gives that $H^2(\Gamma_c, E(R)) \simeq H^0(\Gamma_c, E(R))^\vee$. Clearly $H^0(\Gamma_c, E(R)) \simeq E(R)^{\Gamma_c}$ is torsion-free. This implies that its dual and hence $H^2(\Gamma_c, E(R))$ is torsion-free. The claim that there can only be small torsion in $H^2(\Gamma_c, E(\mathcal{O}))$ now follows as $R = \mathcal{O}[\frac{1}{t!}]$. □

Proposition 2.6. *Let G be the Bianchi group $\mathrm{PSL}_2(\mathcal{O}_d)$. Let π be prime element of $\mathcal{O}$ over the rational prime p and put κ_π for its residue field. Let Γ be a torsion-free finite index subgroup of G. Then*

$$H^2_{cusp}(\Gamma, E_{k,\ell}(\mathcal{O})) \otimes \kappa_\pi \simeq H^2_{cusp}(\Gamma, E_{k,\ell}(\kappa_\pi))$$

for every $k, \ell < p$.

Proof. Put $E = E_{k,\ell}(\mathcal{O})$. Now consider the commutative diagram

$$\begin{array}{ccccccc}
 & H^2(\Gamma, E) & \xrightarrow{\cdot\pi} & H^2(\Gamma, E) & \longrightarrow & H^2(\Gamma, E(\kappa_\pi)) & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & H^2(U(\Gamma), E) & \xrightarrow{\cdot\pi} & H^2(U(\Gamma), E) & \longrightarrow & H^2(U(\Gamma), E(\kappa_\pi)) &
\end{array}$$

Here the vertical maps are given by the usual restriction maps.

The horizontal lines are exact. The exactness of the first line comes from Proposition 2.2 part (d). The exactness of the second line amounts to Proposition 2.5.

Observe that the cokernel of the restriction map $H^2(\Gamma, E) \to H^2(U(\Gamma), E)$ is isomorphic to $H^3_{cusp}(\Gamma, E) \subset H^3(\Gamma, E)$. Since the virtual cohomological dimension of a Bianchi group is two and Γ is torsion-free, we have $H^3(\Gamma, E) = 0$. Now the claim follows by the Snake Lemma. □

Remark 2.7. If F is a field in which 6 is invertible, then it is known that

$$H^1_{cusp}(\Gamma, E(F))^\vee \simeq H^2_{cusp}(\Gamma, E(F)^\vee)$$

as Hecke modules where $-^\vee$ denotes the dual, see [AS] Lemma 1.4.3. I want to continue our discussion of lifting Hecke eigenvalue systems in Remark 2.3.

Fix an imaginary quadratic field K with ring of integers $\mathcal{O}$ and a prime element $\pi \in \mathcal{O}$ over the rational prime p. Let us give ourselves a torsion-free finite index subgroup Γ of $\mathrm{PSL}_2(\mathcal{O})$. Say we have a Hecke eigenvalue system Φ living in $H^1_{cusp}(\Gamma, E_{k,\ell}(\kappa_\pi))$ which does not lift to $H^1_{cusp}(\Gamma, E_{k,\ell}(\mathcal{O}_\pi))$. Here $\mathcal{O}_\pi$ is the ring of integers of the completion K_π of K at the prime π. If $p > 3$, then by the duality result I mentioned, we deduce that $\Phi^\vee$ lives in $H^2_{cusp}(\Gamma, E_{k,\ell}(\kappa_\pi))$. Note that our coefficient modules are self-dual. Furthermore, if $k, \ell < p$, then using Proposition 2.6 and the lifting theorem of Ash and Stevens ([AS], Prop.1.2.2), we infer that there is a Hecke eigenvalue system Ψ living in $H^2_{cusp}(\Gamma, E_{k,\ell}(\mathcal{O}_\pi))$ lifting $\Phi^\vee$.

The conclusion now is that the Hecke eigenvalue system Ψ is realized by a *torsion* Hecke eigenclass $c \in H^2_{cusp}(\Gamma, E_{k,\ell}(\mathcal{O}_\pi))$. Because if not, then we can realize Ψ in $H^2_{cusp}(\Gamma, E_{k,\ell}(K_\pi))$ and using the duality again, we infer the existence of the integral system $\Psi^\vee$ living in $H^1_{cusp}(\Gamma, E_{k,\ell}(K_\pi))$. Clearly $\Psi^\vee$ is a lift of Φ and this would contradict our starting assumption of non-liftibility.

3. the first cohomology

I will now describe a method, first observed by Fox [F], that allows us to compute H^1 of any finitely presented group with coefficients in a finite dimensional module. It is well known that Bianchi groups are finitely presented. Presentations for many Bianchi groups are in the literature, see, for example, [FGT].

Let me illustrate the method through an example. A formal exposition is contained in [FGT]. Let $w = \sqrt{-2}$ and $G = \mathrm{PSL}_2(\mathbb{Z}[w])$. It is known that

$$G =< A, B, U \mid B^2 = (AB)^3 = [A, U] = (BU^2BU^{-1})^2 = 1 >$$

where A, B, U can be realized as $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & w \\ 0 & 1 \end{smallmatrix}\right)$ respectively.

Let E be any G-module. Given any cocycle $f : G \to E$, any value $f(X)$ can be expressed linearly in terms of the images $f(A), f(B), f(U)$ of the generators of G, e.g.

$$f(ABU) = f(A) \cdot BU + f(B) \cdot U + f(U).$$

Moreover, $f(A), f(B), f(U)$ satisfy the linear equations coming from the relations of the presentation. For example,

$$B^2 = 1 \Longrightarrow f(B)(B + 1) = 0.$$

Conversely, any pair $(x, y, z) \in E^3$ satisfying the linear equations coming the presentation gives uniquely a cocycle. Thus the space of cocycles can be seen as the kernel of the matrix corresponding to this linear system. One gets the coboundaries similarly and hence computes $H^1(G, E)$ as the quotient of the two spaces.

Note that to compute with a finite index subgroup Γ of G, it is not practical to apply the method to a presentation of Γ (which can be derived from that of G once the coset representatives are known). It is best to use Shapiro's lemma and compute $H^1(G, \mathrm{Coind}_\Gamma^G(E))$.

3.1. data on the integral first cohomology

I have implemented the above algorithm in MAGMA for the five Euclidean imaginary quadratic fields $K = \mathbb{Q}(\sqrt{-d})$ with $d = 1, 2, 3, 7, 11$. In the following, let $\mathcal{O}_d$ denote the corresponding ring of integers.

By the theory of modules over principal ideal domains, we know that our $\mathcal{O}$-module $H^1(\Gamma, E(\mathcal{O}))$ has a decomposition

$$H^1(\Gamma, E(\mathcal{O})) \simeq \mathcal{O}/(a_1) \oplus \ldots \oplus \mathcal{O}/(a_m) \oplus \mathcal{O}^r.$$

with $a_i \neq 0, 1$ and $a_i | a_{i+1}$. The a_i are called elementary divisors and are unique up to multiplication by units. The exponent r is called the rank.

Below I report on my computations. I have exclusively worked with parallel weights $E_{n,n}$ as only these weights carry information about the cuspidal Bianchi modular forms.

Observe that the torsion is always "small" as stated by the proposition above. The only exception to this is the ramifying prime which always appears in the torsion. We show the rank in a seperate column as it provides a means to check our work against the dimension computations of [FGT].

data for $H^1(\mathrm{PSL}_2(\mathcal{O}_1), E_{n,n}(\mathcal{O}_1))$

n	norms of elt. divisors	primes	rank
0			0
1	[4]	(2)	1
2	[2, 16]	(2)	0
3	[2, 2, 4]	(2)	1
4	[2, 2, 2, 8, 1152]	(2,3)	0
5	[2, 2, 2, 2, 4, 4]	(2)	2
6	[2, 2, 2, 2, 2, 2, 8, 8, 800]	(2,5)	0
7	[2, 2, 2, 2, 2, 2, 2, 2, 4, 4, 4]	(2)	3
8	[2, 2, 2, 2, 2, 2, 2, 2, 2, 4, 8, 8, 32, 225792]	(2,3,7)	0
9	[2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 4, 4, 4, 4, 4]	(2)	3
10	[2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 8, 8, 8, 8, 16, 288]	(2,3)	1

data for $H^1(\mathrm{PSL}_2(\mathcal{O}_2), E_{n,n}(\mathcal{O}_2))$

n	norms of elt. divisors	primes	rank
1	[8]	(2)	1
2	[2, 32]	(2)	1
3	[2, 2, 8]	(2)	2
4	[2, 2, 2, 8, 1152]	(2,3)	1
5	[2, 2, 2, 2, 8, 8]	(2)	3
6	[2, 2, 2, 2, 2, 2, 8, 8, 7200]	(2,3,5)	2
7	[2, 2, 2, 2, 2, 2, 2, 2, 8, 8, 8]	(2)	4
8	[2, 2, 2, 2, 2, 2, 2, 2, 2, 4, 8, 8, 32, 225792]	(2,3,7)	2
9	[2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 8, 8, 8, 8, 8]	(2)	5
10	[2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 8, 8, 8, 8, 32, 288]	(2,3)	3

data for $H^1(\mathrm{PSL}_2(\mathcal{O}_3), E_{n,n}(\mathcal{O}_3))$

n	norms of elt. divisors	primes	rank
0			0
1	[3]	(3)	0
2	[3]	(3)	1
3	[3, 108]	(2,3)	0
4	[3, 3, 12]	(2,3)	0
5	[3, 3, 12]	(2,3)	1
6	[3, 3, 3, 3, 10800]	(2,3,5)	1
7	[3, 3, 3, 3, 3, 12]	(2,3)	1
8	[3, 3, 3, 3, 3, 12, 2352]	(2,3,7)	1
9	[3, 3, 3, 3, 3, 3, 3, 108, 972]	(2,3)	1

data for $H^1(\mathrm{PSL}_2(\mathcal{O}_7), E_{n,n}(\mathcal{O}_7))$

n	norms of elt. divisors	primes	rank
0			1
1	[7]	(7)	1
2	[28]	(2,7)	1
3	[28]	(2,7)	1
4	[1008]	(2,3,7)	2
5	[28]	(2,7)	2
6	[4, 4, 4, 2800]	(2,5,7)	2
7	[1372]	(2,7)	3
8	[4, 4, 28, 40320]	(2,3,7)	3
9	[4, 4, 28, 28]	(2,7)	3
10	[4, 4, 28, 1008]	(2,3,7)	4
11	[4, 4, 28, 28]	(2,7)	4

data for $H^1(\mathrm{PSL}_2(\mathcal{O}_{11}), E_{n,n}(\mathcal{O}_{11}))$

n	norms of elt. divisors	primes	rank
0			1
1	[11]	(11)	1
2	[11]	(11)	2
3	[44]	(2,11)	2
4	[396]	(2,3,11)	2
5	[44]	(2,11)	3
6	[39600]	(2,3,5,11)	4
7	[44]	(2,11)	4
8	[4, 77616]	(2,3,7,11)	4
9	[4, 44]	(2,11)	5
10	[4, 4, 1584]	(2,3,11)	8
11	[4, 5324]	(2,11)	6

4. the second cohomology

The main method I employ for computing the second cohomology is based on reduction theory as used in [SV]. The cohomological dimension of Bianchi groups is 2 and the symmetric space they act on, namely the hyperbolic 3-space $\mathbb{H} \simeq \mathbb{C} \times \mathbb{R}^+$, is 3 dimensional. Reduction theory gives us a contractible 2 dimensional CW-complex inside $\mathbb{H}$ which is a deformation retract for the action of the Bianchi group. Moreover, the cellular action of the Bianchi group on the CW-complex is cocompact. This makes the CW-complex a suitable tool for cohomological computations.

I will continue to focus on the Euclidean imaginary quadratic fields. The reduction theory for Bianchi groups has been worked out for these fields by Mendoza [Me] and Flöge [Fl]. See also [BLW, FR].

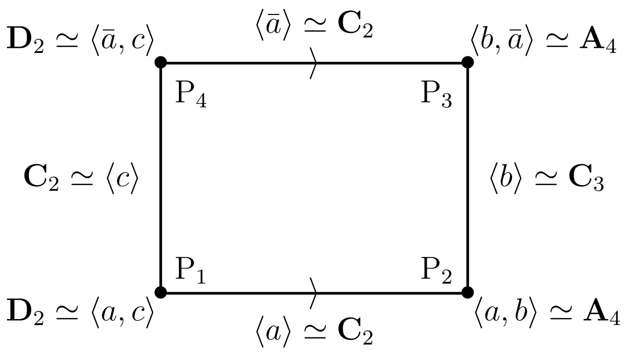
For an overview of Mendoza's construction, I refer readers to [SV]. I will exhibit the method for the case of the Bianchi group $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[w])$ with $w = \sqrt{-2}$.

Let $\mathcal{C}$ be the 2-dimensional CW-complex constructed by Mendoza for Γ. Then a fundamental cellular domain $\mathcal{F}$ for the action of Γ on $\mathcal{C}$ is given by the area on the unit hemisphere centered at the origin of $\mathbb{H}$ above the rectangle in $\mathbb{C}\times\{0\}$ with vertices $(\pm\frac{w}{2}, 0)$ and $(\frac{1}{2}\pm\frac{w}{2}, 0)$.

Let

$$a := \left(\begin{smallmatrix} 1 & w \\ w & -1 \end{smallmatrix}\right), \quad b := \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right), \qquad c := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right).$$

The stabilizers of the edges (1-cells) and the vertices (0-cells) of $\mathcal{F}$ are shown in the following picture



The horizontal edges are identified by the element $g = \left(\begin{smallmatrix} 1 & w \\ 0 & 1 \end{smallmatrix}\right)$, that is $gP_1P_2 = P_4P_3$. Thus the quotient by Γ is a cylinder. Moreover, the stabilizer of the whole rectangle (2-cell) is trivial.

Next one feeds this data into the equivariant cohomology spectral sequence

$$E_1^{p,q}(M) = \bigoplus_{\sigma\in\Sigma_p} H^q(\Gamma_\sigma, M) \Longrightarrow H^{p+q}(\Gamma, M).$$

where M is any $\mathbb{Z}\Gamma$-module and Σ_p is a set of representatives of all the Γ-orbits of the p-cells of $\mathcal{C}$. See page 164 of [B] for a description. The homological version of this spectral sequence has been used in [SV], [FR].

Let Γ_i, Γ_{ij} stand for the stabilizers of the vertex P_i and the edge between P_i and P_j respectively. Let M be a right Γ-module over $\mathbb{Z}[w][\frac{1}{6}]$. As primes above 2 and 3 are inverted, the cohomology of the (finite) stabilizers vanish in degree greater than 0. Hence, we have $E_1^{p,q}(M) = 0$ for all $q > 0$. Therefore, the spectral sequence is concentrated on the horizontal axis $q = 0$ and the cohomology of the cochain complex

$$E_1^{0,0} \xrightarrow{d_1^{0,0}} E_1^{1,0}(M) \xrightarrow{d_1^{1,0}} E_1^{2,0}(M)$$

gives $H^*(\Gamma, M)$, that is

$$H^0(\Gamma, M) = Ker(d_1^{0,0}) \quad H^1(\Gamma, M) = Ker(d_1^{1,0})/Im(d_1^{0,0}), \quad H^2(\Gamma, M) = M/Im(d_1^{1,0})$$

Now with the appropriate substitutions, the cochain complex reads

$$\bigoplus_{vertex\ i} H^0(\Gamma_i, M) \xrightarrow{d_1^{0,0}} \bigoplus_{edge\ ij} H^0(\Gamma_{ij}, M) \xrightarrow{d_1^{1,0}} H^0(\langle Id \rangle, M).$$

Here $\langle Id \rangle$ is the trivial stabilizer of the 2-cell $\mathcal{F}$.

To explicitly compute H^2, it remains to describe the differential $d_1^{1,0}$. One can choose the orientation on $\mathcal{F}$ so that the differential map becomes as follows

$$M^{\Gamma_1} \oplus M^{\Gamma_2} \xrightarrow{d_1^{0,0}} M^{\Gamma_{12}} \oplus M^{\Gamma_{23}} \oplus M^{\Gamma_{34}} \xrightarrow{d_1^{1,0}} M.$$

$$d_1^{1,0}(m_{12}, m_{23}, m_{34}, m_{41}) = m_{12} + m_{23} + m_{34} - m_{12} \cdot g^{-1}$$

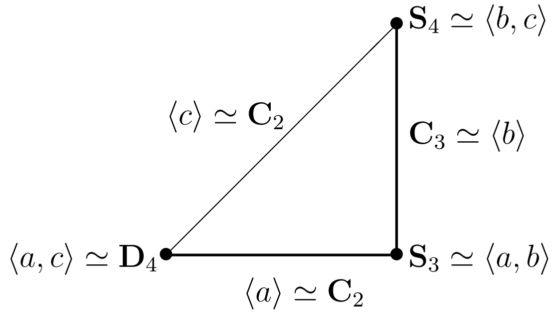
The information on the fundamental 2-cell for the groups $\mathrm{PSL}_2(\mathcal{O}_d)$ with $-d = 1, 2, 3, 7, 11$ is included in the article [SV], so I do not repeat it here. The same information for the groups $\mathrm{PGL}_2(\mathcal{O}_d)$ with $-d = 1, 2, 3, 7, 11$ is not included in that article and one needs to go to the above mentioned thesis of Mendoza (although a few of them are contained in [BLW] as well) which is very hard to access from outside of Germany. So I will now describe the information on these groups in the pictorial form as above.

- $\mathrm{PGL}_2(\mathcal{O}_1)$

Put $w = \sqrt{-1}$. Let

$$a := \left(\begin{smallmatrix} 0 & w \\ w & 0 \end{smallmatrix}\right), \quad b := \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right), \qquad c := \left(\begin{smallmatrix} 0 & 1 \\ w & 0 \end{smallmatrix}\right).$$

The stabilizers of the edges and the vertices of $\mathcal{F}$ are shown in the following picture



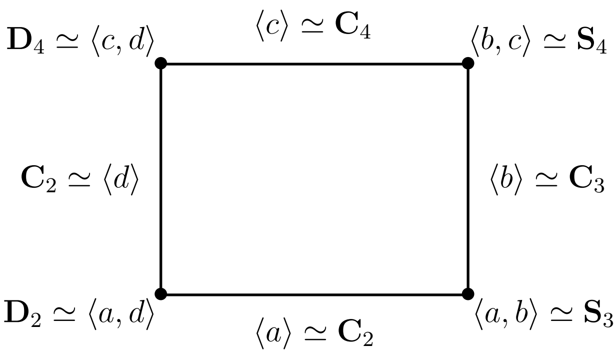
There are no identifications and the stabilizer of the triangle (2-cell) is trivial.

- $\mathrm{PGL}_2(\mathcal{O}_2)$

Put $w = \sqrt{-2}$. Let

$$a := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad b := \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \qquad c := \begin{pmatrix} w & 1 \\ 1 & 0 \end{pmatrix}, \quad d := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The stabilizers of the edges and the vertices of $\mathcal{F}$ are shown in the following picture



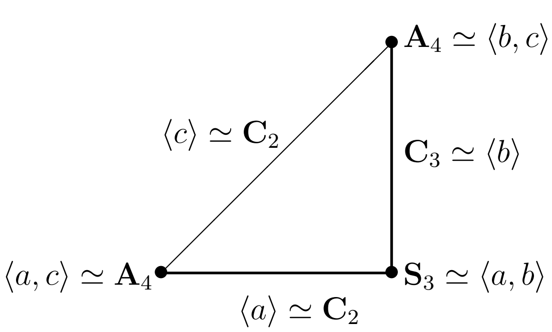
There are no identifications and the stabilizer of the rectangle (2-cell) is trivial.

- $\mathrm{PGL}_2(\mathcal{O}_3)$

Put $w = \frac{1+\sqrt{-3}}{2}$. Let

$$a := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad b := \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \qquad c := \begin{pmatrix} 0 & w \\ 1 & 0 \end{pmatrix}.$$

The stabilizers of the edges and the vertices of $\mathcal{F}$ are shown in the following picture



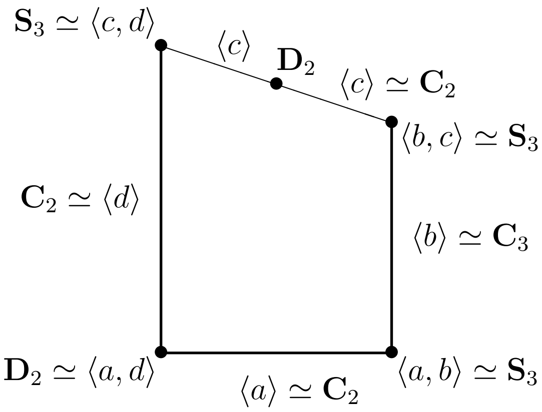
There are no identifications and the stabilizer of the triangle (2-cell) is trivial.

- $\mathrm{PGL}_2(\mathcal{O}_7)$

Put $w = \frac{1+\sqrt{-7}}{2}$. Let

$$a := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \quad b := \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right), \qquad c := \left(\begin{smallmatrix} 1 & -w \\ \bar{w} & -1 \end{smallmatrix}\right), \quad d := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right).$$

The stabilizers of the edges and the vertices of $\mathcal{F}$ are shown in the following picture



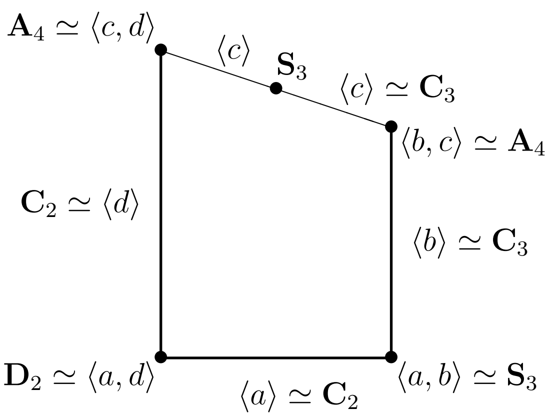
The two adjacent short edges on the top are identified via $g = \left(\begin{smallmatrix} 1 & -w \\ 0 & -1 \end{smallmatrix}\right)$ which fixes the vertex between them. Thus these two edges are oppositely oriented and the stabilizer of the vertex between them is $\mathbf{D}_2 \simeq \langle c, g \rangle$. Again the stabilizer of the whole 2-cell is trivial.

- $\mathrm{PGL}_2(\mathcal{O}_{11})$

Put $w = \frac{1+\sqrt{-11}}{2}$. Let

$$a := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \quad b := \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right), \qquad c := \left(\begin{smallmatrix} 1 & -w \\ \bar{w} & -2 \end{smallmatrix}\right), \quad d := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right).$$

The stabilizers of the edges and the vertices of $\mathcal{F}$ are shown in the following picture



The two adjacent short edges on the top are identified via $g = \left(\begin{smallmatrix} 1 & -w \\ 0 & -1 \end{smallmatrix}\right)$ which fixes the vertex between them. Thus these two edges are oppositely oriented and the stabilizer of the vertex between them is $\mathbf{S}_3 \simeq \langle c, g \rangle$. Again the stabilizer of the whole 2-cell is trivial.

4.1. data on the integral second cohomology: level 1

I have implemented the above algorithm in MAGMA. I have not inverted the primes above 2,3 and thus the computations may not give correct data on 2,3 torsion. Below I give a complete list of the primes that appear in the torsion part of the second cohomology of both PSL and PGL. The large primes are highlighted in bold face. The data imply that if (p) ramifies in $\mathcal{O}$ then there is p-torsion in the integral second cohomology (except the case $k = \ell = 0$) but I have not been able to prove that this is always the case.

$H^2(\mathrm{PSL}_2(\mathcal{O}_1), E_{n,n}(O_1))$

n	primes	rank
1	[ ]	1
2	[2]	1
3	[2, 3]	1
4	[2, 3]	1
5	[2]	2
6	[2, 3 5]	1
7	[2, 3, 7]	3
8	[2, 3, 5, 7]	1
9	[2, 3]	3
10	[2, 3, 5, 7]	2
11	[2, 3, 5, 11]	4
12	[2, 3, 5, 7, 11]	1
13	[2, 3, 5]	5
14	[2, 3, 5, 7, 11, 13]	2
15	[2, 3, 5, 7]	5
16	[2, 3, 5, 7, 11, 13]	2
17	[2, 3, 5, 7]	6
18	[2, 3, 5, 7, 11, 13, 17, **19**, **23**]	2
19	[2, 3, 5, 7, 13, 19]	7
20	[2, 3, 5, 7, 11, 13, 17, 19, **409**, **6997**]	2
21	[2, 3, 5, 7, **59**]	7
22	[2, 3, 5, 7, 11, 13, 17, 19, **13707791**]	3
23	[2, 3, 5, 7, 11, **23**, **113**]	8

$H^2(\mathrm{PGL}_2(\mathcal{O}_1), E_{n,n}(O_1))$

n	primes	rank
1	[ ]	1
2	[2]	1
3	[2, 3]	1
4	[2]	1
5	[2]	2
6	[2, 3, 5]	1
7	[2, 3, 7]	2
8	[2, 3, 5, 7]	1
9	[2, 3]	3
10	[2, 3, 5, 7]	2
11	[2, 3, 11]	3
12	[2, 3, 5, 11]	1
13	[2, 3, 5]	4
14	[2, 3, 5, 7, 11, 13]	2
15	[2, 3, 5, 7]	4
16	[2, 3, 5, 7, 13]	2
17	[2, 3, 5, 7]	5
18	[2, 3, 5, 7, 11, 17]	2
19	[2, 3, 5, 19]	5
20	[2, 3, 5, 7, 13, 17, 19, **409**]	2
21	[2, 3, 5, 7]	6
22	[2, 3, 5, 7, 11, 19]	3
23	[2, 3, 5, 7, 11, 23]	6
24	[2, 3, 5, 7, 11, 13, 17, 23, **1033**]	2
25	[2, 3, 5, 7, 11, 17]	7

$H^2(\mathrm{PSL}_2(\mathcal{O}_2), E_{n,n}(O_2))$

n	primes	rank
1	[ ]	1
2	[2]	1
3	[2, 3]	2
4	[2, 3]	1
5	[2, 3, 5]	3
6	[2, 3, 5]	2
7	[2, 3, 5, 7]	4
8	[2, 3, 5, 7]	2
9	[2, 3, 5, 7, **31**]	5
10	[2, 3, 5, 7]	3
11	[2, 3, 5, 7, 11]	6
12	[2, 3, 5, 7, 11, **37**]	3
13	[2, 3, 5, 7, 11, 13, **547**]	7
14	[2, 3, 5, 7, 11, 13, **439**, **110281**]	4
15	[2, 3, 5, 7, 11, 13, **61**, **163**]	8

$H^2(\mathrm{PGL}_2(\mathcal{O}_2), E_{n,n}(O_2))$

n	primes	rank
1	[ ]	1
2	[2]	1
3	[2]	2
4	[2, 3]	1
5	[2, 5]	3
6	[2, 3, 5]	1
7	[2, 3, 7]	4
8	[2, 3, 5, 7]	1
9	[2, 3]	5
10	[2, 3, 5, 7]	2
11	[2, 3, 5]	6
12	[2, 3, 5, 7, 11, **37**]	1
13	[2, 3, 5, 13]	7
14	[2, 3, 5, 7, 11, 13, **110281**]	2
15	[2, 3, 5, 7]	8
16	[2, 3, 5, 7, 11, 13, **1671337**]	2
17	[2, 3, 5, 7, **103**]	9
18	[2, 3, 5, 7, 11, 13, 17, **3812807473**]	2
19	[2, 3, 5, 7, **907**]	10
20	[2, 3, 5, 7, 11, 13, 17, 19, **3511**, **879556698451244053**]	2

$H^2(\mathrm{PSL}_2(\mathcal{O}_3), E_{n,n}(O_3))$

n	primes	rank
1	[ ]	1
2	[3]	1
3	[3]	1
4	[2, 3]	1
5	[2, 3, 5]	1
6	[2, 3, 5]	2
7	[2, 3, 5]	2
8	[2, 3, 7]	1
9	[2, 3, 7]	2
10	[2, 3, 5]	3
11	[2, 3, 5, 11]	2
12	[2, 3, 5, 7, 11]	2
13	[2, 3, 5, 11]	3
14	[2, 3, 5, 7, 13]	3
15	[2, 3, 5, 7, 13]	3
16	[2, 3, 5, 7, 11]	3
17	[2, 3, 5, 7, 17]	3
18	[2, 3, 5, 7, 13, 17]	4
19	[2, 3, 5, 7, 11, 17, **61**]	4
20	[2, 3, 5, 7, 19]	3
21	[2, 3, 5, 7, 13, 19, **151**]	4
22	[2, 3, 5, 7, 11, 17]	5
23	[2, 3, 5, 7, 11, 23, **103**]	4
24	[2, 3, 5, 7, 11, 13, 17, 19, 23, **53**]	4
25	[2, 3, 5, 7, 11, 17, 23, **29**, **947**]	5
26	[2, 3, 5, 7, 11, 13]	5

$H^2(\mathrm{PGL}_2(\mathcal{O}_3), E_{n,n}(O_3))$

n	primes	rank
1	[ ]	1
2	[ ]	1
3	[2, 3]	1
4	[2, 3]	1
5	[2, 3, 5]	1
6	[2, 3, 5]	1
7	[2, 3]	2
8	[2, 3, 7]	1
9	[2, 3]	2
10	[2, 3, 5]	2
11	[2, 3, 5, 11]	2
12	[2, 3, 5, 7, 11]	1
13	[2, 3]	3
14	[2, 3, 5, 7, 13]	2
15	[2, 3, 5]	3
16	[2, 3, 5, 7, 11]	2
17	[2, 3, 5, 17]	3
18	[2, 3, 5, 7, 13, 17]	2
19	[2, 3, 5, 7]	4
20	[2, 3, 5, 7, 19]	2
21	[2, 3, 5, 7]	4
22	[2, 3, 5, 7, 11, 17]	3
23	[2, 3, 5, 11, 23]	4
24	[2, 3, 5, 7, 11, 13, 19, 23, **53**]	2
25	[2, 3, 5, 7]	5

$H^2(\mathrm{PSL}_2(\mathcal{O}_7), E_{n,n}(O_7))$

n	primes	rank
1	[ ]	1
2	[2, 7]	1
3	[2, 3, 7]	1
4	[2, 3, 7]	2
5	[2, 3, 5, 7]	2
6	[2, 3, 5, 7]	2
7	[2, 3, 5, 7]	3
8	[2, 3, 5, 7]	3
9	[2, 3, 5, 7]	3
10	[2, 3, 5, 7]	4
11	[2, 3, 5, 7, 11]	4
12	[2, 3, 5, 7, 11, **127**]	6
13	[2, 3, 5, 7, 11, 13, **31**]	5
14	[2, 3, 5, 7, 11, 13, **73**]	5
15	[2, 3, 5, 7, 11, 13, **271**, **431**]	5
16	[2, 3, 5, 7, 11, 13]	6
17	[2, 3, 5, 7, 11, 13, 17, **37**, **67**, **89**, **101**, **277**]	6
18	[2, 3, 5, 7, 11, 13, 17, **43**, **457**, **2069**, **3323**]	6

$H^2(\mathrm{PGL}_2(\mathcal{O}_7), E_{n,n}(O_7))$

n	primes	rank
1	[ ]	1
2	[2]	1
3	[2, 3, 7]	1
4	[2, 3, 7]	1
5	[2, 5, 7]	2
6	[2, 3, 5, 7]	1
7	[2, 3, 7]	3
8	[2, 3, 5, 7]	1
9	[2, 3, 7]	3
10	[2, 3, 5, 7]	2
11	[2, 3, 5, 7]	4
12	[2, 3, 5, 7, 11, **127**]	1
13	[2, 3, 5, 7, 13]	5
14	[2, 3, 5, 7, 11, 13, **73**]	2
15	[2, 3, 5, 7, **431**]	5
16	[2, 3, 5, 7, 11, 13]	2
17	[2, 3, 5, 7, 17, **37**]	6
18	[2, 3, 5, 7, 11, 13, 17, **43**, **457**, **2069**, **3323**]	2
19	[2, 3, 5, 7, 13, 19, **311**]	7
20	[2, 3, 5, 7, 11, 13, 17, 19, **42197**, **12272815271**]	2

$H^2(\mathrm{PSL}_2(\mathcal{O}_{11}), E_{n,n}(O_{11}))$

n	primes	rank
1	[ ]	1
2	[2]	2
3	[2, 3, 11]	2
4	[2, 3, 11]	2
5	[2, 3, 5, 11]	3
6	[2, 3, 5, 11]	4
7	[2, 3, 5, 7, 11]	4
8	[2, 3, 5, 7, 11]	4
9	[2, 3, 5, 7, 11, **23**]	5
10	[2, 3, 5, 7, 11]	8
11	[2, 3, 5, 7, 11, **37**]	6
12	[2, 3, 5, 7, 11]	6
13	[2, 3, 5, 7, 11, 13, **43**, **19973**]	7
14	[2, 3, 5, 7, 11, 13]	8
15	[2, 3, 5, 7, 11, 13, **31**, **47**, **1409**, **30817**]	8
16	[2, 3, 5, 7, 11, 13, 17, 19, **41**, **281**]	8

$H^2(\mathrm{PGL}_2(\mathcal{O}_{11}), E_{n,n}(O_{11}))$

n	primes	rank
1	[ ]	1
2	[2]	1
3	[2]	2
4	[2, 3, 11]	1
5	[2, 11]	3
6	[2, 3, 5, 11]	1
7	[2, 3, 7, 11]	4
8	[2, 3, 5, 7, 11]	1
9	[2, 3, 11]	5
10	[2, 3, 5, 7, 11]	4
11	[2, 3, 5, 11]	6
12	[2, 3, 5, 7, 11]	1
13	[2, 3, 5, 11, 13]	7
14	[2, 3, 5, 7, 11, 13]	2
15	[2, 3, 5, 7, 11, **47**]	8
16	[2, 3, 5, 7, 11, 13, **41**, **281**]	2
17	[2, 3, 5, 7, 11, 17, **67**]	9
18	[2, 3, 5, 7, 11, 13, 17, **449**, **20147**, **201797**]	2

4.2. congruence subgroups

Now let us focus on the second cohomology of congruence subgroups. For computational considerations, we will focus on the subgroups of the type

$$\Gamma_0(\mathfrak{a}) : \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G : c \equiv 0 \mod \mathfrak{a} \right\}.$$

Here G is either $\mathrm{PSL}_2(\mathcal{O}_d)$ or $\mathrm{PGL}_2(\mathcal{O}_d)$ and $\mathfrak{a}$ is an ideal of $\mathcal{O}_d$, which is called the level.

4.2..1 trivial weight: torsion

To compute the second cohomology with trivial weight, the approach employed in [EGM] is more efficient than the reduction theory approach used above. The idea is to compute the abelianization of the congruence subgroup Γ using a (finite) presentation for the Bianchi group G and the knowledge of the permutation action of the generators of G on a set of coset representatives of Γ in G.

The relationship between first homology and the second cohomology is given by the Lefschetz duality. Let R be any module in which 6 is invertible. Then for any $R[\Gamma]-$module E, we have

$$H_1(\Gamma, E) \simeq H^2_c(\Gamma, E)$$

where the right hand side is the cohomology with compact support, see Section 2.

We are only interested in the case of the trivial coefficients $R = \mathbb{Z}[1/6]$. We need to study the exact sequence

$$H^1(\Gamma, R) \to H^1(U(\Gamma), R) \to H^2_c(\Gamma, R) \to H^2_{cusp}(\Gamma, R) \to 0$$

Assume that K has class number one and that $\Gamma = \Gamma_0(\mathfrak{p})$ where $\mathfrak{p}$ is a prime ideal of residue degree one. It is easy to see that Γ has only two cusps; $\{0, \infty\}$. That is $|\Gamma\backslash\mathbb{P}| = 2$ and the two classes are represented by the elements $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. If $-d \neq 1, 3$, then the stabilizers of the cusps are free abelian of rank 2 and by Corollaire 2 of [S1] p.517, the image of $H^1(\Gamma, R)$ has rank 2 in $H^1(U(\Gamma), R)$ which has rank 4. For $-d = 1, 3$, the situation is complicated by the existence of torsion in the stabilizers of cusps. In this case, let $\Gamma_c^+ := \Gamma \cap U_{D_c}$ for each cusp c of Γ in the terminology of Section 2.1. Then $\Gamma_c/\Gamma_c^+ \simeq \mu := \langle \left(\begin{smallmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{smallmatrix}\right) \rangle$ where ε is a generator of the roots of unity in $\mathcal{O}_d$. The inflation-restriction sequence then gives

$$H^1(\Gamma_c, R) \simeq H^0(\mu, H^1(\Gamma_c^+, R))$$

Now direct computation shows that the latter is trivial.

I wrote programs to compute $\Gamma_0(\mathfrak{p})^{ab}$ in MAGMA for the Euclidean $\mathcal{O}_d$, that is, for $-d = 1, 2, 3, 7, 11$. In a couple of minutes the program computed up to primes of norm 5000. It is easily seen that the torsion get very large very quickly compared to the norm of the level of the congruence subgroup. Recall from the introduction that Grunewald and Schwermer speculated in [GS] that no p-torsion appearing in abelianization of a finite index subgroup Γ is greater then half of the index of Γ inside the Bianchi group. In the case of $\Gamma_0(\mathfrak{p})$, their speculation says that any p-torsion that appear in $\Gamma_0(\mathfrak{p})^{ab}$ should satisfy

$$p \leq \frac{\mathbf{N}\mathfrak{p} + 1}{2}$$

where $\mathbf{N}\mathfrak{p}$ is the norm of the prime ideal $\mathfrak{p}$.

The smallest primes that provide counter-examples to this speculation for our five Bianchi groups $\mathrm{PSL}_2(\mathcal{O}_d)$ are listed below in Table 1.

d	Norm of $\mathfrak{p}$	rank of $\Gamma_0(\mathfrak{p})^{ab}$	prime torsion of $\Gamma_0(\mathfrak{p})^{ab}$
1	401	0	[2, 5, 41, 271]
2	193	2	[2, 3, 23, 251]
3	937	0	[2, 3, 13, 599]
7	137	2	[2, 17, 83]
11	103	2	[2, 3, 17, 19, 71]

Table 1: smallest counter examples to Grunewald-Schwermer

My computations of $\Gamma_0(\mathfrak{p})^{ab}$ agree perfectly with that of [EGM] mentioned in the introduction. It is easily observed that the primes in the torsion get to astronomical sizes even within the range $\mathbf{N}\mathfrak{p} \leq 5000$. Below in Table 2 is a sample of the primes that appear in the torsion of $\Gamma_0(\mathfrak{p})^{ab}$ with $4900 \leq \mathbf{N}\mathfrak{p} \leq 5000$ for the five Euclidean imaginary quadratic fields. The complete list (where I computed up to much bigger norms) is available on my website. It is interesting to observe that in the cases $-d = 1, 3$, where we have nontrivial units, the torsion grows much slower than the other three cases.

4.2..2 trivial weight: rank

In this section I will report on the rank of $H^2_{cusp}(\Gamma_0(\mathfrak{a}), \mathcal{O})$. This rank is clearly equal to the dimension of $H^2_{cusp}(\Gamma_0(\mathfrak{a}), \mathbb{C})$ and thus its nonvanishing is conjecturally connected to abelian varieties of GL_2−type over imaginary quadratic fields (see [Cr, EGM, Se1]). Moreover, the vanishing of this rank in certain cases is equivalent to the existence of rational homology spheres (see [LMR]). In this section I will report on my computations related to these two aspects.

As explained in the previous subsection, the rank of $\Gamma_0(\mathfrak{a})^{ab}$ is related to the rank r of $H^2_{cusp}(\Gamma_0(\mathfrak{a}), \mathcal{O})$. More precisely, when $\mathfrak{p}$ is a prime ideal of residue degree one, our discussion above shows that

$$r = \mathrm{rank}(\Gamma_0(\mathfrak{p})^{ab}), \quad \text{for} \quad -d = 1, 3$$

and

$$r + 2 = \mathrm{rank}(\Gamma_0(\mathfrak{p})^{ab}), \quad \text{for} \quad -d \neq 1, 3$$

I have computed the rank of $\Gamma_0(\mathfrak{p})^{ab}$ for prime ideals $\mathfrak{p}$ of residue degree one and norm about ≤ 35000 for $-d = 1, 3$. I was only able to go up to norms about 15000 for $-d = 2, 7, 11$. I report on the distribution of prime levels according to the ranks in Table 3 where I use $N(r, x)$ to denote the number of primes of residue degree one with norm $< x$ and such that $H^2_{cusp}(\Gamma_0(\mathfrak{a}), \mathcal{O})$ has rank r. In [FGT], Finis, Grunewald and Tirao used an efficient method which works with finite fields and approximated the ranks up to norm 60000 for $-d = 1$. My computations showed that their approximations for the most part were indeed exact but not always.

It is believed that there are infinitely many prime ideals $\mathfrak{p}$ of residue degree one such that $H^2_{cusp}(\Gamma_0(\mathfrak{p}), \mathbb{C})) = 0$. On the other hand, in analogy with the conjecture that there

N$\mathfrak{p}$	**some** of the primes that appear in the torsion of $\Gamma_0(\mathfrak{p})^{ab}$
	PSL($\mathcal{O}_1$)
4909	[2, 3, 7, 13, 409, 10691, 22871, 29423, 56980673, 71143433]
4933	[2, 3, 37, 101, 137, 577, 947, 21169, 194981]
4937	[2, 7, 37, 617, 10859, 108893, 4408530403, 157824962047]
4957	[2, 3, 7, 13, 31, 59, 14347, 3051863, 9405667, 23132267]
4969	[2, 3, 23, 71, 373, 191299, 39861006443, 8672729371087]
4973	[2, 11, 13, 47, 71, 113, 127, 331, 6317, 7949, 39023, 628801, 2995319]
4993	[2, 3, 5, 7, 11, 13, 101, 173, 798569, 5995036891, 18513420749]
	PSL($\mathcal{O}_2$)
4931	[. . . , 3772418780827, 67462419379713541, 4425411062257370822320521799]
4937	[. . . , 1889149903, 7397090738497, 880941232181841675673769]
4969	[. . . , 2728733329370698225919458399, 1145255958474009403487881957882603818711]
4987	[. . . , 1354882997352809, 167973141926075800477, 1092106383035778134156299]
4993	[. . . , 15997185593, 146336789672061572439301877, 48440175547438146744626201933]
	PSL($\mathcal{O}_3$)
4903	[3, 7, 19, 29, 37, 43, 61, 137, 191, 733]
4909	[2, 3, 7, 13, 19, 47, 67, 409, 1409]
4933	[2, 3, 5, 137, 173, 383, 719, 1451, 100057]
4951	[3, 5, 7, 11, 271, 3797, 6696049]
4957	[2, 3, 5, 7, 23, 43, 59, 233, 823, 62207]
4969	[2, 3, 5, 7, 23, 181, 2591, 516336433]
4987	[2, 3, 11, 71, 277, 619, 21977, 1971691]
4993	[2, 3, 11, 13, 29, 727, 4153, 27127]
4999	[2, 3, 7, 17, 29, 41, 83, 38593, 179623]
	PSL($\mathcal{O}_7$)
4909	[. . . , 3354447021713, 666100957349057134013, 1336355737543020209509]
4937	[. . . , 836083247742263, 600017487726483699711, 1344885261548364695671]
4943	[. . . , 94861335404089, 157213239530981, 345644733766517, 714087340201211]
4951	[. . . , 42137202713,117560966195702656377, 4774583154593351353]
4957	[. . . , 6803766726937001299, 21088956680308937473, 34130091188757085391]
4967	[. . . , 42061245937, 3414861551033731, 3857868721737476417]
4993	[. . . , 16112554517, 22230923149, 47405513059, 17179435084786759]
4999	[. . . , 47183940647, 47747826462797, 176725513764138170761817312541116531]
	PSL($\mathcal{O}_{11}$)
4909	[. . . , 49160270015318479411503, 31607539487402198903985237411069250311]
4931	[. . . , 59242366654994144915737, 397153057377536493107457514082773]
4933	[. . . , 471591580131222099301009, 753357254439534230416253]
4937	[. . . , 77460612005670238441079011896069980573813]
4943	[. . . , 496859062013858727411, 7533150099701393721041, 1806172579157695730540919793]
4951	[. . . , 325612994479665364754902328362211, 5758585827071565173844533348539011]
4973	[. . . , 668079334182971453623, 222335612071745269867644006471711]
4987	[. . . , 266855965325604420491069696711, 1217080095020051647103747260931]
4999	[. . . , 35270997998154652004835942597708494620078410433635847]

Table 2: a sample of large torsion occuring in $\Gamma_0(\mathfrak{p})^{ab}$

	$d=-1$	$d=-2$	$d=-3$	$d=-7$	$d=-11$
r	$N(r,35000)$	$N(r,15000)$	$N(r,36000)$	$N(r,15000)$	$N(r,13000)$
0	1636	802	1654	780	685
1	145	50	150	65	67
2	57	12	70	21	16
3	9	3	19	2	2
4	4	1	3	0	0
5	1	0	0	1	0
6	1	0	0	0	1
7	1	0	1	0	0
≥ 8	0	0	0	0	0

Table 3: number of primes of degree one with $\dim H^2_{cusp}(\Gamma_0(\mathfrak{p}),\mathbb{C}))=r$

$x/3000$	$R(x)/L_1(x)$	$R(x)/L_2(x)$	$R(x)/L_3(x)$	$R(x)/L_7(x)$	$R(x)/L_{11}(x)$
1	1.793	3.654	1.827	2.294	2.099
2	1.650	3.172	1.540	2.489	2.101
3	1.720	3.334	1.435	2.462	2.281
4	1.828	3.608	1.534	2.617	2.495
5	1.927	3.610	1.532	2.662	-
6	1.950	3.292	1.489	-	-
7	1.912	3.114	1.581	-	-
8	1.801	-	1.549	-	-
9	1.782	-	1.599	-	-
10	1.781	-	1.596	-	-
11	1.830	-	1.637	-	-
12	1.831	-	1.631	-	-

Table 4: data related to the distribution of prime level cohomology classes

are infinitely many elliptic curves over $\mathbb{Q}$ with prime conductor, see [BS] p.97, it is reasonable to expect that there are infinitely many prime ideals $\mathfrak{p}$ of residue degree one and $H^2_{cusp}(\Gamma_0(\mathfrak{p}),\mathbb{C}))\neq 0$. In [FGT], in analogy with the distribution questions for elliptic curves (see [BM]), the following question was posed (stated here in a slightly more general form).

Question 4.1. Let $\mathcal{O}$ be the ring of integers of an imaginary quadratic number field. Is there constant C such that the asymptotic relation

$$\sum_{\mathfrak{p},\ \mathbf{N}\mathfrak{p}\leq x} \dim\, H^1_{cusp}(\Gamma_0(\mathfrak{p}),\mathbb{C}) \sim C\frac{x^{\frac{5}{6}}}{\log\, x}$$

holds as x goes to infinity, where the sum ranges over residue degree one prime ideals $\mathfrak{p}\lhd\mathcal{O}$.

Let us put $L_d(x) := \sum_{\mathfrak{p},\ \mathbf{N}\mathfrak{p}\leq x} \dim\, H^1_{cusp}(\Gamma_0(\mathfrak{p}),\mathbb{C})$ for the ring $\mathcal{O}_d$ and $R(x) := x^{\frac{5}{6}}/\log\, x$. Table 4 compares the two functions $L(x)$ and $R(x)$ within the range of my computations.

In [CD], Calegari and Dunfield constructed a family of commensurable arithmetic rational homology 3-spheres, that is, commensurable arithmetic Kleinian groups Γ such that $H_1(\Gamma\backslash\mathbb{H},\mathbb{Q}) \simeq H_1(\Gamma,\mathbb{Q}) = 0$. In [LMR], Long, Maclahlan and Reid asked the question that whether there are infinitely many commensurability classes of arithmetic rational homology 3-spheres. In the same paper, they posed the following two conjectures (they are slightly rephrased in an equivalent form that fits better with this paper).

Conjecture 4.1.

1. *There exist infinitely many pairs of prime ideals* $\{\mathfrak{p}_1,\mathfrak{p}_2\} \subset \mathbb{Z}[i]$ *such that*
$$H^2_{cusp}(\Gamma_0(\mathfrak{p}_1\mathfrak{p}_2),\mathbb{Q}) = 0.$$

2. *Let* $\mathfrak{p} = (1+i)$. *There are infinitely many prime ideals* $\mathfrak{q} \subset \mathbb{Z}[i]$ *with* $\mathbf{N}\mathfrak{q} = 1 \mod 12$ *such that*
$$H^2_{cusp}(\Gamma_0(\mathfrak{p}\mathfrak{q}),\mathbb{Q}) = 0.$$

If the second conjecture holds, then using Jacquet-Langlands correspondence we get a positive answer to their question stated above.

I computed the ranks of $\Gamma_0(\mathfrak{p}\mathfrak{q})^{ab}$ where $\mathfrak{p} = (1+i)$ and $\mathfrak{q} \subset \mathbb{Z}[i]$ prime with $\mathbf{N}\mathfrak{q} = 1 \mod 12$ of norm ≤ 5821. There are 180 such prime ideal $\mathfrak{q}$, 102 of them satisfied the desired property that
$$H^2_{cusp}(\Gamma_0(\mathfrak{p}\mathfrak{q}),\mathbb{Q}) = \Gamma_0(\mathfrak{p}\mathfrak{q})^{ab} \otimes \mathbb{Q} = 0.$$
Although this is clearly not a large enough sample pool, the uniform distribution of the primes of vanishing rank makes the second conjecture seem plausible nevertheless.

4.3. asymptotics of torsion as the level increases

Very recently Müller [Mü] and Bergeron-Venkatesh [BV] obtained significant results that relate the asymptotic behaviour of the size of the torsion in the homology of certain cocompact lattices in $\mathrm{SL}_2(\mathbb{C})$ to that of the volume of the associated 3-folds. Although Bianchi groups and their congruence subgroups are outside of the scope of these results as they are not cocompact and it is very interesting to numerically investigate whether similar asymptotic relations hold for them.

In this section, I will compare the size of the torsion in the first homology and the volume of the associated 3-folds. As it is very costly to increase the weight, I will concentrate on increasing the level.

Let $\Gamma_0(\mathfrak{p})$ where $\mathfrak{p}$ is a prime ideal of $\mathcal{O}_d$ with residue degree one. Let $H_1(\Gamma_0(\mathfrak{p}),\mathbb{Z})_{tor}$ denote the torsion part of the first homology of $\Gamma_0(\mathfrak{p})$ with coefficients in Z. Let $\mathrm{vol}(\Gamma_0(\mathfrak{p})\backslash\mathbb{H})$ denote the volume of the 3-fold $\Gamma_0(\mathfrak{p})\backslash\mathbb{H}$ where $\mathbb{H}$ is the hyperbolic 3-space. In light of the result of Bergeron-Venkatesh, the following is a reasonable question.

Question 4.2. With the notation of the above paragraph, is there a constant C_d such that the asymptotic relation
$$\log|H_1(\Gamma_0(\mathfrak{p}),\mathbb{Z})_{tor}| \sim C_d \cdot \mathrm{vol}(\Gamma_0(\mathfrak{p})\backslash\mathbb{H})$$

holds as norm of the ideals $\mathfrak{p} \triangleleft \mathcal{O}_d$, which are prime with residue degree one, tends to infinity?

To computationally investigate the question, we need to approximate the volumes first. Using the well-known formula

$$\mathbb{V}_d := \mathrm{vol}(\mathrm{PSL}_2(\mathcal{O}_d)\backslash\mathbb{H}) = \frac{|\triangle_d^{3/2}|}{4\pi^2}\, \zeta_{K_d}(2)\ ,$$

where $\triangle_d$ is the discriminant of the field K_d and ζ_{K_d} is the Dedekind zeta function of K_d, we get

$$\begin{aligned}
\mathbb{V}_1 &\simeq 0.305321864725739671684867838311\\
\mathbb{V}_2 &\simeq 1.00384100334119813727236488577\\
\mathbb{V}_3 &\simeq 0.169156934401608937503533759046\\
\mathbb{V}_7 &\simeq 0.888914927816353263598904154202\\
\mathbb{V}_{11} &\simeq 1.38260830790264587367165334450
\end{aligned}$$

Now for $\mathfrak{p} \triangleleft \mathcal{O}_d$ prime of residue degree one over the rational prime p, we have

$$\mathrm{vol}(\Gamma_0(\mathfrak{p})\backslash\mathbb{H}) = (p+1)\cdot \mathbb{V}_d.$$

I have collected data on the ratio of $\log|H_1(\Gamma_0(\mathfrak{p}), \mathbb{Z})_{tor}|$ to $\mathrm{vol}(\Gamma_0(\mathfrak{p})\backslash\mathbb{H})$ in the case of the five Euclidean $\mathcal{O}_d$. It is very significant that the ratio is always very close to 1/20 within the range of my computations. A sample is given in Table **??** where I use the convention $T_{\mathfrak{p}} := \log|H_1(\Gamma_0(\mathfrak{p}), \mathbb{Z})_{tor}|$ and $V_{\mathfrak{p}} := \mathrm{vol}(\Gamma_0(\mathfrak{p})\backslash\mathbb{H})$.

$\mathcal{O}_1$		$\mathcal{O}_2$		$\mathcal{O}_3$		$\mathcal{O}_7$		$\mathcal{O}_{11}$	
$\mathbf{N}\mathfrak{p}$	$T_{\mathfrak{p}}/V_{\mathfrak{p}}$	$\mathbf{N}\mathfrak{p}$	$T_{\mathfrak{p}}/V_{\mathfrak{p}}$	$\mathbf{N}\mathfrak{p}$	$T_{\mathfrak{p}}/V_{\mathfrak{p}}$	$\mathbf{N}\mathfrak{p}$	$T_{\mathfrak{p}}/V_{\mathfrak{p}}$	$\mathbf{N}\mathfrak{p}$	$T_{\mathfrak{p}}/V_{\mathfrak{p}}$
34033	0.05341	19249	0.05295	34123	0.05200	17189	0.05381	15269	0.05317
34057	0.05338	19259	0.05446	34129	0.05115	17203	0.05155	15271	0.05453
34061	0.05461	19267	0.05124	34141	0.05656	17207	0.05384	15277	0.05241
34129	0.05581	19273	0.05334	34147	0.05641	17231	0.05184	15299	0.05377
34141	0.05366	19289	0.05136	34159	0.05534	17257	0.05239	15313	0.05440
34157	0.05069	19379	0.05202	34171	0.05471	17291	0.05457	15349	0.05410
34213	0.05314	19387	0.05314	34183	0.05528	17299	0.05198	15359	0.05158
34217	0.05494	19403	0.05358	34213	0.05297	17327	0.05116	15361	0.05141
34253	0.05325	19417	0.05424	34231	0.05510	17333	0.05326	15383	0.05341
34261	0.05462	19427	0.05410	34261	0.05559	17341	0.05445	15401	0.05379
34273	0.05184	19433	0.05235	34267	0.05553	17383	0.05052	15427	0.05418
34297	0.05571	19441	0.05333	34273	0.05423	17389	0.05228	15467	0.05243
34301	0.05622	19457	0.05300	34297	0.05544	17417	0.05427	15493	0.05373
34313	0.05433	19483	0.05471	34303	0.05393	17431	0.05386	15497	0.05412
34337	0.05498	19489	0.05372	34327	0.05392	17467	0.05328	15511	0.05277
34361	0.05497	19507	0.05316	34351	0.05399	17483	0.05289	15541	0.05418
34369	0.05594	19531	0.05201	34369	0.05606	17497	0.05357	15559	0.05358
34381	0.05171	19553	0.05240	34381	0.05153	17509	0.05257	15569	0.05365
34421	0.05447	19571	0.05220	34429	0.05649	17539	0.05461	15581	0.05377
34429	0.05278	19577	0.05295	34471	0.05280	17551	0.05329	15601	0.05091
34457	0.05422	19603	0.05539	34483	0.05598	17579	0.05468	15607	0.05524
34469	0.05355	19609	0.05182	34501	0.05680	17581	0.05639	15629	0.05505
34501	0.05308	19681	0.05124	34513	0.05412	17599	0.05531	15643	0.05142
34513	0.05282	19697	0.05294	34519	0.05481	17609	0.05303	15647	0.05439
34537	0.05294	19699	0.05430	34537	0.05277	17623	0.05124	15667	0.05386
34549	0.05308	19739	0.05127	34543	0.05490	17627	0.05550	15679	0.05475
34589	0.05314	19753	0.05434	34549	0.05539	17669	0.05478	15731	0.05280
34613	0.05368	19763	0.05146	34591	0.05321	17683	0.05342	15733	0.05438
34649	0.05349	19777	0.05254	34603	0.05434	17707	0.05295	15739	0.05401
34673	0.05409	19793	0.05224	34651	0.05788	17747	0.05375	15761	0.05302
34693	0.05112	19801	0.05391	34687	0.05571	17749	0.05346	15767	0.05418
34721	0.05155	19819	0.05493	34693	0.05646	17761	0.05250	15797	0.05407
34729	0.05223	19841	0.05358	34729	0.05574	17789	0.05339	15823	0.05027
34757	0.05913	19843	0.05317	34747	0.05830	17791	0.05173	15877	0.05427
34781	0.05375	19867	0.05351	34759	0.05264	17837	0.05274	15887	0.05290
34841	0.05442	19889	0.05145	34807	0.05057	17851	0.05226	15889	0.05331
34849	0.05354	19891	0.05451	34819	0.05620	17903	0.05317	15907	0.05366
34877	0.05501	19913	0.05378	34843	0.05454	17921	0.05355	15937	0.05236
34897	0.05212	19937	0.05458	34849	0.05250	17929	0.05326	15959	0.05094
34913	0.05479	19961	0.05167	34897	0.05741	17957	0.05374	15973	0.05215
34949	0.05231	19963	0.05267	34939	0.05214	17959	0.05347	16061	0.05428
34961	0.05493	19979	0.05295	34963	0.05557	17971	0.05208	16063	0.05349
34981	0.05354	19993	0.05488	34981	0.05409	17977	0.05524	16069	0.05240

Table 5: the ratio the size of the torsion to the volume as level grows

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