$\mathbb{O}P^2$ bundles in M-theory

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Abstract

Ramond has observed that the massless multiplet of eleven-dimensional supergravity can be generated from the decomposition of certain representation of the exceptional Lie group F_4 into those of its maximal compact subgroup Spin(9). The possibility of a topological origin for this observation is investigated by studying Cayley plane, $\mathbb{O}P^2$, bundles over eleven-manifolds Y^{11}. Consequently, the origin of the massless fields and their supersymmetry in M-theory are characterized geometrically and topologically. The lift of the topological terms gives constraints on the cohomology of Y^{11} which are derived. The effect of the construction on the partition function and the compatibility with other physical theories is discussed. The various genera of $\mathbb{O}P^2$ are calculated and higher structures on Y^{11} are related to higher structures on the total space M^{27}. The latter, being 27-dimensional, might provide a candidate for 'bosonic M-theory'. The discussion leads to a connection with an octonionic version of Kreck-Stolz elliptic homology theory.

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Contents

1 Introduction

The relation between M-theory and type IIA string theory leads to very interesting connections to K-theory [28, 27] and twisted K-theory [68] [15] [16]. Exceptional groups have also long appeared in physics. In particular, the topological part of the M-theory action is encoded in part by an E_8 gauge theory in eleven dimensions [101]. This captures the cohomology of the C-field. Models for the M-theory C-field were proposed in [27] with and without using E_8. The E_8 bundle leads to a loop bundle on the type IIA base of the circle bundle [3] [68]. The role of E_8 and LE_8 was emphasized in [86, 88]. In particular, in [86] an important role for the string orientation was found within the E_8 construction. It is in the case when the base X^{10} is string-oriented that the topological action has a WZW-like interpretation and the degree-two component of the eta-form [68] is identified with the Neveu-Schwarz B-field [86].

In this paper we study another side of the problem, by including the whole eleven-dimensional supermultiplet (g, C_3, Ψ), i.e. the metric, the C-field, and the Rarita-Schwinger field, and not just the C-field. This turns out to be related to another exceptional Lie group, namely F_4, the exceptional Lie group of rank 4. Ramond [76] [78] [79] gave evidence for F_4 coming from the following two related observations:

1. F_4 appears explicitly [79] in the light-cone formulation of supergravity in eleven dimensions [26]. The generators $T^{\mu\nu}$ of the little group $SO(9)$ of the Poincaré group $ISO(1,10)$ in eleven dimensions and the spinor generators T^a combine to form the 52 operators that generate the exceptional Lie algebra $\mathfrak{f}_4$ such that the constants $f^{\mu\nu ab}$ in the commutation relation

$$[T^{\mu\nu}, T^a] = i f^{\mu\nu ab} T^b \tag{1.1}$$

are the structure constants of $\mathfrak{f}_4$. The 36 generators $T^{\mu\nu}$ are in the adjoint of $SO(9)$ and the 16 T^a generate its spinor representation. This can be viewed as the analog of the construction of E_8 out of the generators of $SO(16)$ and of $E_8/SO(16)$ in [38].

2. The identity representation of F_4, i.e. the one corresponding to Dynkin index $[0,0,0,0]$, generates the three representations of Spin(9) [76]

$$\mathrm{Id}(F_4) \longrightarrow (44, 128, 84)\ , \tag{1.2}$$

the numbers on the right hand side correctly matching the number of degrees of freedoms of the massless bosonic content of eleven-dimensional supergravity with the individual summands corresponding, respectively, to the graviton, the gravitino, and the C-field (see the beginning of section 2).

It is the purpose of this paper is to expand on Ramond's observations by investigating the possibility of having an actual $\mathbb{O}P^2 = F_4/\mathrm{Spin}(9)$ bundle over Y^{11} through which the above observations can be explained geometrically and topologically. Since F_4 is the isometry group of the Cayley plane, the $\mathbb{O}P^2$ bundle will be the bundle associated to a principal F_4 bundle. We analyze some conditions under which this is possible.

In physics, the lifting of M-theory via the sixteen-dimensional manifold $\mathbb{O}P^2$ brings us to 27 dimensions. Given a Kaluza-Klein interpretation, this suggests the existence of a theory in 27 dimensions, whose dimensional reduction over $\mathbb{O}P^2$ leads to M-theory. The higher dimensional theory involves spinors, and it is natural to ask whether or not the theory can be supersymmetric. In one form we propose this as a candidate for the 'bosonic M-theory' sought after in [48], from gravitational geometric arguments, and in [80], from matrix model arguments.

We consider the point of view of eleven-dimensional manifolds in M-theory with extra structure. Since any Y^{11} with a String structure is zero bordant in $\Omega_{11}^{\langle 8 \rangle}$ then this raises the question of whether there is an equivalence with a total space of a bundle in which Y^{11} is a base. For the Spin case, Kreck and Stolz [60] constructed an elliptic homology theory in which a spin manifold of dimension $4k$ is Spin bordant to the total space of an $\mathbb{H}P^2$ bundle over a zero-bordant base if and only if its elliptic genus $\Phi_{\rm ell} \in \mathbb{Q}[\delta, \varepsilon]$ vanishes, where the generators δ, ε have degree 4 and 8, respectively. The same authors also expected the existence of a homology theory based on $\mathbb{O}P^2$ bundles for the String case, i.e. for manifolds such that $\frac{1}{2}p_1 = 0$, where p_1 is the first Pontrjagin class. So in our case, we ask whether there is a manifold M^{27} which is an $\mathbb{O}P^2$ bundle over a zero bordant base and what consequence that has on the elliptic and the Witten genus.

Some aspects of the connection to this putative homology theory are

1. The elliptic homology theory requires the fundamental class [1] $[\mathbb{O}P^2]$ of $\mathbb{O}P^2$ to be inverted. This suggests connecting the lower-dimensional theory, in our case eleven-dimensional M-theory, to a higher dimensional one obtained by increasing the dimension by 16.
2. Previous works have used elliptic cohomology. We emphasize that in this paper we make use of a *homology* theory. Thus this not only provides further evidence for the relation between elliptic (co)homology and string/M-theory, but it also provides a new angle on such a relationship.

In previous work [61] [62] [63] [85] [87] evidence from various angles for a connection between string theory and elliptic cohomology was given. These papers relied heavily on analogies with the case in string theory, and were thus not intrinsically M-theoretic. In [82] [83] [84] a program was initiated to make the relation directly with M-theory. Thus, from another angle, the general purpose of this paper is two-fold:

- to point out further connections between elliptic cohomology and M-theory
- to make the connection more M-theoretic, i.e. without reliance on any arguments from string theory.

$\mathbb{O}P^2$ is the Cayley, or octonionic projective, plane. For an extensive description see [81] [40] [12]. The group F_4 acts transitively on $\mathbb{O}P^2$, from which is follows that $\mathbb{O}P^2 \cong F_4/\text{Spin}(9)$. In fact F_4 is the isometry group of $\mathbb{O}P^2$. The tangent space to $\mathbb{O}P^2$ at a point is the coset of the corresponding Lie algebras $\mathfrak{f}_4/\mathfrak{so}(9)$, which is $\mathbb{O}^2 \cong \mathbb{R}^{16}$.

We use the Lorentz signature in studying the spectrum in section 2, and then resort to the Euclidean signature when discussing the geometric and topological aspects in the rest of the paper.

2 The Fields in M-theory

The low energy limit of M-theory (cf. [100] [99] [30]) is eleven-dimensional supergravity [26], whose field content on an eleven-dimensional spin manifold Y^{11} with Spin bundle SY^{11} is

- Two bosonic fields: The metric g and the three-form C_3. It is often convenient to work with Cartan's moving frame formalism so that the metric is replaced by the 11-bein e_M^A such that $e_M^A e_N^B = g_{MN}\eta^{AB}$, where η is the flat metric on the tangent space.
- One fermionic field: The Rarita-Schwinger vector-spinor Ψ_1, which is classically a section of $SY^{11} \otimes TY^{11}$, i.e. a spinor coupled to the tangent bundle.

The count of the on-shell degrees of freedom, i.e. components, of the fields is done by eliminating the redundant gauge degrees of freedom. This could be done for example by choosing the light cone gauge: decompose Minkowski space $\mathbb{R}^{1,10}$ into $\mathbb{R}^{1,1} \oplus \mathbb{R}^9$, with $\mathbb{R}^{1,1} = \text{Span}(\mathbf{v}_1, \mathbf{v}_2)$ where the vectors $\mathbf{v}_i$ satisfy $|\mathbf{v}_1|^2 = |\mathbf{v}_2|^2 = 0$ and $\mathbf{v}_1 \cdot \mathbf{v}_2 \neq 0$.

[1] viewed as a generator.

The Poincaré group $\mathbb{R}^{1,10} \ltimes SO(1,10)$ corresponds to the algebra $\mathbb{R}^{1,10} \widetilde{\oplus}\ \mathfrak{so}(1,10)$ where the brackets $[\mathbb{R}^{1,10}, \mathfrak{so}(1.10)]$ are given by the vector representation of $\mathfrak{so}(1,10)$ on $\mathbb{R}^{1,10}$. Since the latter is abelian then the irreducible representations are one-dimensional, and hence given by the characters $(\mathbb{R}^{1,10})^*$. This is acted upon by $\mathfrak{so}(1,10)$, which decomposes the space of characters into orbits characterized by the mass $m^2 = |\mathbf{v}|^2$ for $\mathbf{v} \in (\mathbb{R}^{1,10})^*$. Let H be the stabilizer of a point. H is called the little group. An irreducible representation of the Poincaré algebra is the space of sections of a homogeneous vector bundle $E = SO(1,10) \times_H K$ over the orbit $SO(1,10)/H$, where K is a representation of H. The representations, by the Wigner classification, are as follows:

- *Massive fields*: For $|\mathbf{v}|^2 \neq 0$ the little group is $H = SO(10)$.
- *Massless fields*: For $|\mathbf{v}|^2 = 0$ the little group is $H = SO(9)$.

The states for eleven-dimensional supergravity are massless and hence form irreducible representations of the little group $SO(9)$. The count is is as follows (with $D = 11$):

1. *The 11-bein* e_M^A: Traceless symmetric $(D-2) \times (D-2)$ matrix gives $\frac{1}{2}D(D-3) = 44$ [53].
2. *The C-field* C_3: A 3-form in $\mathbb{R}^9$ gives $\binom{D-2}{3} = \frac{(D-2)!}{3!(D-2-3)!} = 84$.
3. *The Rarita-Schwinger field* Ψ_1: $2^{\frac{1}{2}(D-1)-1}(D-3) = 128$, where the factor of -1 in the exponent comes from the fact that Ψ_1 is a Majorana, i.e. real, fermion.

2.1 The Euler Triplet

In this section we review Ramond's observation we mentioned in the introduction and state the main theme of this paper. We will basically 'geometrize' and 'topologize' the representation-theoretic observation, hence making room for dynamics from kinematics. Therefore, the appearance of the F_4 representation and the decomposition under the maximal compact subgroup Spin(9) to give the degrees of freedom of the fields will be taken to originate from an $\mathbb{O}P^2$ bundle over Y^{11}.

There are anomalous embeddings of certain groups into an orthogonal group in which the vector representation of the bigger group is identified with the spinor of the smaller group. For example, for $SO(9)$ we have [53]

$$\begin{aligned} SO(16) \quad &\supset \quad SO(9) \\ \text{vector} \quad &= \quad \text{spinor}, \end{aligned} \tag{2.1}$$

both of dimension 16. In fact this explains the emergence of supersymmetry for the supermultiplet of eleven-dimensional supergravity [53] [29] [76]. Furthermore, in [29] it was conjectured that $SO(16)$ is a local symmetry of 11-dimensional supergravity. This was proved in [72]. One of the goals in this paper will be to attempt to provide a geometric origin for the above observation (eqn. (2.1)) via $\mathbb{O}P^2$ bundles, as Spin(16) will be the spin group of the projective plane fiber. We hope this might also shed some light on the enlarged local symmetry in the theory since the symmetry groups coming from bundles on $\mathbb{O}P^2$ will act locally (at least on the space itself).

Since $\text{rank}(F_4) = \text{rank}(\text{Spin}(9))$ then $\mathbb{O}P^2$ is an equal rank symmetric space. A generalization to homogeneous spaces of the Weyl character formula, with maximal torus replaced by the equal rank maximal compact subgroup, is the Gross-Kostant-Ramond-Sternberg character formula [39]

$$V_\lambda \otimes S^+ - V_\lambda \otimes S^- = \sum_c \text{sgn}(c) U_{c\bullet\lambda}, \tag{2.2}$$

which can be applied as follows [76] to the pair $(F_4, \mathrm{Spin}(9))$. The left hand side involves the differences of tensor products of representations V_λ of F_4 with highest weight λ written in terms of its Spin(9) subgroup, and $S^\pm$, the two semi-spinor representations of Spin(16) written in terms of its embedded subgroup Spin(9), i.e. the spin representation associated to the complement of $\mathfrak{spin}(9) = \mathrm{Lie}(\mathrm{Spin}(9))$ in $\mathfrak{f}_4 = \mathrm{Lie}(F_4)$. The right hand side involves the sum over c, the elements of the Weyl group which map the Weyl chamber of F_4 into that of Spin(9). The number of such elements is three, given by the ratio of the orders of the Weyl groups (2.6), i.e. the subset $C \in W_{F_4}$ has one representative from each coset of $W_{\mathrm{Spin}(9)}$. $U_{c\bullet\lambda}$ denotes the Spin(9) representation with highest weight $c \bullet \lambda = c(\lambda + \rho_{F_4}) - \rho_{\mathrm{Spin}(9)}$, with ρ the sum of fundamental weights. For F_4, as mentioned above, there corresponds three equivalent ways of embedding Spin(9) into F_4. This implies that for each representation of F_4, there are $\chi(F_4/\mathrm{Spin}(9)) = 3$ irreducible representations of Spin(9) generated, called the *Euler triplet*.

The consequence for eleven-dimensional supergravity is that the fields satisfy the character formula exactly for the pair $(F_4, \mathrm{Spin}(9))$ [76]. Under the decomposition $\mathrm{Spin}(16) \supset \mathrm{Spin}(9)$, one of the semi-spinor representations, S^+, stays the same, $128 = 128$, while the other, S^-, decomposes as $128' = 44 + 84$. For a highest weight $\lambda = 0$, one gets $c(\rho_{F_4}) = \rho_{SO(9)}$ the character formula is then clearly satisfied [76] as

$$\mathrm{Id} \otimes S^+ - \mathrm{Id} \otimes S^- = 0, \tag{2.3}$$

i.e.

$$128 - (44 + 84) = 128 - 44 - 84. \tag{2.4}$$

The Dynkin labels of the fields in the representation of Spin(9) are [2000] for the graviton as a symmetric second rank tensor, [0010] for the 3rd rank antisymmetric tensor C_3, and [1001] for the Rarita-Schwinger spinor-vector.

Remarks

1. There is a very interesting Dirac operator whose index is not zero on $\mathbb{O}P^2$. This is Kostant's cubic Dirac operator [59]

$$\not{\mathcal{K}}\xi := \sum_{a=1}^{16} \Gamma^a T^a \xi = 0, \tag{2.5}$$

where Γ^a, $a, b = 1, 2, \cdots, 16$ are $2^8 \times 2^8$ gamma matrices that generate the Clifford algebra $\left\{\Gamma^a, \Gamma^b\right\} = 2\delta^{ab}$. Solutions of the Kostant equation (2.5) consists of all Euler triplets, including the supergravity multiplet [79]. The right hand side of (2.2) is the kernel of (2.5). We will deal with other Dirac operators in section 3.1.

2. The Euler characteristic of $\mathbb{O}P^2$ can be calculated as the ratio of the orders of the Weyl groups

$$\chi(\mathbb{O}P^2) = \chi\left(F_4/\mathrm{Spin}(9)\right) = \frac{|W(F_4)|}{|W(B_4)|} = \frac{|W(F_4)|}{\mathbb{Z}_2^4 \odot S_4} = \frac{2^7.3^2}{2^4.4!} = 3. \tag{2.6}$$

Such a formula holds for general equal rank symmetric spaces G/H, by a classic result of Hopf and Samelson.

We now give the main theme around which this paper is centered.

Main Idea: *We interpret Ramond's triplets as arising from $\mathbb{O}P^2$ bundles with structure group F_4 over our eleven-dimensional manifold Y^{11}, on which M-theory is 'defined'.*

We will deal with $\mathbb{O}P^2$ bundles systematically and in detail in section 3, but first we proceed with the geometric interpretation of the main idea, as well as propose a geometric interpretation for the observation (2.1).

2.2 Spin(9)-structures and the M-theory fields

Before putting $\mathbb{O}P^2$ as a fiber, we start with just the space $\mathbb{O}P^2$ itself.

2.2.1 Spin(9) bundles

We start with the Spin structure on the Cayley plane.

Lemma 2.1. *$\mathbb{O}P^2$ admits a unique Spin structure.*

Over the homogeneous space $\mathbb{O}P^2 = F_4/\mathrm{Spin}(9)$ we always have the canonical Spin(9) bundle, which we call $\wp$. Let $\Delta : \mathrm{Spin}(9) \to U(16)$ be the spinor representation. We can thus form associated vector bundles with structure group $U(16)$ over $\mathbb{O}P^2$. To investigate these we should look at the K-theory of $\mathbb{O}P^2$. This has been done for general equal rank symmetric spaces G/H in [10]. The group $K^1(G/H)$ is zero, whereas $K^0(G/H)$ is a free abelian group of rank equal to the Euler number, so that $K^0(\mathbb{O}P^2) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. Furthermore, $K^0(\mathbb{O}P^2)$ has no torsion and the Chern character map ch : $K^0(\mathbb{O}P^2) \to H^{\mathrm{even}}(\mathbb{O}P^2; \mathbb{Q})$ is injective. Since $H^*(\mathbb{O}P^2; \mathbb{Z})$ has no torsion, K^0 is isomorphic to the cohomology of $\mathbb{O}P^2$. Therefore,

Proposition 2.2. *A complex vector bundle over $\mathbb{O}P^2$ is uniquely characterized by the classes in degrees 0, 8, and 16.*

Let $\Re(\mathrm{Spin}(9))$ be the representation ring of Spin(9) and let $\beta : \Re(\mathrm{Spin}(9)) \to K^0(\mathbb{O}P^2)$ be the map that assigns vector bundles over $\mathbb{O}P^2$ to representations of Spin(9), so that we have the composite map

$$\mathrm{Spin}(9) \xrightarrow{\Delta} \Re(\mathrm{Spin}(9)) \xrightarrow{\beta} K^0(\mathbb{O}P^2) \xrightarrow{\mathrm{ch}} H^{\mathrm{even}}(\mathbb{O}P^2; \mathbb{Q}) \ . \tag{2.7}$$

In fact the map β is surjective, which can be seen as follows [10]. Let s_j be the jth elementary symmetric function in the x_i^2, where x_i, $i = 1, 2, 3, 4$, are elements of the maximal torus of Spin(9), as in [20]. Then, using $s_2 = s_2(x_1^2, x_2^2, x_3^2, x_4^2) = \sum_{i<j} x_i x_j$ and $s_4 = s_4(x_1^2, x_2^2, x_3^2, x_4^2) = \prod_{i=1}^4 x_i^2$, the Chern character

$$\begin{aligned} \mathrm{ch}(\beta\Delta) &= 2^4 \prod_{i=1}^{4} \cosh\left(\frac{x_i}{2}\right) \\ &= \mathrm{rk} + \frac{s_2}{6} + \text{higher terms} \\ &= 16 + u + \text{higher terms} \ , \end{aligned} \tag{2.8}$$

has u, the generator of $H^8(\mathbb{O}P^2; \mathbb{Z}) = \mathbb{Z}$, as a summand. Therefore we have

Proposition 2.3. *Every complex vector bundle over $\mathbb{O}P^2$ is an associated vector bundle for the* Spin(9) *principal bundle $\wp$.*

2.2.2 Spin(9)-structures

Let $\mathfrak{f}_4$ and $\mathfrak{spin}(9)$ be the Lie algebras of F_4 and Spin(9), respectively. The adjoint action of F_4 is given by

$$\mathrm{Ad}_{F_4} : F_4 \longrightarrow \mathrm{Aut}_{\mathrm{Lie}}(\mathfrak{f}_4). \tag{2.9}$$

Consider the restriction to Spin(9)

$$\mathrm{Ad}_{F_4,\mathrm{Spin}(9)} := \mathrm{Ad}_{F_4}|_{\mathrm{Spin}(9)} : \mathrm{Spin}(9) \longrightarrow \mathrm{Aut}_{\mathrm{Lie}}(\mathfrak{f}_4), \tag{2.10}$$

which is given by

$$\mathrm{Ad}_{F_4}|_{\mathrm{Spin}(9)}(k)X = \mathrm{Ad}_{F_4}(k)X = \mathrm{Ad}_{\mathrm{Spin}(9)}(k)X \in \mathfrak{spin}(9), \tag{2.11}$$

for $X \in \mathfrak{spin}(9)$ and $k \in \mathrm{Spin}(9)$. This means that $\mathfrak{spin}(9)$ is an invariant subspace for the respresentation $\mathrm{Ad}_{F_4}|_{\mathrm{Spin}(9)}$ of Spin(9) in $\mathfrak{f}_4$, and there is the factor representation

$$\mathrm{Ad}^{\perp} : \mathrm{Spin}(9) \longrightarrow GL(\mathfrak{f}_4/\mathfrak{spin}(9)). \tag{2.12}$$

The sequence

$$0 \longrightarrow \mathfrak{spin}(9) \longrightarrow \mathfrak{f}_4 \longrightarrow \mathfrak{f}_4/\mathfrak{spin}(9) \longrightarrow 0 \tag{2.13}$$

is exact and Spin(9)-equivariant. Consider the principal fiber bundle

$$\begin{array}{ccc} \mathrm{Spin}(9) & \longrightarrow & F_4 \\ & & \Big\downarrow p \\ & & F_4/\mathrm{Spin}(9)\ . \end{array} \tag{2.14}$$

Using the representations (2.9) and (2.10) we form the associated bundles E_1

$$\begin{array}{ccc} \mathfrak{spin}(9) & \longrightarrow & F_4 \times_{\mathrm{Spin}(9)} \mathfrak{f}_4/\mathfrak{spin}(9) = E_1 \\ & & \Big\downarrow \pi_1 \\ & & F_4/\mathrm{Spin}(9) \end{array} \tag{2.15}$$

and E_2

$$\begin{array}{ccc} \mathfrak{spin}(9) & \longrightarrow & F_4 \times_{\mathrm{Spin}(9)} \mathfrak{spin}(9) = E_2 \\ & & \Big\downarrow \pi_2 \\ & & F_4/\mathrm{Spin}(9)\ , \end{array} \tag{2.16}$$

respectively. Then we have the following characterization of the tangent bundle of the Cayley plane.

Proposition 2.4. *$T(\mathbb{O}P^2)$ is the associated vector bundle E_1. Furthermore, $E_1 \oplus E_2$ is a trivial vector bundle.*

Results for general G/K are proved in [70].

Denote by $\mathcal{F}(\mathbb{O}P^2)$ the frame bundle of the Cayley plane with structure group $\mathrm{S}O(16)$. A Spin(9)-structure is a reduction $\mathcal{R} \subset \mathcal{F}(\mathbb{O}P^2)$ of the $\mathrm{S}O(16)$-bundle $\mathcal{F}(\mathbb{O}P^2)$ via the homomorphism $\kappa_9 : \mathrm{Spin}(9) \to \mathrm{S}O(16)$. A Spin(9)-structure defines certain other geometric structures [35]. In particular, it induces a 9-dimensional real, oriented Euclidean vector bundle V^9 with Spin structure

$$V^9 := \mathcal{R} \times_{\mathrm{Spin}(9)} \mathbb{R}^9. \tag{2.17}$$

Lemma 2.5. *$\mathbb{O}P^2$ admits a* Spin(9)*-structure.*

Proof. Due to the topology of $\mathbb{O}P^2$, the only nontrivial cohomology, with any coefficients, is in the top and the middle dimension (see Appendix). Then the only possible obstruction to reducing the structure group from Spin(16) to Spin(9) is

$$H^8\left(\mathbb{O}P^2; \pi_{8-1}\left(\frac{\mathrm{Spin}(16)}{\mathrm{Spin}(9)}\right)\right). \tag{2.18}$$

From the homotopy exact sequence for the fibration

$$\mathrm{Spin}(9) \longrightarrow \mathrm{Spin}(16) \longrightarrow \mathrm{Spin}(16)/\mathrm{Spin}(9), \tag{2.19}$$

and the fact that the homotopy groups of Spin(i), $i = 9, 16$ are

$$\pi_{3\leq n\leq 15}(\mathrm{Spin}(16)) = (\mathbb{Z},0,0,0,\mathbb{Z},\mathbb{Z}_2,\mathbb{Z}_2,0,\mathbb{Z},0,0,0,\mathbb{Z}\oplus\mathbb{Z}) \tag{2.20}$$

$$\begin{aligned}\pi_{3\leq n\leq 15}(\mathrm{Spin}(9)) = (&\mathbb{Z},0,0,0,\mathbb{Z},\mathbb{Z}_2\oplus\mathbb{Z}_2,\mathbb{Z}_2\oplus\mathbb{Z}_2,\mathbb{Z}_8,\mathbb{Z}\oplus\mathbb{Z}_2,\\ &0,\mathbb{Z}_2,\mathbb{Z}_2\oplus\mathbb{Z}_8,\mathbb{Z}\oplus Z_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2)\,,\end{aligned} \tag{2.21}$$

we get that $\pi_7(\mathrm{Spin}(16)/\mathrm{Spin}(9)) = 0$. Therefore, there are no obstructions to reducing the structure group from Spin(16) to Spin(9). □

Lemma 2.6. *(Properties of V^9)*

1. *Spinors: The tangent bundle $T(\mathbb{O}P^2)$ is isomorphic to the bundle $\Delta_9(V^9)$ of real spinors of the vector bundle V^9.*

2. *Stiefel-Whitney classes: The Stiefel-Whitney classes of $\mathbb{O}P^2$ are related to the corresponding classes of V^9 by the formula*
$$w_8(\mathbb{O}P^2) = w_4^2(V^9) + w_8(V^9). \tag{2.22}$$

3. *Pontrjagin classes:*

$$\begin{aligned} p_1(V^9) &= 0 = p_3(V^9) && (2.23)\\ p_2(V^9) &= -p_2(\mathbb{O}P^2) = -6u && (2.24)\\ p_4(V^9) &= -\frac{1}{13}p_4(\mathbb{O}P^2) = -3u^2\,. && (2.25)\end{aligned}$$

Proof. Part (1) follows from the definition. It is known that $\mathfrak{f}_4 = \mathfrak{so}(9)\oplus S^+$ [2] [12]. The isotropy group Spin(9) acts on the tangent space $T_x\mathbb{O}P^2 = \mathfrak{f}_4/\mathfrak{spin}(9)$ as a sixteen-dimensional representation, the spinor representation Δ_9 of Spin(9).
Part (2) follows from an application of the discussion in [37] for a general 16-manifold with Spin(9)-structure. We just show how to get the Stiefel-Whitney classes of $\mathbb{O}P^2$. We use the Wu classes $\nu_i \in H^i(\mathbb{O}P^2;\mathbb{Z}_2)$ defined by

$$\langle\, \nu_i \cup u\ ,\ [\mathbb{O}P^2]\,\rangle = \langle\, Sq^i u\ ,\ [\mathbb{O}P^2]\,\rangle. \tag{2.26}$$

Since $Sq^8u = u^2$ then the total Wu class of $\mathbb{O}P^2$ is $\nu = 1+u+u^2$, so that, by (2.26), the total Stiefel-Whitney class is

$$w(\mathbb{O}P^2) = Sq\ \nu = 1 + u + u^2. \tag{2.27}$$

For part (3) we apply theorem 2 (or corollary 3) of [35] to the case of $\mathbb{O}P^2$ so that we have the following:
First $p_1(\mathbb{O}P^2) = 2p_1(V^9) = 0$.
Second, $p_2(\mathbb{O}P^2) = \frac{7}{4}(V^9) - p_2(V^9)$ so that $p_2(V^9) = -p_2(\mathbb{O}P^2)$ since $p_1(V^9)$ is zero.
Third, $p_3(\mathbb{O}P^2) = \frac{1}{8}\left(7p_1^3(V^9) - 12p_1(V^9)p_2(V^9) + 16P_3(V^9)\right)$, which gives that $p_3(V^9) = 0$ since $p_2(V^9) = 0$ and $p_3(\mathbb{O}P^2) = 0$.
Fourth, $p_4(\mathbb{O}P^2) = \frac{1}{128}\left(35p_1^4(V^9) - 120p_1^2(V^9)p_2(V^9) + 400p_1(V^9)p_3(V^9) - 1664p_3(V^9)\right)$, which gives $p_4(V^9) = -\frac{1}{13}p_4(\mathbb{O}P^2)$ upon using $p_1(V^9) = 0$. □

Lemma 2.7. *The Euler class and the fourth L-polynomial of $\mathbb{O}P^2$ are given in terms of the Pontrjagin classes of V^9 as*

$$e(\mathbb{O}P^2) = \frac{p_2^2(V^9) - 4p_4(V^9)}{16} \tag{2.28}$$

$$L_4(\mathbb{O}P^2) = -\frac{1}{3^4.5^2.7}\left(19p_2^2(V^9) + 4953p_4(V^9)\right) \tag{2.29}$$

Proof. The formula for the Euler class follows either from substitution of the Pontrjagin classes of V^9 in terms of the Pontrjagin classes of $\mathbb{O}P^2$ in the Euler class formula of $\mathbb{O}P^2$ or directly by observing that, with $p_1(V^9) = 0$,

$$e(\mathbb{O}P^2) = \frac{1}{256}p_1^4(V^9) - \frac{1}{32}p_1^2(V^9)p_2(V^9) + \frac{1}{16}p_2^2(V^9) - \frac{1}{4}p_4(V^9) \tag{2.30}$$

gives the answer. Finally, the formula for L_4 follows by direct substitution into

$$L_4 = \frac{1}{3^4.5^2.7}\left(381p_4 - 71p_3p_1 - 19p_2^2 + 22p_2p_1^2 - 3p_1^4\right), \tag{2.31}$$

so that

$$L_4(\mathbb{O}P^2) = -\frac{1}{3^4.5^2.7}\left(19p_2^2(V^9) + 4953p_4(V^9)\right) \tag{2.32}$$

□

Remark. Using V^9 we can recover the signature of $\mathbb{O}P^2$ given in (3.6)

$$\sigma(\mathbb{O}P^2) = -\frac{1}{3}\int_{\mathbb{O}P^2} p_4(V^9) = \frac{1}{39}\int_{\mathbb{O}P^2} p_4(\mathbb{O}P^2), \tag{2.33}$$

which is related to the Euler class by $e(\mathbb{O}P^2) = 3\sigma(\mathbb{O}P^2)$.

2.2.3 Consequences for the M-theory fields

One major advantage of the introduction of an $\mathbb{O}P^2$ bundle is that in this picture the bosonic fields of M-theory, namely the metric and the C-field, can be unified.

Theorem 2.8. *The metric and the C-fields are orthogonal components of the positive spinor bundle of $\mathbb{O}P^2$.*

Proof. The spinor bundle $S^+(\mathbb{O}P^2)$ of the Cayley plane is isomorphic to

$$S^+(\mathbb{O}P^2) = S_0^2(V^9) \oplus \Lambda^3(V^9), \tag{2.34}$$

where S_0^2 denotes the space of traceless symmetric 2-tensors. This follows from an application of proposition 3 in [35] which requires the study the 16-dimensional spin representations $\Delta_{16}^{\pm}$ as Spin(9)-representations. The element $e_1 \cdots e_{16}$ belongs to the subgroup $\widetilde{\text{Spin}(9)} \subset \text{Spin}(16)$ and acts on $\Delta_{16}^{\pm}$ by multiplication by (± 1). This means that Δ_{16}^{+} is an $SO(9)$-representation, but Δ_{16}^{-} is a Spin(9)-representation [2]. Both representations do not contain non-trivial Spin(9)-invariant elements. Such an element would define a parallel spinor on $F_4/\text{Spin}(9)$ but, since the Ricci tensor of $\mathbb{O}P^2$ is not zero (see section 3.3.1), the spinor must vanish by the Lichnerowicz formula [66] $D^2 = \nabla^2 + \frac{1}{4}R_{\text{scal}}$. Then Δ_{16}^{+} as a Spin(9) representation is given by equation (2.34), and Δ_{16}^{-} is the unique irreducible Spin(9)-representation of dimension 128. □

Remarks
1. From the above we see that the Rarita-Schwinger field is given by the negative spinor bundle of $\mathbb{O}P^2$.
2. The 11-bein can also be seen from the nine-dimensional bundle in another way. It is an element of $SL(9)/\text{Spin}(9)$, which indeed has dimension 44.
3. In [57] it was shown that the bosonic degrees of freedom, g and C, can be assembled into an $E_{8(+8)}$-valued vielbein in eleven dimensions. As $E_{8(+8)}$ is the global symmetry of the two factors in the symmetry group $E_{8(+8)} \times SO(16)$, it would be interesting to see whether the discussion of the second factor here might be related to [57].

Thus we have

Theorem 2.9. *The massless fields of M-theory are encoded in the spinor bundle of* $\mathbb{O}P^2$.

Next we have the following observation

Proposition 2.10. *There is no obstruction to having sections of the* Spin(9) *bundle on a manifold of dimension greater than or equal to 9.*

Proof. This has been observed in [37] and [52]. The real dimension of the spinor representation S is $d = 2^{\frac{m}{2}}\alpha$, where α depends on the dimension and consequently on the condition on the spinors (i.e. Majorana, Weyl), so that the maximum dimension m of the manifold M for which $d = m$ is $m = 8$. When $m > 8$ the dimensions cease to be equal anymore, $\dim S > \dim M$. The obstruction bundle is the bundle of spinors of unit norm whose fiber is $SO(d)$. As the only nontrivial homotopy group of the sphere S^{d-1} in degrees less than or equal to $d-1$ is $\pi_{d-1}(S^{d-1}) = \mathbb{Z}$, the primary– and only– obstruction lies in $H^d(M^m;\mathbb{Z})$. For $n \geq 9$ one has $d > m$, so that the obstruction is zero. □

Remark. We can use the twisted geometric Dirac operator introduced in [65] to give another interpretation of the the Euler triplet in M-theory. Since $\mathbb{O}P^2$ is Spin, the identity representation of F_4 is the index of the the Dirac operator on $\mathbb{O}P^2$ twisted by the homogeneous vector bundle induced by the representation of Spin(9). Calling this representation $\mathcal{V}$ and consider the representations S_+^* and S_-^*, dual to half-Spin representations S^+ and S^-, respectively. Applying [65], we have the twisted Dirac operator

$$D_{S(\mathbb{O}P^2)\otimes\mathcal{V}} : L^2\left(F_4 \times_{\text{Spin}(9)} (S_+^* \otimes \mathcal{V}\right) \longrightarrow L^2\left(F_4 \times_{\text{Spin}(9)} (S_-^* \otimes \mathcal{V}\right) \;, \tag{2.35}$$

whose index is

$$\text{Index}\, D_{S(\mathbb{O}P^2)\otimes\mathcal{V}} = \text{Id}\,(F_4)\;. \tag{2.36}$$

2.3 Supersymmetry

We have seen that supersymmetry is created from bundles on $\mathbb{O}P^2$. More precisely, this is really due to parallel spinors on $\mathbb{R}^9$. In fact, this can be seen from another angle. There is a supersymmetric structure inside of V^9, which makes $\mathfrak{f}_4$ into a Lie superalgebra. The connection comes from the relation between real Killing spinors on a space and the parallel spinors on the cone over that space [13]. Let us see how this works, following [33]. The eight-sphere S^8 with the standard round metric g has a Spin bundle $S(S^8)$ on which there is an action of the Clifford bundle $C\ell(TS^8)$ and a Spin(8) invariant inner product. A Killing spinor over S^8 is a nonzero section ϵ of $S(S^8)$ which satisfies, for all vector fields X,

$$\nabla_X \epsilon = \lambda X \cdot \epsilon, \tag{2.37}$$

with Killing constant $\lambda \in \mathbb{R}$. In local coordinates, using $\lambda = \frac{1}{2}$, this is

$$(\nabla_\mu - \frac{1}{2}\gamma_\mu)\epsilon = 0. \tag{2.38}$$

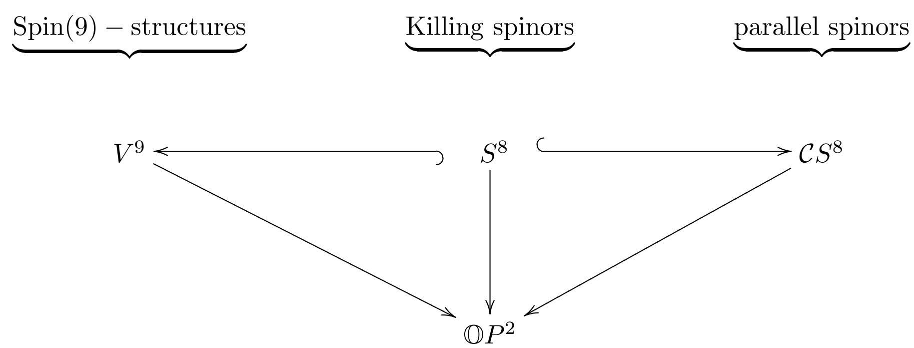
The cone on S^8 is $\mathcal{C}S^8 = \mathbb{R}^9 \setminus \{0\}$. The metric $dr^2 + r^2 g$, however, can be extended to the origin, so that we can take the cone to be $\mathbb{R}^9$. Thus

$$\begin{array}{ccc} \text{Parallel spinors on } \mathbb{R}^9 & \Longleftrightarrow & \text{Real Killing spinors on } S^8 \\ \nabla_\mu \hat{\epsilon} = 0 & \longleftrightarrow & (\nabla_\mu - \frac{1}{2}\gamma_\mu)\epsilon = 0\ . \end{array} \tag{2.39}$$

The observation in [33] is that this decomposition, written as $\mathfrak{l} = \mathfrak{l}_0 \oplus \mathfrak{l}_1$, has the interpretation in terms of Killing superalgebras on S^8: $\mathfrak{l}_0 = \mathfrak{so}(9)$ is the Lie algebra of isometries of S^8 and $\mathfrak{l}_1 = S^+$ is the space of Killing spinors on S^8. The latter comes, via the cone construction, from real Killing spinors on the cone $\mathbb{R}^9$. Hence

$$\mathfrak{f}_4 = \left\{\text{Even isometries on } S^8\right\} \oplus \left\{\text{Odd isometries on } S^8\right\}, \tag{2.40}$$

and the Lie brackets for the super Lie algebra are satisfied [33]. Schematically (abusing notation of fiber vs. bundle), we have



(2.41)

From this and the earlier discussion we therefore have

Proposition 2.11. *$\mathfrak{f}_4$ is the Lie superalgebra of a sphere inside V^9. Hence the unification of the fields in M-theory and their supersymmetry can be seen from the eight-sphere over $\mathbb{O}P^2$.*

We can give another interpretation to the Euler triplet in terms of spinors. We have seen in the Remark containing equation (2.36) that the Euler triplet can be interpreted as an index of a twisted Dirac operator. The kernel of the operator (2.35) is the space of harmonic spinors, which is the desired representation up to sign [65]. Therefore, we get another characterization of the supergravity multiplet.

Proposition 2.12. *The identity representation of F_4 encoding the supergravity multiplet is the space of twisted harmonic spinors on $\mathbb{O}P^2$.*

Comparison to generation of supersymmetry from lattices. Next we discuss the relation, similarities and differences between the above process of generating fermions and supersymmetry and the one through which the various closed superstring theories are derived starting from the closed bosonic string [24]. The spectrum of the bosonic string contains no fermions and so these are generated on a lattice in internal space. In [24] the following procedure was created:

(1) Seek an internal symmetry group G containing the little group Spin(8). This is achieved by a torus compactification T/Λ_G, with Λ_G the root lattice of a simply-laced group G of rank 8.
(2) Declare the diagonal subgroup $SO(8)_{\rm diag} \subset SO(8) \times {\rm Spin}(8)$ as the new transverse group. This implies that the spinor representations of Spin(8) describe fermionic states.
(3) Extend $SO(8)_{\rm diag}$ to the full Lorentz group $SO(1,9)_{\rm diag}$.
(4) Impose the supersymmetry requirement that a consistent truncation on the spectrum of the bosonic string be performed. This requires a regular embedding so that the root lattice $\Lambda_{\rm Spin(8)}$ is contained in Λ_G.

The only simply-laced groups which contain Spin(8) as a subgroup in a regular embedding are E_6, E_7 and E_8. Requiring the rank to be 8 then singles out $G = E_8 \times E_8$. Then [24]:
(i) the choice $G_L = G_R = E_8 \times E_8$ for the groups in the left and right sector gives the two type II string theories;
(ii) the same choice with a truncation on the left-moving sector gives the $E_8 \times E_8$ heterotic string;
(iii) the choice $G_L = E_8 \times E_8$, $G_R = {\rm Spin}(32)/\mathbb{Z}_2$ together with a truncation on the left-moving sector gives the ${\rm Spin}(32)/\mathbb{Z}_2$ heterotic string theory.

Now let us compare the similarities and the differences of our case with the above formalism of [24]. We record this in the following remarks.

Remarks
1. The M-theory case is geometric and involves nontrivial topology. This is in contrast to the torus in a vertex-operator-like construction in the string case.
2. F_4 is not simply-laced and hence cannot be involved in the internal torus construction.
3. In both cases, the fermions are generated from the internal space. However, in [24], fermionic states are generated from bosonic states. In fact, in our case, the whole massless spectrum of eleven-dimensional supergravity is generated from the two Spin bundles in dimension sixteen. This method of generating fermions is very different from the string formalism of generating fermions from torus compactification.
4. The signature $\sigma(M^{4k})$ of an oriented $4k$-dimensional manifold M^{4k} is an invariant of the manifold. Moreover, the signature of $-M^{4k}$, which is M^{4k} with the orientation reversed, is equal to the negative of the signature of M^{4k}: $\sigma(-M^{4k}) = -\sigma(M^{4k})$. Since $\sigma(\mathbb{O}P^2) \neq 0$, from (3.6), this means that there is no orientation-reversing homeomorphism $f : \mathbb{O}P^2 \to \mathbb{O}P^2$ such that $f_*[\mathbb{O}P^2] = -[\mathbb{O}P^2]$. The implication is, in particular, that we cannot impose any such involution on the fermions.
5. The construction in M-theory using F_4 involves the Spin bundle of $\mathbb{O}P^2$. This means that in twenty-seven dimensions the theory will have fermions. This is a major difference from the bosonic string case, which has no fermions in its spectrum. How can this be compatible with the bosonic string and with the classification of supersymmetry in general? In relation to the bosonic string, it could be that there is an involution that kills the fermions in a way similar to what happens to the C-field in going from M-theory to the heterotic string, or from the conjectural bosonic M-theory in [48] to bosonic string theory. Let us now consider the second part of the question related to the classification of supersymmetry. The action in twenty-seven dimensions might involve fermions, and so the question is whether this will/can be supersymmetric. That is something to be investigated. However, for now we can say that being supersymmetric does not contradict the no-go theorems in supersymmetry as those involve the Lorentz condition. The sixteen-dimensional internal space can be taken to have either all time or all space signature, i.e. $(16,0)$ or $(0,16)$, respectively. We then get for the signature (t,s) of the 27-dimensional space

$$(1,10) + (0,16) = (1,26) \quad (2.42)$$
$$(1,10) + (16,0) = (17,10). \quad (2.43)$$

The first one obviously wildly violates the no-go theorems but the second does not as $t-s=7$. Note that a version of eleven-dimensional M-theory with $s-t=7$ was constructed by Hull [50]. While supersymmetry seems mathematically admissible, it is far from obvious what to make physically of so many such time directions. We do not address this here.

3 $\mathbb{O}P^2$ Bundles

Having motivated $\mathbb{O}P^2$ bundles in M-theory, we now carry on with our proposal and construct such bundles in eleven dimensions. We study the properties of the $\mathbb{O}P^2$ bundle as well as the associated F_4 bundle and give some consistency conditions. As bundles are characterized by characteristic classes and genera, we 'compare' the structure of the base and that of the total space. For that purpose we start with discussing the relevant genera of the fiber.

3.1 Genera of $\mathbb{O}P^2$

A genus is a function on the cobordism ring Ω (see section 5 for cobordism). More precisely, it is a ring homomorphism $\varphi : \Omega \otimes R \to R$, where R is any integral domain over $\mathbb{Q}$. It could be $\mathbb{Z}$, $\mathbb{Z}_p$ or $\mathbb{Q}$ itself. Genera in general have expressions given in terms of characteristic classes. Two important 'modern' genera are the elliptic genus Φ_{ell} and the Witten genus Φ_W. The first is characterized by two parameters, denoted ε and δ, whose various values give different specializations of Φ_{ell}. Special values of the parameters correspond to more 'classical' genera. The values $\delta=\varepsilon=1$ leads to the L-genus $L : \Omega \otimes \mathbb{Q} \to \mathbb{Q}$, and the values $\delta = -\frac{1}{8}$, $\varepsilon = 0$ leads to the $\widehat{A}$-genus $\widehat{A} : \Omega \otimes \mathbb{Q} \to \mathbb{Q}$. Depending on the type of cobordism considered, Ω and also R can vary. For instance, when the manifolds are Spin then the $\widehat{A}$-genus is an integer and so $\widehat{A} : \Omega^{\text{Spin}} \otimes \mathbb{Z} \to \mathbb{Z}$. The Witten genus is defined for special manifolds, namely ones with a String structure or $BO\langle 8\rangle$-structure, and those are the manifolds that satisfy $\frac{1}{2}p_1 = 0$, where p_1 is the first Pontrjagin class of the tangent bundle. The Witten genus is a map $\Phi_W : \Omega^{BO\langle 8\rangle} \otimes R \to MF = R[E_4, E_6]$, where MF is the ring of modular forms generated by the Eisenstein series E_4 and E_6, and R is usually $\mathbb{Q}$ or $\mathbb{Z}$. We describe this more precisely below.

It is natural to ask what the values of the elliptic genus and of the Witten genus of $\mathbb{O}P^2$ are. First, however, we consider the classical genera.

3.1.1 The classical genera

We give the following specialization.

Lemma 3.1. *1. The $\widehat{A}$-genus of $\mathbb{O}P^2$ is zero, $\widehat{A}[\mathbb{O}P^2] = 0$.*

2. The L-genus of $\mathbb{O}P^2$ is u^2, where u is the generator of $H^8(\mathbb{O}P^2; \mathbb{Z})$.

Proof. We start with the $\widehat{A}$-genus. There is more than one proof for this. The first one uses the Lichnerowicz theorem [66] which states that manifolds with positive scalar curvature have zero $\widehat{A}$-genus. We will verify this for $\mathbb{O}P^2$ by direct calculation. The degree 16 part of the $\widehat{A}$-genus is (see [44] or [4])

$$\widehat{A}_{16} = \frac{1}{2^8.3^3.5^2.7}\left(\frac{127}{2^6}p_1^4 - \frac{113}{2^4.3}p_1^2 p_2 + \frac{4}{3}p_1 p_3 + \frac{13}{2^3.3}p_2^2 - \frac{1}{2}p_4\right). \tag{3.1}$$

Inserting the values of the Pontrjagin classes of $\mathbb{O}P^2$, namely $p_1 = 0 = p_3$, $p_2 = 6u$ and $p_4 = 39u^2$ (see the Appendix), we get

$$\widehat{A}_{16}[\mathbb{O}P^2] = \frac{1}{2^{11}.3^4.5^2.7}\left(13(36u^2) - 12(39u^2)\right), \tag{3.2}$$

which indeed gives zero.

For the L-genus, we also verify by direct calculation. The degree 16 part of L is (see [44] or [4])

$$L_{16} = \frac{1}{3^4.5^2.7}\left(381p_4 - 71p_3p_1 - 19p_2^2 + 22p_2p_1^2 - 3p_1^2\right). \tag{3.3}$$

Again, inserting the values of the Pontrjagin classes for $\mathbb{O}P^2$ we get

$$L_{16}[\mathbb{O}P^2] = \frac{u^2}{3^2.5^2.7}(13.127 - 2^2.19), \tag{3.4}$$

which is indeed equal to u^2. □

Remarks From Lemma 3.1 we deduce the following.
1. The index of the Dirac operator for spinors – as sections of the spinor bundle $S\mathbb{O}P^2$ – is zero

$$\text{Index}\, D = \langle\, \widehat{A}_{16}(\mathbb{O}P^2)\ ,\ [\mathbb{O}P^2]\,\rangle := \int_{\mathbb{O}P^2} \widehat{A} = 0\ , \tag{3.5}$$

where $[\mathbb{O}P^2]$ is the fundamental cycle and $\langle\ ,\ \rangle$ the Kronecker pairing of cohomology with homology. Positive scale curvature implies that there are no zero modes.
2. The signature of $\mathbb{O}P^2$ is

$$\sigma(\mathbb{O}P^2) = \langle\, L_{16}(\mathbb{O}P^2)\ ,\ [\mathbb{O}P^2]\,\rangle := \int_{\mathbb{O}P^2} L = \int_{\mathbb{O}P^2} u^2 = 1, \tag{3.6}$$

since u^2 is the integral generator for the top cohomology.

3.1.2 The Witten genus

Next we consider another genus. The Witten genus can be defined in such a way that $\Phi_W(M^{4k}) = \text{num}_k E_{2k}$ for $k > 1$ and $\Phi_W(M^4) = 0$, where $\text{num}_n/d_n = B_{2n}/4n$ is the given numerator, with num_n and d_n relatively prime, and B_{2n} the even Bernoulli numbers. This means that we use

$$\sigma(z,\tau) = z \exp\left(-\sum_{k\geq 2} \frac{2}{(2k)!}\frac{B_{2k}}{4k} E_{2k}(\tau) z^{2k}\right). \tag{3.7}$$

This then comes from a logarithm of some series $Q(z)$, which is found from

$$\begin{aligned}
\log(Q(z)) &= \sum_{k\geq 2} 2\frac{\text{num}_k}{d_k} E_{2k}\frac{z^{2k}}{(2k)!} \\
&= \sum_{k\geq 2} 2\frac{G_{2k}z^{2k}}{(2k)!} \\
&= \log\left(\frac{z}{\sigma(z,\tau)}\right),
\end{aligned} \tag{3.8}$$

which shows that $Q(z) = z/\sigma(z,\tau)$. The ring of modular forms for the full modular group is (cf. [6]) $MF = \mathbb{Z}[E_4, E_6, \Delta]/(E_4^3 - E_6^2 - 1728\Delta)$, where $\Delta = q\prod_n (1-q^n)^{24}$. There is a convenient collection of manifolds $\{M^{4n}\}$ that generate the rational cobordism ring $\Omega \otimes \mathbb{Q}$ [67]. The advantage of this basis is that each M^{4n} has a single nonzero Pontrjagin class, the top one $p_n = d_n(2n-1)!m$ where m generates $H^{4n}(M^{4n})$. By inspecting the Bernoulli numbers we can see that the first four terms in d_n are $24, 240, 504, 480$. This is enough for working up until real dimension 16. Let us illustrate this for $\mathbb{H}P^2$, for which the answer is known [67].

Lemma 3.2. *The Witten genus of the quaternionic projective plane is* $\Phi_W(\mathbb{H}P^2) = E_4/288$.

Proof. The basis spaces in dimension 8 are M^8 and $(M^4)^2 = M^4 \times M^4$. Form the linear combination $X = aM^8 + b(M^4)^2$. The idea is to match the Pontrjagin numbers of X with those of $\mathbb{H}P^2$, which determines the values of the constants a and b. The top Pontrjagin classes of M^4 and M^8 are given by

$$\begin{aligned} p_1(M^4) &= d_1 m = 2^3 \cdot 3m \\ p_2(M^8) &= 3! d_2 m = 2^5 \cdot 3^2 \cdot 5m^2, \end{aligned} \tag{3.9}$$

so that

$$p_2((M^4)^2) = p_1(M^4)p_1(M^4) = 2^6 \cdot 3^2 m^2 \tag{3.10}$$

from which we can deduce the Pontrjagin numbers for X

$$\begin{aligned} p_2[X] &= ap_2[M^8] + bp_2[(M^4)^2] = a2^6 \cdot 3^2 + b2^5 \cdot 3^2 \cdot 5. \\ p_1^2[X] &= ap_1^2[M^8] + bp_1^2[(M^4)^2] = 0 + b2^7 \cdot 3^2. \end{aligned} \tag{3.11}$$

The zero in the second equation is due to the fact that, by definition, the manifold M^8 has nono-zero Pontrjagin class only in top degree. Now setting $p_2[X] = p_2[\mathbb{H}P^2] = 7$ and $p_1^2[X] = p_1^2[\mathbb{H}P^2] = 4$ in (3.11) gives the values $a = b = (2^5 \cdot 3^2)^{-1}$, so that $X = \frac{1}{288}\left((M^4)^2 + M^8\right)$.

We now calculate the Witten genus of X, which is the same as that of $\mathbb{H}P^2$. Since the coefficient of $(M^4)^2$ is not zero, then by [67], $\Phi_W(X)$ takes its values in $\mathbb{Q}[E_4, E_6]$, i.e. without the discriminant. In dimension 8, modular forms are generated by E_4 and not E_6. Since the Witten genus is linear then one gets $\Phi_W(\mathbb{H}P^2) = \Phi_W(X) = E_4/288$. ☐

Note that, in contrast, the Witten genus of the Cayley plane is zero.

Theorem 3.3. *The Witten genus of* $\mathbb{O}P^2$ *is zero,* $\Phi_W(\mathbb{O}P^2) = 0$.

Proof. $\mathbb{O}P^2$ has positive scalar curvature, so its $\widehat{A}$-genus is zero $\widehat{A}(\mathbb{O}P^2) = 0$. $\mathbb{O}P^2$ is also a String manifold, so its Witten genus $\Phi_W(\mathbb{O}P^2) : \Omega_{16}^{BO\langle 8\rangle} = \pi_{16}MO\langle 8\rangle \to \pi_* eo_2 = MF_*$ must be a modular form for $SL(2,\mathbb{Z})$ of weight equals to half its dimension [104], i.e. 8. What modular forms do we have? The ring of integral modular forms is (cf. [6])

$$MF_* = \mathbb{Z}[E_4, E_6, \Delta]/(2^6 \cdot 3^3\Delta - E_4^3 + E_6^2) \tag{3.12}$$

where $E_4 \in MF_4$, $E_6 \in MF_6$, and $\Delta \in MF_{12}$. Thus the only modular form of weight 8 is E_4^2. However the form of the Eisenstein series is $E_4 = 1+$ higher terms, so that c_4 does not start with zero. Therefore $\Phi_W(\mathbb{O}P^2) = 0$. ☐

Remarks

1. It is a conjecture of Stolz [93] that if a smooth closed String manifold M admits a Riemannian metric with positive Ricci curvature then the *rational* Witten genus $\Phi_W(M)$ vanishes. Thus $\mathbb{O}P^2$ satisfies the statement of this conjecture. However, the integral Witten genus of $\mathbb{O}P^2$ vanishes. This integral case is left open in [93].

2. The rational version of the result of Theorem 3.3 could also be deduced from the fact that $\mathbb{O}P^2$ is a homogeneous space of a compact semi-simple Lie group for which the Witten genus vanishes [93].

3. The Witten genus is, a priori, associated with the power series

$$\Phi_W(x) = \exp\left(\sum_{k>0,\ 2|k} \frac{2}{k!} G_k x^k \right), \tag{3.13}$$

where

$$G_k(\tau) = \frac{-B_k}{2k} + \sum_{n=1}^{\infty} \left(\sum_{d|n} d^{k-1} \right) q^k \ . \tag{3.14}$$

For $k=2$, $G_2 = -\frac{1}{24} + q + 3q^2 + \cdots$, and is not modular but rather

$$G_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 G_2(\tau) + \frac{i}{4\pi} c(c\tau+d)\ . \tag{3.15}$$

The String condition, $\frac{1}{2}p_1(X)=0$, ensures that the series begins with G_4, hence avoiding problems with the non-modularity of G_2.

We give some consequences of Theorem 3.3. The expansion of the Witten genus in terms of the $\widehat{A}$-genus is given by

$$\begin{aligned} \Phi &= q^{-8} \cdot \widehat{A}\left(M, \bigotimes_{n=2m+1>0} \Lambda_{-q^n}(T\mathbb{O}P^2) \otimes \bigotimes_{n=2m>0} S_{q^n}(T\mathbb{O}P^2) \right) \\ &= q^{-8}\left(\widehat{A}(\mathbb{O}P^2) - \widehat{A}(\mathbb{O}P^2, T\mathbb{O}P^2)\cdot q + \widehat{A}(\mathbb{O}P^2, \Lambda^2\mathbb{O}P^2 + T\mathbb{O}P^2)\cdot q^2 + \cdots \right), \end{aligned} \tag{3.16}$$

where $\widehat{A}(\mathbb{O}P^2, V) := \langle \widehat{A}(\mathbb{O}P^2)\cdot \mathrm{ch}(V\otimes\mathbb{C}), [\mathbb{O}P^2]\rangle$. Using the fact that $\Phi_W(\mathbb{O}P^2)=0$ we set each term in the q-expansion to zero. In particular, the first two terms give $\widehat{A}(\mathbb{O}P^2)=0$, which we already know, as well as $\widehat{A}(\mathbb{O}P^2, T\mathbb{O}P^2)=0$. Since $\mathbb{O}P^2$ is a Spin manifold then we can use the Atiyah-Singer index theorem to deduce

Proposition 3.4. *The index of the Dirac operator and the index of the Rarita-Schwinger operator on $\mathbb{O}P^2$ are both zero, i.e.* Index $D=0$ *and* Index $D_{RS}=0$.

Remark. Even though the index of the Dirac and the Rarita-Schwinger operators (and all higher Dirac operators coupled to powers of the tangent bundle) are zero, the index of the Kostant operator (2.5) is not zero. It is in fact given by the left hand side of the formula (2.2).

3.1.3 The elliptic genus

Next we consider the elliptic genus $\Phi_{\mathrm{ell}} : \Omega_*^{BO\langle 8\rangle}\otimes\mathbb{Q} \to \mathbb{Q}[\delta,\varepsilon]$, where the generators δ and ε have degrees 4 and 8, respectively. We start with the known result.

Lemma 3.5. *For the quaternionic projective plane, $\Phi_{\mathrm{ell}}(\mathbb{H}P^2)=\varepsilon$.*

Proof. The calculation of the elliptic genus relies on a standard idea where $\mathbb{H}P^2$ is identified with another space X such that the two spaces have the same image in $\Omega_8 \otimes \mathbb{Q}$. The advantage is that X is built out of 'basis spaces', whose genera are easily computable. The spaces $\mathbb{C}\mathrm{P}^2, \mathbb{C}\mathrm{P}^4, \mathbb{C}\mathrm{P}^6, \cdots$, form a basis sequence for $\Omega \otimes \mathbb{Q}$, i.e.

$$\Omega \otimes \mathbb{Q} = \mathbb{Q}[\,\mathbb{C}P^2, \mathbb{C}P^4, \mathbb{C}P^6, \cdots] \tag{3.17}$$

as a graded polynomial ring, where $\mathbb{C}P^{2k}$ has weight given by the dimension $4k$. For eight real dimensions we thus have

$$\Omega_8 \otimes \mathbb{Q} = \left\{\mathbb{C}P^2 \times \mathbb{C}P^2, \mathbb{C}P^4\right\}. \tag{3.18}$$

Now we have to write $\mathbb{H}P^2$ in terms of the basis (3.18)

$$X = a(\mathbb{C}P^2 \times \mathbb{C}P^2) + b\mathbb{C}P^4. \tag{3.19}$$

To determine the constants a and b we equate the Pontrjagin numbers of $\mathbb{H}P^2$ and X. For the former we know that $p_1 = 2$ and $p_2 = 7$, so that the Pontrjagin numbers are

$$p_1^2[\mathbb{H}P^2] = 4, \qquad\qquad p_2[\mathbb{H}P^2] = 7. \tag{3.20}$$

To calculate the Pontrjagin numbers for X we first calculate those for each of the basis spaces. From $p_1(\mathbb{C}P^2) = 3$, $p_1(\mathbb{C}P^4) = 5$, and $p_2(\mathbb{C}P^4) = 10$, and using the Whitney product formula for the Pontrjagin classes, which for two four-manifolds M and M reads

$$\begin{aligned} p_2[M \times N] &= p_1[M]\; p_1[N] \\ p_1^2[M \times N] &= 2p_1[M]\; p_1[N], \end{aligned} \tag{3.21}$$

so that we get

$$p_2[\mathbb{C}P^2 \times \mathbb{C}P^2] = 9, \qquad\qquad p_1^2[\mathbb{C}P^2 \times \mathbb{C}P^2] = 18. \tag{3.22}$$

Now we compare the Pontrjagin numbers

$$\begin{aligned} p_1^2[X] &= ap_1^2[\mathbb{C}P^2 \times \mathbb{C}P^2] + bp_1^2[\mathbb{C}P^4] = 18a + 25b \\ p_2[X] &= ap_2[\mathbb{C}P^2 \times \mathbb{C}P^2] + bp_2[\mathbb{C}P^4] = 9a + 10b. \end{aligned} \tag{3.23}$$

Equating these numbers to the ones in (3.20) gives $a = 3$ and $b = -2$. Now we compute the elliptic genus of $X = 3(\mathbb{C}P^2 \times \mathbb{C}P^2) - 2\mathbb{C}P^4$. The elliptic genera of $\mathbb{C}P^{2k}$ are computable via the generating formula

$$\sum_{k=0}^{\infty} \Phi_{\mathrm{ell}}(\mathbb{C}P^{2k}) t^{2k} = \frac{1}{\sqrt{1 - 2\delta t^2 + \varepsilon t^4}}. \tag{3.24}$$

Using the expansion of the right hand side as $1 + \delta t^2 + \frac{1}{2}(3\delta^2 - \varepsilon)t^4 + \cdots$, we identify the elliptic genera

$$\Phi_{\mathrm{ell}}(\mathbb{C}P^2) = \delta, \qquad\qquad \Phi_{\mathrm{ell}}(\mathbb{C}P^4) = \frac{1}{2}(3\delta^2 - \varepsilon). \tag{3.25}$$

With this we now can calculate the elliptic genus of $\mathbb{H}P^2$

$$\begin{aligned} \Phi_{\mathrm{ell}}([\mathbb{H}P^2]) &= 3\Phi_{\mathrm{ell}}\left([\mathbb{C}P^2 \times \mathbb{C}P^2]\right) - 2\Phi_{\mathrm{ell}}\left([\mathbb{C}P^4]\right) \\ &= 3\delta^2 - 2 \cdot \frac{1}{2}(3\delta^2 - \varepsilon), \end{aligned} \tag{3.26}$$

which indeed is ε. □

Theorem 3.6. *The elliptic genus of the Cayley plane is* $\Phi_{\rm ell}(\mathbb{O}P^2) = \varepsilon^2$.

Proof. There are several ways to prove this. The first one is to use the idea of cobordism as in the proof of Lemma 3.5. However, we can simply apply a result from [45]. Since $\mathbb{O}P^2$ is a connected homogeneous space of a compact connected Lie group F_4, and since $\mathbb{O}P^2$ is oriented and admits a Spin structure, then the normalized elliptic genus $\Phi_{\rm norm} := \Phi_{\rm ell}/\varepsilon^2$ is a constant modular function

$$\Phi_{\rm norm}(\mathbb{O}P^2) = \sigma(\mathbb{O}P^2). \tag{3.27}$$

Thus we immediately get the result. □

Remark. The F_4-equivariant elliptic genus of $\mathbb{O}P^2$, i.e. the index of the loop signature operator $\mathcal{S}$

$$\text{Index}_{F_4}\mathcal{S} = L(\mathbb{O}P^2)\text{ch}\left\{\bigotimes_{n\geq 1} S_{q^n}(T\mathbb{O}P^2)\bigotimes_{m\geq 1}\Lambda_{q^m}(T\mathbb{O}P^2)\right\}[\mathbb{O}P^2]\;, \tag{3.28}$$

is then just the signature $\sigma(\mathbb{O}P^2)$.

3.1.4 The Ochanine genus

We next consider the Ochanine genus [75], which is a generalization of the elliptic genus in such a way that it involves q-expansions. The Ochanine genus is a ring homomorphism

$$\Phi_{\rm och} : \Omega_*^{\rm spin} \longrightarrow KO_*(\text{pt})[[q]], \tag{3.29}$$

from the Spin cobordism ring to the ring of power series with coefficients in

$$KO_*(\text{pt}) = \mathbb{Z}\left[\eta, \omega, \mu, \mu^{-1}\right] / \left(2\eta, \eta^3, \eta\omega, \omega^2 - 2^2\mu\right), \tag{3.30}$$

where $\eta \in KO_1, \omega \in KO_4$, and $\mu \in KO_8$ are generators of degrees 1, 4, and 8, respectively, and are given by the normalized Hopf bundles $\gamma_{_{\mathbb{R}P^1}} - 1$, $\gamma_{_{\mathbb{H}P^1}} - 1$, $\gamma_{_{\mathbb{O}P^1}} - 1$ (viewed as real vector bundles) over the real, quaternion, and Cayley projective lines $\mathbb{R}P^1 = S^1$, $\mathbb{H}P^1 = S^4$, and $\mathbb{O}P^1 = S^8$.

For a manifold M^m of dimension m, corresponding to the projection map $\pi^{M^m} : M^m \to \text{pt}$ there is the Gysin map $\pi_!^{M^m} : KO(M^m) \to KO^m(\text{pt}) = KO_m(\text{pt})$. Now consider a real vector bundle E on M^m and form the following combination of exterior powers and symmetric powers of E

$$\Theta_q(E) = \sum_{i\geq 0}\Theta^i(E)q^i = \bigotimes_{n\geq 1}\left(\Lambda_{-q^{2n-1}}(E)\otimes S_{q^n}(E)\right), \tag{3.31}$$

which, since it is multiplicative under Whitney sum, can be considered as an exponential map $\Theta_q : KO(M^m) \to KO(M^m)[[q]]$. Now specialize E to be the reduced tangent bundle $\widetilde{TM^m}$, which is $TM^m - m$. Then the Ochanine genus is defined to be [75] [60]

$$\begin{aligned}\Phi_{\rm och}(M^m) &:= \sum_{i\geq 1}\Phi^i_{\rm och}(M^m)q^i \\ &= \sum_{i\geq 0}\pi_!^{M^m}\left(\Theta^i(\widetilde{TM^m})\right)q^i \\ &= \theta(q)^{-m}\langle\, \Theta_q(TM^m), [M^m]_{KO}\,\rangle \in KO_m(\text{pt})[[q]],\end{aligned} \tag{3.32}$$

where $[M^m]_{KO} \in KO_m(M^m)$ denotes the Atiyah-Bott-Shapiro orientation [9] of M^m, $\langle\ ,\ \rangle : KO^i(X) \otimes KO_j(X) \to KO_{j-i}$ is the Kronecker pairing, and

$$\theta(q) := \Theta_q(1) = \prod_{n\geq 1} \frac{1-q^{2n-1}}{1-q^{2n}} = 1 - q + q^2 - 2q^3 \pm \cdots \quad \in \mathbb{Z}[[q]], \tag{3.33}$$

is the Ochanine genus of the trivial line bundle.

The degree zero part $\Theta^0(E)$ is a trivial real line bundle, and corresponds to the Atiyah invariant $\Phi^0_{\text{och}}(M) = \pi_!^{M^m}(1) = \langle 1, [M^m]_{KO}\rangle = \alpha(M^m)$. The cobordism invariant $\alpha \in KO_m$ [8] can be interpreted as the index of a family of operators associated to M^m parametrized by S^m [46]. Thus the α-invariant is the classical value of the Ochanine genus in the same way that the $\widehat{A}$-genus and the L-genus are the classical values of the elliptic genus corresponding, respectively, to

$$\begin{aligned} \delta = \widehat{A}(\mathbb{C}P^2) &= -\frac{1}{8}, \qquad & \varepsilon = \widehat{A}(\mathbb{H}P^2) = 0, \quad \text{and} \\ \delta = L(\mathbb{C}P^2) &= 0, & \varepsilon = L(\mathbb{H}P^2) = 1. \end{aligned} \tag{3.34}$$

The Ochanine genus is related to the restriction $\Phi_{\text{ell,int}}$ to Ω_*^{spin} of the universal elliptic genus $\Phi_{\text{ell,uni}} : \Omega_*^{SO} \to \mathbb{Q}[[q]]$, whose parameters are

$$\begin{aligned} \delta &= -\frac{1}{8} - 3\sum_{n\geq 1}\left(\sum_{d|n,\ d \text{ odd}} d\right) q^n = -\frac{1}{8} + q - \text{expansion}, \\ \varepsilon &= \sum_{n\geq 1}\left(\sum_{d|n,\ \frac{n}{d} \text{ odd}} d^3\right) q^n = 0 + q - \text{expansion}. \end{aligned} \tag{3.35}$$

More precisely, $\Phi_{ell,int} = Ph \circ \Phi_{\text{och}} : \Omega_*^{\text{spin}} \to \mathbb{Z}[[q]]$, where Ph is the Pontrjagin character

$$Ph : KO^*(X) \xrightarrow{\otimes\mathbb{C}} K^*(X) \xrightarrow{\text{ch}} H^{**}(X;\mathbb{Q}), \tag{3.36}$$

which can be thought of as the analog for real vector bundles of the Chern character for complex vector bundles.

We now check the value of Φ_{och} for $\mathbb{O}P^2$.

Theorem 3.7. *The Ochanine genus of $\mathbb{O}P^2$ is $\Phi_{\text{och}}(\mathbb{O}P^2) = \varepsilon^2\mu^2$.*

Proof. The Ochanine genus $\Phi_{\text{och}}(\mathbb{O}P^2)$ is the map $\Omega_{16}^{\text{spin}} \to KO_{16}[[q]]$. Note that $\Omega_{16}^{\text{spin}} = \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$ and that $KO_{16}(\text{pt}) = \mathbb{Z}$ with generator μ^2. The image $\Phi_{\text{och}}(\Omega_{16}^{\text{spin}})$ is the set of all modular forms of degree 16 and weight 8 over $KO_{16} = \mathbb{Z}$. Let $M^\Gamma(KO_{16})$ be the graded ring of modular forms over KO_{16} for Γ, a subgroup of finite index in $SL(2;\mathbb{Z})$. For $M_*^\Gamma(\mathbb{Z}) = \mathbb{Z}[\delta_0, \varepsilon]$, where $\delta_0 = -8\delta \in M_2^\Gamma(\mathbb{Z})$ and δ and $\varepsilon \in M_4^\Gamma(\mathbb{Z})$ are the generators in (3.35), we have

$$\begin{aligned} M^\Gamma(KO_{16}) &\cong KO_{16} \otimes M_*^\Gamma(\mathbb{Z}) \\ &= \mathbb{Z} \otimes \mathbb{Z}[\delta_0, \varepsilon] \,. \end{aligned} \tag{3.37}$$

Then a modular form of degree 16 and weight 8 can be written in a unique way as a polynomial $P(\delta_0, \varepsilon)$ of weight 8 with integer coefficients. Still applying the construction in [75], the Ochanine genus in our case is

$$\Phi_{\text{och}}(\mathbb{O}P^2) = \left(a_0(\mathbb{O}P^2)\delta_0^4 + a_1(\mathbb{O}P^2)\delta_0^2\varepsilon + a_2(\mathbb{O}P^2)\varepsilon^2\right)\mu^2, \tag{3.38}$$

with uniquely defined homomorphisms , for $i = 1, 2, 3$,

$$a_i \cdot \mu^2 : \Omega_{16}^{\text{spin}} = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \longrightarrow KO_{16} = \mathbb{Z} \,. \tag{3.39}$$

The integers a_i can be determined as follows. We have already seen that the lowest coefficient is given by the Atiyah invariant. Since $\mathbb{O}P^2$ admits a Riemannian metric of positive scalar curvature (see section 3.3.1) then, from [46], $\alpha(\mathbb{O}P^2) = 0$, and hence we have determined that $a_0(\mathbb{O}P^2) = 0$. Another way of seeing this is to notice that for manifolds of dimension $4k$, the Atiyah invariant is essentially the $\widehat{A}$-genus, which, by Lichnerowicz theorem [66], vanishes for a manifold with positive scalar curvature. The highest coefficient, $a_2(\mathbb{O}P^2)$, is given by the Ochanine k-invariant, which in this case is just the signature

$$a_2(\mathbb{O}P^2) = \sigma(\mathbb{O}P^2) = 1. \tag{3.40}$$

It remains to calculate a_1. This is given by the first KO-Pontrjagin class Π_1

$$a_1(\mathbb{O}P^2) = \Pi_1(T\mathbb{O}P^2) = -\Lambda^1(T\mathbb{O}P^2 - 16), \tag{3.41}$$

which is just $-(T\mathbb{O}P^2 - 16)$. The KO-Pontjagin classes are defined as follows [5]. For an n-dimensional vector bundle ξ over a space X, $\Pi_u(\xi) \in KO^0(X)$ are defined by

$$(1+t)^n \sum_{k=0}^{\infty} \frac{t^k}{(1+t)^{2k}} \Pi_k(\xi) = \sum_{k=0}^{\infty} t^k \Lambda^k(\xi) \,. \tag{3.42}$$

For $k = 1$ this gives the first KO-Pontrjagin class used in (3.41). Alternatively, we can look at the q-components of Φ_{och} from the first line of equation (3.32) and get

$$\begin{aligned} \Phi^0_{\text{och}}(\mathbb{O}P^2) &= \langle 1 \,,\, [\mathbb{O}P^2]_{KO} \rangle = \alpha(\mathbb{O}P^2) \\ \Phi^1_{\text{och}}(\mathbb{O}P^2) &= \langle -\Pi_1(\mathbb{O}P^2) \,,\, [\mathbb{O}P^2]_{KO} \rangle = \langle -(T\mathbb{O}P^2 - 16) \,,\, [(\mathbb{O}P^2)]_{KO} \rangle. \end{aligned} \tag{3.43}$$

We still have to calculate a_1. We use the topological Riemann-Roch theorem (see [95]) which states that for M a closed Spin manifold and $x \in \widetilde{KO}^*(M)$, then $Ph\langle x, [M]_{KO}\rangle = \langle \widehat{A}(M)Ph(x), [M]\rangle_H$, where $\langle \ ,\ \rangle_H$ is the Kronecker pairing on cohomology. Taking $M = \mathbb{O}P^2$ and $x = T\mathbb{O}P^2$, we get for a_1

$$\langle \, \widehat{A}(\mathbb{O}P^2) Ph(T\mathbb{O}P^2) \,,\, [\mathbb{O}P^2] \, \rangle_H, \tag{3.44}$$

which is zero because, as we have seen, $\widehat{A}(\mathbb{O}P^2) = 0$. □

Corollary 3.8. *The q-expansion of $\Phi_{\text{och}}(\mathbb{O}P^2)$ is $\mu^2 q^2(1 + 16q + 120q^2 + 576q^3 + \cdots)$.*

Proof. We expand ε from (3.35) to get

$$\varepsilon = q + 8q^2 + 28q^3 + 64q^4 + 126q^5 + 224q^6 + 344q^7 + \cdots \,, \tag{3.45}$$

so that

$$\varepsilon^2 = q^2 + 16q^3 + 120q^4 + 576q^5 + \cdots \,. \tag{3.46}$$

□

Remarks

1. The $\widehat{A}$-genus is obtained from the Ochanine genus by setting ε to zero, or equivalently setting q to zero in our case. We can see from the above expressions in either Theorem 3.7 or from Corollary 3.8 that indeed we do reproduce $\widehat{A}(\mathbb{O}P^2) = 0$.

2. The signature is obtained from the Ochanine genus by replacing δ by -1, ε by 1, ω by 2, and μ by 1. Applying these transformations to $\Phi_{\text{och}}(\mathbb{O}P^2)$ gives 1, which is indeed $\sigma(\mathbb{O}P^2)$, cf. (3.6).

3.2 $\mathbb{O}P^2$ bundles over eleven-manifolds

Consider the fiber bundle $E \to Y^{11}$ with fiber $\mathbb{O}P^2$ and structure group F_4. There is a universal bundle of this type. $\mathbb{O}P^2$ bundles over Y^{11} are pullbacks of the universal bundle

$$\mathbb{O}P^2 = F_4/\mathrm{Spin}(9) \longrightarrow B\mathrm{Spin}(9) \longrightarrow BF_4 \tag{3.47}$$

by the classifying map $f : Y^{11} \to BF_4$. In this paper we will consider the diagram

$$\begin{array}{ccccc} \mathbb{O}P^2 & \longrightarrow & M^{27} & & \\ & & \downarrow{\scriptstyle \pi} & \searrow & \\ & & Y^{11} & \xrightarrow[f]{} & BF_4\,. \end{array} \tag{3.48}$$

Note that the map from M^{27} to BF_4 can be $f\pi$ and this will be useful later in section 5. We first have the following.

Proposition 3.9. *The obstruction to existence of a section of an $\mathbb{O}P^2$ fiber bundle over an eleven-dimensional manifold Y^{11} lies in $H^9(Y^{11};\mathbb{Z})$, $H^{10}(Y^{11};\mathbb{Z}_2)$ and $H^{11}(Y^{11};\mathbb{Z}_2)$.*

Proof. For a fiber bundle $F \to E \to B$, the existence to having a section lies in the groups $H^r(B;\pi_{r-1}(F))$ for all nonzero $r \in \mathbb{N}$. In our case, $\mathbb{O}P^2$ has $\pi_i = 0$ for $i \leq 7$, so that the first obstruction is in $H^9\left(Y^{11};\pi_8(\mathbb{O}P^2)\right)$, which is $H^9(Y^{11};\mathbb{Z})$. The next two nontrivial homotopy groups of $\mathbb{O}P^2$, both are $\mathbb{Z}_2$, in dimension 9 and 10 so that the obstructions are in $H^{10}(Y^{11};\mathbb{Z}_2)$ and $H^{11}(Y^{11};\mathbb{Z}_2)$. $\mathbb{O}P^2$ has further nontrivial homotopy groups but that would bring us to $H^{\geq 12}$, which are zero for an eleven-manifold. □

Remarks
1. The first obstruction $H^9(Y^{11};\mathbb{Z})$ is called the primary obstruction.
2. Note that the primary obstruction is a $\mathbb{Z}$-class whereas the secondary obstructions are $\mathbb{Z}_2$-classes.

In forming bundles with $\mathbb{O}P^2$ as fibers, we are forming bundles of $BO\langle 8\rangle$-manifolds over Y^{11}. We will next investigate the relation between structures on Y^{11}, on the fiber $\mathbb{O}P^2$, and on the total space M^{27}.

3.3 Relating Y^{11} and M^{27}

3.3.1 Geometric consequences: the curvatures

We start with the Riemannian geometry of $\mathbb{O}P^2$. Consider the following three subsets of $\mathbb{O}^3$

$$U_1 = \{1\} \times \mathbb{O} \times \mathbb{O}, \qquad U_2 = \mathbb{O} \times \{1\} \times \mathbb{O}, \qquad U_3 = \mathbb{O} \times \mathbb{O} \times \{1\}, \tag{3.49}$$

and form the union $\mathcal{U} := U_1 \cup U_2 \cup U_3$. Define the following relation $\sim$ on $\mathbb{O}^3$:

$$[a,b,c] \sim [d,e,f] \longleftrightarrow \text{ there exists } \lambda \in \mathbb{O} - \{0\} \text{ such that } a = d\lambda, b = e\lambda, c = f\lambda. \tag{3.50}$$

The relation $\sim$ on $\mathcal{U}$ is an equivalence relation [7]. The Cayley projective plane is the set of equivalence classes of $\mathcal{U}$ by the equivalence relation $\sim$,

$$\mathbb{O}P^2 = \mathcal{U}/\sim\ . \tag{3.51}$$

Keeping in mind $\mathbb{O} \cong \mathbb{R}^8$, an atlas on $\mathbb{O}P^2$ can be taken to be $(U_i/\sim, \phi_i)$, $i = 1, 2, 3$, where the homeomorphisms ϕ_i are given by

$$\begin{aligned}
\phi_1 &: U_1/\sim \longrightarrow \mathbb{R}^{16}, \qquad \phi_1([a,b,c]) = (b,c),\\
\phi_2 &: U_2/\sim \longrightarrow \mathbb{R}^{16}, \qquad \phi_2([a,b,c]) = (a,c),\\
\phi_3 &: U_3/\sim \longrightarrow \mathbb{R}^{16}, \qquad \phi_3([a,b,c]) = (a,b)\ .
\end{aligned} \tag{3.52}$$

The transition functions $\phi_i \circ \phi_j^{-1} : \mathbb{R}^{16} \to \mathbb{R}^{16}$

$$\begin{aligned}
\phi_1 \circ \phi_2^{-1}(a,b) &= (a^{-1}, ba^{-1}) = \phi_2 \circ \phi_1^{-1}(a,b),\\
\phi_1 \circ \phi_3^{-1}(a,b) &= (ba^{-1}, a^{-1}) = \phi_3 \circ \phi_1^{-1}(a,b),\\
\phi_2 \circ \phi_3^{-1}(a,b) &= (b^{-1}, ab^{-1}) = \phi_3 \circ \phi_2^{-1}(a,b)
\end{aligned} \tag{3.53}$$

are diffeomorphisms and hence we get a smooth 16-dimensional manifold structure for $\mathbb{O}P^2$ [41].

The metric on $\mathbb{O}P^2$ can be obtained from the metrics on the charts which are compatible with respect to transition maps. The metric, with (u,v) coordinate functions, is [41]

$$ds^2 = \frac{|du|^2(1+|v|^2) + |dv|^2(1+|u|^2) - 2\mathrm{Re}[(u\overline{v})(dvd\overline{u})]}{(1+|u|^2+|v|^2)^2}. \tag{3.54}$$

In terms of a coordinate frame $\{e_1, \cdots, e_8, f_1, \cdots, f_8\}$ where $e_i = \partial_i$ and $f_i = \partial_{i+8}$ for $1 \leq i \leq 8$, the unmixed components of the metric are

$$\begin{aligned}
g(e_i, e_j) &= \delta_{ij} \frac{1+|v|^2}{(1+|u|^2+|v|^2)^2},\\
g(f_i, f_j) &= \delta_{ij} \frac{1+|u|^2}{(1+|u|^2+|v|^2)^2}.
\end{aligned} \tag{3.55}$$

The mixed components, in terms of the standard orthonormal basis $\{x_1, \cdots, x_8\}$ of $\mathbb{O}$ are

$$g(e_i, f_j) = g(f_i, e_j) = -\frac{\langle (u\overline{v})x_j, x_i\rangle}{(1+|u|^2+|v|^2)^2}. \tag{3.56}$$

Using the identity

$$R_{\mu\nu\lambda\sigma} = R_{\mu\nu\lambda}{}^{\sigma} = \Gamma^{\sigma}_{\mu\lambda;\nu} - \Gamma^{\sigma}_{\nu\lambda;\mu} = \frac{1}{2}\left[g_{\nu\lambda;\mu\sigma} + g_{\mu\sigma;\nu\lambda} - g_{\mu\lambda;\nu\sigma} - g_{\nu\sigma;\mu\lambda}\right], \tag{3.57}$$

the only non-vanishing components of the Riemann tensor are [41]

$$\begin{aligned}
&R(e_i, e_j, e_i, e_j) = -R(e_i, e_j, e_j, e_i) = 4,\\
&R(f_i, f_j, f_i, f_j) = -R(f_i, f_j, f_i, f_j) = 4,\\
&R(e_i, e_j, f_k, f_l) = R(f_k, f_l, e_i, e_j) = -\langle x_i\overline{x}_l, x_j\overline{x}_k\rangle + \langle x_j\overline{x}_l, x_i\overline{x}_k\rangle,\\
&R(e_i, f_j, e_k.f_l) = R(f_i, e_j, f_k, e_l) = \langle x_i\overline{x}_j, x_k\overline{x}_l\rangle,\\
&R(f_i, e_j, e_l, f_k) = -\langle x_i\overline{x}_j, x_k\overline{x}_l\rangle\ .
\end{aligned} \tag{3.58}$$

It can now be easily seen that both the Ricci curvature tensor $R_{\mu\nu}$ and the Ricci scalar R are both positive.

Taking M^{27} to be the total space of an $\mathbb{O}P^2$ bundle over Y^{11} then the Ricci curvatures of the two spaces are related. In particular, since $\mathbb{O}P^2$ is a compact Riemannian manifold which has a metric of positive Ricci curvature on which the Lie group F_4 acts by isometries, and the base Y^{11} is a compact manifold, it follows from O'Neill's formulae for submersions (see [18]) that

Proposition 3.10. *If the base Y^{11} admits a metric of positive Ricci curvature, then so does the 27-dimensional space.*

This is shown by taking a certain metric on M^{27} with totally geodesic fibers ([18]) and then shrinking the $\mathbb{O}P^2$ fibers à la Kaluza-Klein. This is a specific case of the $\mathbb{O}P^2$ analog of Proposition 3.6 in [93].

3.3.2 Topological consequences: the higher structures

We ask the question whether topological conditions on Y^{11}, namely having Spin, String, or Fivebrane structure [90] [91], will lead to (similar) structures on M^{27}. The answer to such a question is possible because we know about the (non-)existence of these structures on $\mathbb{O}P^2$.

The condition $\lambda := \frac{1}{2}p_1 = 0$ for lifting the structure group of the tangent bundle to String(n) is related to the condition $W_7 = 0$ for orientation with respect to either the $p = 2$ integral Morava K-theory $K(2)$ or Landweber's elliptic cohomology theory $E(2)$ [61]. The first condition implies the second, but the converse is not true, a counterexample being $X^{10} = S^2 \times S^2 \times \mathbb{C}\mathrm{P}^3$ [61]. Thus if we assume the String orientation, then we are already guaranteed the W_7 orientation, and so the discussion and constructions in [61] [62] [63] [85] for ten-dimensional string theory apply. The condition $\lambda = 0$ can be extended from ten to eleven dimensions and vice versa. This is because for $Y^{11} = X^{10} \times S^1$ the first Pontrjagin classes are related as (using bundle notation) $p_1(TX^{10} \oplus TS^1) = p_1(TX^{10}) + p_1(TS^1)$, but for dimensional reasons $p_1(TS^1) = 0$ so that we have $p_1(Y^{11}) = p_1(X^{10})$. Thus the String condition can be translated from M-theory to string theory and back as desired.

There is no cohomology in degree four for $\mathbb{O}P^2$, so we immediately have

Proposition 3.11. *$\mathbb{O}P^2$ admits a $BO\langle 8\rangle$-structure.*

Remark. If Y^{11} is a $BO\langle 8\rangle$-manifold, i.e. is $MO\langle 8\rangle$-orientable, then it has an $MO\langle 8\rangle$ homology fundamental class,

$$[Y^{11}]_{MO\langle 8\rangle} \in MO\langle 8\rangle_{11}(Y^{11}). \tag{3.59}$$

Any integral expression will involve this class. This would also enter the construction of the $BO\langle 8\rangle$ partition function.

We would like to check to what extent we can know the cohomology of the total space M^{27} in terms of the cohomology of the base Y^{11}, given that we know the cohomology of the fiber $\mathbb{O}P^2$. One way to detect this is by using the Serre spectral sequence for the bundle

$$E_2^{p,q} = H^p\left(Y^{11}, H^q(\mathbb{O}P^2)\right) \Rightarrow H^{p+q}(M^{27})\ . \tag{3.60}$$

Consider the case of a product $M^{27} = \mathbb{O}P^2 \times Y^{11}$, i.e. when the bundle is trivial. In this case, we can use the Künneth theorem which for a field F (e.g. $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}_p$) is

$$H^n(M^{27}; F) = H^n(\mathbb{O}P^2 \times Y^{11}; F) \cong \bigoplus_{i+j=n} H^i(\mathbb{O}P^2; F) \otimes_F H^j(Y^{11}; F), \tag{3.61}$$

and for a ring R (e.g. $\mathbb{Z}$), using the universal coefficient theorem, is the split sequence

$$0 \longrightarrow \bigoplus_i \left(H^i(\mathbb{O}P^2; R) \otimes_R H^{n-i}(Y^{11}; R)\right) \longrightarrow H^n(\mathbb{O}P^2 \times Y^{11}; R) \longrightarrow$$

$$\bigoplus_i \mathrm{Tor}_R\left(H^i(\mathbb{O}P^2; R); H^{n-i-1}(Y^{11}; R)\right) \longrightarrow 0\ . \tag{3.62}$$

Then we have, for any coefficients C,

$$H^n(\mathbb{O}P^2 \times Y^{11}; C) \cong \bigoplus_p H^p\left(Y^{11}; H^{n-p}(\mathbb{O}P^2; C)\right) . \tag{3.63}$$

Since the cohomology of $\mathbb{O}P^2$ is nonzero only in degrees 8 and 16, we therefore get get

Proposition 3.12.

$$H^n(\mathbb{O}P^2 \times Y^{11}; C) \cong H^{n-8}(Y^{11}; C) \oplus H^{n-16}(Y^{11}; C) . \tag{3.64}$$

We next consider the case when the bundle is not trivial. A simplification is made if coefficients are taken so that the cohomology of the fiber is trivial in those coefficients. The torsion ('bad') primes for F_4 are 2 and 3, so that one might expect that those are the primes that do not cause such a simplification. It will turn out that this is true only for $p = 3$, as we now show. We first show that $p = 3$ occurs and then that it is the only one.

The cohomology of the classifying spaces of Spin(9) and F_4 with $\mathbb{Z}_p$ coefficients, $p = 2, 3$, are as follows. The cohomology ring of BF_4 with coefficients in $\mathbb{Z}_2$ is given by the polynomial ring [19]

$$H^*(BF_4; \mathbb{Z}_2) = \mathbb{Z}_2\left[x_4, x_6, x_7, x_{16}, x_{24}\right] , \tag{3.65}$$

where x_i are polynomial generators of degree i related by the Steenrod square operation $Sq^i : H^n(BF_4; \mathbb{Z}_2) \to H^{n+i}(BF_4; \mathbb{Z}_2)$ as

$$x_6 = Sq^2 x_4, \qquad x_7 = Sq^3 x_4, \qquad x_{24} = Sq^8 x_{16} . \tag{3.66}$$

$H^*(BF_4; \mathbb{Z}_3)$ is generated by x_i for i = 4, 8, 9, 20, 21, 25, 26, 36, 48, with the structure of a polynomial algebra [98]. Considering $p = 3$, this is

$$H^*(BF_4; \mathbb{Z}_3) \cong \mathbb{Z}_3[x_{36}, x_{48}] \otimes \left(\mathbb{Z}_3[x_4, x_8] \otimes \left\{1, x_{20}, x_{20}^2\right\} + \Lambda(x_9) \otimes \mathbb{Z}_3[x_{26}] \otimes \{1, x_{20}, x_{21}, x_{25}\}\right) . \tag{3.67}$$

The generators can be chosen to be related by the Steenrod power operations at $p = 3$, $P^i : H^n(BF_4; \mathbb{Z}_3) \to H^{n+4i}(BF_4; \mathbb{Z}_3)$, as

$$\begin{array}{lll} x_8 = P^1 x_4 & x_9 = \beta x_8 = \beta P^1 x_4 & x_{20} = P^3 P^1 x_4 \\ x_{21} = \beta P^3 P^1 x_4 & x_{25} = P^4 \beta P^1 x_4 & x_{26} = \beta P^4 \beta P^1 x_4 \end{array} \tag{3.68}$$

and $x_{48} = P^3 x_{36}$. If we restrict to degrees ≤ 11 then we have the truncated polynomial

$$H^*(BF_4; \mathbb{Z}_3) \cong \mathbb{Z}_3[x_4, x_8] + \Lambda(x_9). \tag{3.69}$$

The classes coming from BSpin(9) are just the Stiefel-Whitney classes in the $\mathbb{Z}_2$ case and the Pontrjagin classes (reduced mod 3) in the integral ($\mathbb{Z}_3$ case). These are actually not much different from the classes of BSpin(11). Explicitly, at $p = 2$ the cohomology ring of BSpin(9) is given by the polynomial ring [77]

$$H^*(B\text{Spin}(9); \mathbb{Z}_2) = \mathbb{Z}_2\left[w_4, w_6, w_7, w_8, w'_{16}\right] , \tag{3.70}$$

where w_i is the restriction of the universal Stiefel-Whitney class, and w'_{16} is the Stiefel-Whitney class $\omega_{16}(\Delta_{\text{Spin}(9)})$ of the spin representation $\Delta_{\text{Spin}(9)} : \text{Spin}(9) \to O(16)$. At $p = 3$, $H^*(B\text{Spin}(9); \mathbb{Z}_3)$ is generated by the first four Pontrjagin classes [98]

$$H^*(B\text{Spin}(9); \mathbb{Z}_3) = \mathbb{Z}_3[p_1, p_2, p_3, p_4], \quad \deg(p_i) = 4i . \tag{3.71}$$

Let us look at $\mathbb{Z}_3$ coefficients. From (3.69) and (3.71) we see that $H^9(B\text{Spin}(9);\mathbb{Z}_3) = 0$ while $H^9(BF_4;\mathbb{Z}_3) \neq 0$, which implies that the map $H^9(BF_4;\mathbb{Z}_3) \to H^9(B\text{Spin}(9);\mathbb{Z}_3)$ cannot be injective. Therefore, at $p = 3$ the Serre spectral sequence is not trivial. In the case of $\mathbb{Z}_2$, the situation is reversed, this time in degree eight: $H^8(B\text{Spin}(9);\mathbb{Z}_2) \neq 0$ and $H^8(BF_4;\mathbb{Z}_2) = 0$.

Now we proceed with the uniqueness by applying the results in [58]. The cohomology of $\mathbb{O}P^2$ is $H^*(\mathbb{O}P^2;C) = C[x]/x^3$, $|x| = \deg x = 8$, as an algebra. Then, requiring that the Serre fibering $\mathbb{O}P^2 \to M^{27} \to Y^{11}$ be trivial over C implies for the E_2-term

$$E_2 = H^*(Y^{11};C) \otimes_C C[x]/x^3. \tag{3.72}$$

Now the E_9 term is $E_{|x|+1} = E_2$ and the fibering is nontrivial if and only if we have a nonzero differential

$$d_9(1 \otimes x) \neq 0. \tag{3.73}$$

If $d_9(1 \otimes x) = a \otimes 1 \neq 0$ then

$$0 = d_9(1 \otimes x^3) = 3(a \otimes x^2). \tag{3.74}$$

Hence the characteristic of C must not be relatively prime to 3, the degree of the ideal in the cohomology ring of $\mathbb{O}P^2$. Therefore, we have

Proposition 3.13. *The Serre spectral sequence for the fiber bundle $\mathbb{O}P^2 \to M^{27} \to Y^{11}$ is nontrivial only for cohomology with $\mathbb{Z}_3$ coefficients.*

We will make use of this and also say more in section 4.1 – see theorem 4.1 and the discussion around it.

Remarks

1. Note that a priori the characteristic of C should divide the order of the Weyl group of F_4. Since $|W(F_4)| = 2^7 \cdot 3^2$ then the candidate primes are 2 and 3 only. We have seen that among these two numbers only the prime 3 gives a nontrivial Serre fibration.

2. Note that the primes 2 and 3 are also the torsion primes of F_4. It is not the case in general that the torsion primes for G are exactly the same primes that appear in the factorization of $|W(G)|$.

The total space of an $\mathbb{H}P^2$ bundle over a Spin manifold is again a Spin manifold. However, the same property is not automatically true for total spaces of $\mathbb{O}P^2$ bundles over $BO\langle 8\rangle$-manifolds. The reason is that while the tangent bundle T along the fibers of the universal bundle (4.23) has a Spin structure — since $H^i(B\text{Spin}(9)) = 0$ for $i = 1, 2, 3$ — it has no $BO\langle 8\rangle$ structure. This can be explained as follows [55]. The complementary roots of $i : \text{Spin}(9) \hookrightarrow F_4$ are the 16 roots $\frac{1}{2}(\pm x_1 \pm x_2 \pm x_3 \pm x_4)$, where x_i denote the standard linear forms on $\mathfrak{so}(9)$. Using Borel-Hirzebruch methods [20], the total Pontrjagin class $p(T) \in H^*(B\text{Spin}(9);\mathbb{Q})$ is given by the product $\frac{1}{4}\prod(\pm x_1 \pm x_2 \pm x_3 \pm x_4)$, so that the first Pontrjagin class is

$$p_1(T) = 2(x_1^2 + x_2^2 + x_3^2 + x_4^2) \in H^4(B\text{Spin}(9);\mathbb{Q}). \tag{3.75}$$

This is of course invariant under the Weyl group of Spin(9). However, it is also invariant under $W(F_4)$, and hence belongs to $H^4(BF_4;\mathbb{Q}) = \mathbb{Q}$ as well. This shows that $p_1(T)$ can be considered as coming from the universal space for Spin(9) or F_4.

Proposition 3.14. *If Y^{11} admits a String structure then so does M^{27} provided that there is no contribution from the degree four class from BF_4.*

Proof. We have the $\mathbb{O}P^2$ bundle over Y^{11} with total space M^{27}

$$\begin{array}{ccc} M^{27} & \xrightarrow{\tilde{f}} & B\mathrm{Spin}(9) \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle Bi} \\ Y^{11} & \xrightarrow[f]{} & BF_4 \end{array}, \tag{3.76}$$

which gives the decomposition $TM^{27} = \pi^* TY^{11} \oplus \tilde{f}^* T$, and so the tangential Pontrjagin class is

$$p_1(M^{27}) = \pi^* \left(p_1(Y^{11}) + f^* p_1(T) \right) . \tag{3.77}$$

In the case Y^{11} is a 3-connected $BO\langle 8\rangle$-manifold, we have that $H^4(Y^{11};\mathbb{Z})$ is free and $\pi^* : H^4(Y^{11};\mathbb{Z}) \to H^4(M^{27};\mathbb{Z})$ is an isomorphism. Thus M^{27} is also a $BO\langle 8\rangle$-manifold if and only if $f^*\overline{x}_4 = 0 \in H^4(Y^{11};\mathbb{Z})$, where $\overline{x}_4 \in H^4(BF_4;\mathbb{Z})$ is the generator. Therefore we have shown that M^{27} is String if and only if G_4 in M-theory gets no contribution from BF_4. □

Remarks

1. The quantization condition for the field strength G_4 in M-theory is known [101]. Since this field does not seem to get a contribution from a class in BF_4, the condition in Proposition 3.14 seems reasonable. In some sense we could view the presence of such a degree four class as an anomaly which we have just cured. Alternatively, one can discover that this is not as serious as it might seem— see the more complete discussion in section 4.2.

2. We connect the above discussion back to cobordism groups. While there is no transfer map from $\Omega_{11}^{\langle 8\rangle}(BF_4)$ to $\Omega_{27}^{\langle 8\rangle}$, there is a transfer map after killing $\overline{x}_4$ [55]. Denoting by [2] $BF_4\langle\overline{x}_4\rangle$ the corresponding classifying space that fibers over BF_4, killing $\overline{x}_4$ is done by pulling back the path fibration $PK(\mathbb{Z},4) \to K(\mathbb{Z},4)$ with a map $\overline{x}_4 : BF_4 \to K(\mathbb{Z},4)$ realizing $\overline{x}_4$. The corresponding transfer map is $\Omega_{11}^{\langle 8\rangle}(BF_4\langle\overline{x}_4\rangle) \to \Omega_{27}^{\langle 8\rangle}$.

Next, for the higher structures we have

Proposition 3.15. **1.** *In order for M^{27} to admit a Fivebrane structure, the second Pontrjagin class of Y^{11} should be the negative of that of $\mathbb{O}P^2$, i.e. $p_2(TY^{11}) = -p_2(T\mathbb{O}P^2) = -6u$.*
2. $\widehat{A}(M^{27}) = 0$*, irrespective of whether or not the $\widehat{A}$-genus of Y^{11} is zero.*
3. $\Phi_W(M^{27}) = 0$.
4. $\Phi_{\mathrm{ell}}(M^{27}) = 0$.

Proof. For part (1) note that if Y^{11} admits a Fivebrane structure then M^{27} does not necessarily admit such a structure. This is because the obstruction to having a Fivebrane structure is $\frac{1}{6}p_2$ [91] but we know that $\frac{1}{6}p_2(\mathbb{O}P^2) = u \neq 0$. However, we can choose Y^{11} appropriately so that it conspires with $\mathbb{O}P^2$ to cancel the obstruction and lead to a Fivebrane structure for M^{27}. Noting that the tangent bundles are related as $TM^{27} = TY^{11} \oplus T\mathbb{O}P^2$, then considering the degree eight part of the formula (see [71])

$$p(E \oplus F) = \sum p(E)p(F) \quad \text{mod 2−torsion.} \tag{3.78}$$

we get for our spaces

$$p_2(TY^{11} \oplus T\mathbb{O}P^2) = p_1(TY^{11})p_1(T\mathbb{O}P^2) + p_2(TY^{11}) + p_2(T\mathbb{O}P^2) \quad \text{mod 2−torsion.} \tag{3.79}$$

[2] This is the analog of the String group when $G = \mathrm{Spin}$, in the sense that it is the 3-connected cover.

Since we have $p_1(T\mathbb{O}P^2)=0$, then requiring that $p_2(TM^{27})=0$ leads to the constraint that $p_2(TY^{11})+p_2(T\mathbb{O}P^2)=0$ modulo 2-torsion.

For part (2) we use the multiplicative property of the $\widehat{A}$-genus for Spin fiber bundles to get

$$\widehat{A}(M^{27})=\widehat{A}(Y^{11})\widehat{A}(\mathbb{O}P^2). \tag{3.80}$$

Since the $\widehat{A}$-genus of $\mathbb{O}P^2$ is zero then the result follows.

For part (3) we use a result of Ochanine [74]. Taking the total space M^{27} and the base Y^{11} to be closed oriented manifolds, and since the fiber $\mathbb{O}P^2$ is a Spin manifold and the structure group F_4 of the bundle is compact, then the multiplicative property of the genus can be applied

$$\Phi_W(M^{27})=\Phi_W(\mathbb{O}P^2)\Phi_W(Y^{11}). \tag{3.81}$$

Now since we proved in Theorem 3.3 that $\Phi_W(\mathbb{O}P^2)=0$, it follows immediately that $\Phi(M^{27})$ is zero regardless of whether or not $\Phi_W(Y^{11})$ vanishes. Even more, $\Phi_W(Y^{11})$ is zero because Y^{11} is odd-dimensional. [3]

For part (4) we use the fact that the fiber is Spin and the structure group F_4 is compact and connected so we can apply the multiplicative property of the elliptic genus [74]

$$\Phi_{\rm ell}(M^{27})=\Phi_{\rm ell}(Y^{11})\Phi_{\rm ell}(\mathbb{O}P^2). \tag{3.82}$$

In this case the genus for the fiber is not zero (see Proposition 3.6) but the elliptic genus of Y^{11} is zero, again because of dimension. Therefore $\Phi_{\rm ell}(M^{27})=0$. □

We next consider the relation between the Ochanine genera of the base and of the total space. Recall from (3.29) that $\Phi_{\rm och}$ is a homomorphism from the Spin cobordism ring $\Omega_*^{\rm spin}$. Note that in order to define this genus for Y^{11} we need to consider $\Omega_{11}^{\rm spin}$. Unfortunately, this is known to be zero. Also, the image would be in $KO_{11}({\rm pt})[[q]]$, but because of the structure of $KO_*({\rm pt})$, being given by $(\mathbb{Z}_2,\mathbb{Z}_2,0,\mathbb{Z},0,0,0,\mathbb{Z})$, then $KO_{11}({\rm pt})=0$. We might conclude that the Ochanine genus is zero. However, we can try to get around this by noticing that for the special class of Y^{11} which are circle bundles over a ten-dimensional Spin manifold X^{10} we can define the Ochanine genus using the multiplicative property for this circle bundle. Then we can use the multiplicative property again but now for the $\mathbb{O}P^2$ bundle, thus obtaining $\Phi_{\rm och}(M^{27})$. This relies on the fact that $\Omega_r^{\rm spin}$ and $KO_r({\rm pt})$ are not zero for $r=1$ and 10, and is possible in the first case because $G=S^1$ and in the second case because the dimension of the fiber $\mathbb{O}P^2$ is divisible by 4.

We already have the Ochanine genus for $\mathbb{O}P^2$ in Theorem 3.7, so we need to find that genus for Y^{11}, taken as the total space of an S^1 bundle over X^{10}. We start with the circle.

Lemma 3.16. *The Ochanine genus of S^1 is $\Phi_{\rm och}(S^1)=\eta$, where η is the degree one generator in $KO_*({\rm pt})$.*

Proof. In dimension one, the classical version of the Ochanine genus is Atiyah's α-invariant

$$\alpha:\Omega_1^{\rm spin}=\mathbb{Z}_2\longrightarrow KO_1({\rm pt})=\mathbb{Z}_2\eta, \tag{3.83}$$

where η is given in (3.30). Since $\Omega_1^{\rm spin}$ is generated by S^1 then

$$\alpha(S^1)=\pi_!^{S^1}(1)=\eta. \tag{3.84}$$

[3] However, see the case when Y^{11} is a circle bundle at the end of this section.

This is the classical part of $\Phi_{\rm och}(S^1)$. We next consider the q-expansions. In dimension one, $\Phi_{\rm och}$ is a modular form over $\mathbb{F}_2$ and, applying the construction in [75] for dimension $8m+1$, it can be expressed as the polynomial

$$\Phi_{\rm och}(S^1) = \alpha(S^1) + q\text{−expansions}. \tag{3.85}$$

However, the expansions start at order $m = 1$, and so they do not contribute for the case of the circle. □

Remark. The circle in Lemma 3.16 is the one with the *nontrivial* Spin structure. This is also called the supersymmetric or the Ramond-Ramond Spin structure.

Next we consider the case of the ten-dimensional manifold X^{10}. In ten dimensions $\Phi_{\rm och}(\Omega_{10}^{\rm spin})$ is the set of all modular forms of degree ten and filtration ≤ 4 over the coefficients $KO_{10}({\rm pt}) = \mathbb{Z}_2\eta^2$ [75].

Proposition 3.17. *The Ochanine invariant of a Spin manifold X^{10} is determined by the signature of a twelve-dimensional manifold.*

Proof. In dimension $8m+2$, $\Phi_{\rm och}$ is a modular form over $\mathbb{F}_2$ and can be expressed as a polynomial in the basis form $\overline{\varepsilon} = \sum_{n\geq 1} q^{(2n-1)^2} \in \mathbb{Z}_2[[q]]$, which is the mod 2 reduction of the ε of equation (3.35) [75]. That is, in these dimensions $\delta_0 := -8\delta$ and ε from (3.35) can be replaced by their mod 2 reductions. Since $\delta_0 = -8\delta = 1 + 24q + \cdots$, so that $\delta_0 \equiv 1 \pmod 2$, then we consider the mod 2 reduction of ε

$$\overline{\varepsilon} = \sum_{n\geq 1} q^{(2n-1)^2} = q + q^9 + q^{25} + \cdots . \tag{3.86}$$

As in the case for the circle, there will be a classical part but also q-expansions in this case

$$\Phi_{\rm och}(X^{10}) = \alpha(X^{10}) + a(X^{10})\overline{\varepsilon}, \tag{3.87}$$

where $a(X^{10})$ is the KO-part of the Brown-Kervaire invariant [22] of X^{10}, and is defined as follows [75]. Since X^{10} is a ten-dimensional compact Spin manifold then $X^{10} \times S^1$ is the boundary of a compact Spin manifold Z^{12}. By [73], the signature of Z^{12} is divisible by 8 and the expression

$$k(X^{10}) = \frac{\sigma(Z^{12})}{8} \in \mathbb{Z}_2 \tag{3.88}$$

is a Spin cobordism invariant satisfying, for all Spin 8-manifolds M^8,

$$k(M^8 \times S^1 \times S^1) = \sigma(M^8) \mod 2. \tag{3.89}$$

The Ochanine k-invariant then takes two values, 0 and 1, depending on whether the signature is an even or odd multiple of 8, respectively. The Brown-Kervaire invariant is then the ring homomorphism $a : \Omega_{10}^{\rm spin} \to KO_{10} \otimes \mathbb{Z}_2$ and is defined to be $k(X^{10})\eta^2\mu \otimes 1 \in KO_{10} \otimes \mathbb{Z}_2$. Therefore,

$$\Phi_{\rm och}(X^{10}) = \alpha(X^{10}) + k(X^{10})\eta^2\mu\overline{\varepsilon}, \tag{3.90}$$

which is again of the form $\alpha + q$-expansions. □

We see that there are two cases to consider, depending on the value of $k(X^{10}) \in \mathbb{Z}_2$.

Proposition 3.18. *If $k(X^{10}) = 0 \in \mathbb{Z}_2$ then $\Phi_{\rm och}(X^{10}) = \alpha(X^{10})$. If $k(X^{10}) = 1 \in \mathbb{Z}_2$ then in $KO_{10} \otimes \mathbb{Z}_2$ we have*

$$\Phi_{\rm och}(X^{10}) = \alpha(X^{10}) + \eta^2\mu(q + q^9 + q^{25} + \cdots) . \tag{3.91}$$

We consider when the Ochanine genus is (non-)zero for Spin manifolds X^{10}.

Remarks

1. First, the Atiyah invariant is zero if the manifold admits a metric of positive scalar curvature [46]. This requires, for example, a flat or hyperbolic factor in X^{10}, for which there are many examples in compactifications in string theory.

2. If X^{10} is the boundary of an eleven-dimensional Spin manifold V^{11}, for instance if we can define the topological term $\int_{V^{11}} G_4 \wedge G_4 \wedge G_4$ as in [86], then we can choose the twelve-manifold to be $W^{12} = V^{11} \times \overline{S}^1$, where $\overline{S}^1$ is the circle with the nontrivial Spin structure. Then the signature of W^{12} is zero and therefore the Ochanine k-invariant is given by the Atiyah invariant.

3. Third, the Ochanine invariant vanishes if $H^5(X^{10};\mathbb{Z}_2) = 0$ [56]. In type IIA string theory there are no non-decomposable five-form field strengths since the Ramond-Ramond (RR) fields F_i are all of even rank and the Neveu-Schwarz field H_3 is of rank 3 (and its dual H_7 is of rank 7). However, we can think of composite fields, such as $\mathsf{F}_2 \cup H_3$, which can, in principle be a $\mathbb{Z}_2$-valued cohomology class. This is important because it is T-dual to the class $\mathsf{F}_3 \cup H_3$ entering the S-duality equation in type IIB string theory. If this is indeed the only such class available then this would have to be zero if we choose to impose $H^5(X^{10};\mathbb{Z}_2) = 0$ as the requirement for the vanishing of k.

4. The Ochanine genus vanishes if there are circle actions on either X^{10} or W^{12} [17].

- Suppose that the Spin manifold X^{10} admits a free or semi-free circle action of odd type, i.e. one that does not lift to the Spin bundle; then the Ochanine genus of X^{10} vanishes.
- Consider the twelve-dimensional Spin manifold W^{12} with $\partial W^{12} = X^{10} \times \overline{S}^1$. Suppose W^{12} admits a circle action α whose restriction to ∂W^{12} is $\lambda \times 1$, where λ is a circle action on X^{10}. Then if λ is an odd type action, then $\Phi_{\rm och}(X^{10}) = 0$.

5. The filtration of $\Phi_{\rm och}(X^{10})$ is exactly 4 if and only if $a(X^{10}) \neq 0$ [75].

6. X^{10} can be taken to be almost complex since, by [75], $\Phi_{\rm och}(\Omega_{10}^{SU}) = \Phi_{\rm och}(\Omega_{10}^{\rm spin})$.

7. A Spin homotopy equivalence between two Spin manifolds N and M is an orientation-preserving homotopy equivalence which maps the Spin structure of N to the Spin structure of M. The Ochanine/Brown-Kervaire invariant $k(X^{10})$ is an invariant of the Spin-homotopy type, but the Atiyah invariant is not, and hence the Ochanine genus is not an invariant of the Spin-homotopy type [56]. For us, this means that we cannot take homotopy and Spin equivalent spaces to X^{10}, Y^{11} and M^{27} without changing $\Phi_{\rm och}$.

There is another description of the Ochanine k-invariant [56], which we will use to make a connection to invariants appearing in M-theory.

Proposition 3.19. *1. The Ochanine invariant of a ten-dimensional closed Spin manifold X^{10} is equal to the mod 2 index of the Dirac operator twisted with the virtual bundle $TX^{10} - 2$.*

Proof. The family index theorem says that for E a real bundle in $KO^0(X^{10})$ an invariant $e \in \mathbb{Z}_2$ was defined by Atiyah and Singer [11] by $\langle E, [X^{10}]_{KO}\rangle = e\eta^2\mu \in KO_{10}$, which turns out to be the mod 2 index of the Dirac operator D_E of X^{10} twisted by the virtual bundle E,

$$e = \dim_{\mathbb{C}} \ker(D_E) \mod 2. \tag{3.92}$$

Applying [55], the k-invariant of X^{10} is the coefficient of q in the expression $f(q)^{-8}\Phi_{\rm och} \in KO_{10}[[q]]$, where

$$f(q) := \sum_{n\geq 1} q^{\binom{n}{2}} = 1 + q + q^3 + q^6 + \cdots , \tag{3.93}$$

since $\varepsilon/q = f(q^8) \bmod 2 = f(q)^8 \bmod 2$. We find the coefficient of q in the expansions. We have

$$f(q)^{-8} = (1+q+\cdots)^{-8} = 1-8q+\cdots . \tag{3.94}$$

The expansion for $\theta(q)$ takes the form

$$\theta(q) = \left(\frac{1-q}{1-q^2}\right)\left(\frac{1-q^3}{1-q^4}\right)\cdots = 1-q+\cdots , \tag{3.95}$$

so that $\theta(q)^{-10} = 1+10q+\cdots$. The expansion of $\Theta_q(E)$ is

$$\begin{aligned}\Theta_q(E) &= \Lambda_{-q}(E)\otimes S_{q^2}(E)+\cdots \\ &= \left(\sum_{k\geq 0}(-q)^k\Lambda^k(E)\right)\otimes\left(\sum_{k\geq 0}(q^2)^k S^k(E)\right) \\ &= 1-qE+\cdots .\end{aligned} \tag{3.96}$$

Putting the expressions (3.94), (3.95), (3.96) together we get

$$\begin{aligned}f(q)^{-8}\theta(q)^{-10}\Theta_q(TX^{10}) &= (1-8q+\cdots)(1+10q+\cdots)(1-qTX^{10}+\cdots) \\ &= 1+(2-TX^{10})q+\cdots .\end{aligned} \tag{3.97}$$

Extracting the coefficient of q we get the desired result.

Note that there is another way of obtaining this which makes use of the grading for $\Phi_{\rm och}$. Instead of looking at θ_q and Θ_q separately, we can look at the coefficient of q in $\Phi_{\rm och}(X^{10})$. This is

$$\Phi^1_{\rm och} = \langle -\Pi_1(TX^{10}), [X^{10}]_{KO}\rangle \in KO_{10} = \mathbb{Z}_2, \tag{3.98}$$

where Π_1 is the first KO-Pontrjagin class (defined by expression (3.42)), which is equal to $\Lambda^1(TX^{10}-10) = TX^{10}-10$. Substituting in (3.98) we get

$$\Phi^1_{\rm och} = \langle -(TX^{10}-10)\ ,\ [X^{10}]_{KO}\rangle, \tag{3.99}$$

which agrees with the product $\theta_q^{-10}\Theta_q(TX^{10}) = 1+(10-TX^{10})q+\cdots$. □

Remarks

1. Note that, interestingly, the bundle we get is the Rarita-Schwinger bundle with the dilatino and the spinor ghosts, as the Rarita-Schwinger field Ψ which leads to gauge invariance is a section of $SX^{10}\otimes(TX^{10}-2\mathcal{O})$, where $\mathcal{O}$ is the trivial line bundle. The (mod 2) index I_{RS} of the corresponding Dirac operator D_{RS} appears in the phase of the partition function [28] through the phase of the Pfaffian

$$Pf(D_{RS}) = (-1)^{I_{RS}/2}\ |Pf(D_{RS}|. \tag{3.100}$$

What is remarkable is that the 'quantum' Rarita-Schwinger operator appears directly in this formulation.

2. In [28] the main focus was the dependence of the partition function on the degree four class a coming from the E_8 gauge theory, but the contribution from I_{RS} was also given. The main example discussed in [28] is $X^{10} = \mathbb{H}P^2\times T^2$. Using the property given in equation (3.89), we can indeed see that the Ochanine k-invariant in this case is not zero. With T^2 taken as the product of two circles with nontrivial Spin structures we have

$$k(S^1\times S^1\times\mathbb{H}P^2) = \sigma(\mathbb{H}P^2) \mod 2, \tag{3.101}$$

which is equal to 1, since $\sigma(\mathbb{H}P^2)=1$.

3. In defining the elliptically refined partition function in M-theory and type IIA string theory, a real-oriented elliptic cohomology theory appears [61]. This is $EO(2)$, the fixed point, with respect to the formal inverse, of the theory $E\mathbb{R}(2)$, the real version of Landweber's theory $E(2)$, which has two generators v_1 and v_2. The orientation in this theory is shown to be given by w_4 [61]. It was also shown that when $w_4(X^{10})=0$, X^{10} has an $EO(2)$-orientation class $[X^{10}]_{EO(2)_{10}} \in EO(2)_{10}(X^{10})$, and for $x\in E^0(X^{10})$, the refined mod 2 index in this theory is

$$j(x)=\langle x\ \overline{x}\ ,\ [X^{10}]_{EO(2)}\rangle \in EO(2)_{10}=\mathbb{Z}_2[v_1^3 v_2^{-1}]. \tag{3.102}$$

Having determined the Ochanine genera for S^1 and X^{10}, we now proceed to determine the corresponding genus for the eleven-dimensional manifold Y^{11}.

Proposition 3.20. *Let Y^{11} be an eleven-dimensional Spin manifold which is the total space of a circle bundle over a ten-dimensional Spin manifold X^{10}. Then the Ochanine genus of Y^{11} is*

$$\Phi_{\rm och}(Y^{11})=\Phi_{\rm och}(X^{10})\cdot\alpha(S^1)\ . \tag{3.103}$$

Proof. Unlike other genera, the Ochanine genus does not in general enjoy a multiplicative property on fiber bundles. However, in the special case when the fiber is the circle with a $U(1)$ action $\Phi_{\rm och}$ does become multiplicative on the circle bundle [60]. We simply apply the result for $S^1\to Y^{11}\to X^{10}$ to get

$$\Phi_{\rm och}(Y^{11})=\Phi_{\rm och}(X^{10})\cdot\Phi_{\rm och}(S^1). \tag{3.104}$$

With $\Phi_{\rm och}(S^1)=\alpha(S^1)$, from Lemma 3.16, the result follows. □

Now that we have the Ochanine genus for Y^{11}, we go back and consider the original questions of finding the Ochanine genus of M^{27}, given that of Y^{11}.

Theorem 3.21. *The Ochanine genus of the total space M^{27} of an $\mathbb{O}P^2$ bundle over an eleven-dimensional compact Spin manifold Y^{11}, which is a circle bundle over a ten-dimensional Spin manifold X^{10}, is*

$$\Phi_{\rm och}(M^{27})=\Phi_{\rm och}(\mathbb{O}P^2)\cdot\Phi_{\rm och}(X^{10})\cdot\alpha(S^1)\ , \tag{3.105}$$

where $\Phi_{\rm och}(\mathbb{O}P^2)$ is given in Theorem 3.7 and $\Phi_{\rm och}(X^{10})$ is given in Proposition 3.18.

Proof. As mentioned in the proof of Proposition 3.20 above, $\Phi_{\rm och}$ is not in general multiplicative for fiber bundles. Again, interestingly, we are in a special case where such a property holds [60]. It is so because the dimension of the fiber $\mathbb{O}P^2$ is a multiple of 4, the structure group F_4 is a compact connected Lie group, and the base Y^{11} is a closed Spin manifold. Applying to the fiber bundle $\mathbb{O}P^2\to M^{27}\to Y^{11}$, and using proposition 3.20, then gives the formula in the theorem. □

Remark. The circle in Theorem 3.21 is the one with the nontrivial/nonbounding/supersymmetric/Ramond-Ramond Spin structure as in Lemma 3.16 (cf. the remark after that Lemma).

4 Terms in the Lifted Action

Having motivated and then constructed $\mathbb{O}P^2$ bundles in M-theory, we now turn to the discussion of some of the consequences. The most obvious question from a physics point of view is to characterize the corresponding 'theory' in 27 dimensions. We will not be able to achieve that, but we will be able to characterize some of the terms in the would-be action up in 27 dimensions. In the absence of a clear handle, we take the most economical approach and concentrate on the topological terms, which in any case are the terms we can trust. We also make some remarks on other terms as well.

4.1 Topological terms

The simplest topological term coming from $\mathbb{O}P^2$ at the rational level would be some differential form of degree sixteen. This could also be decomposable, i.e. a wedge product of differential forms of lower degrees such that the total degree is 16. We should seek forms that naturally occur on $\mathbb{O}P^2$. Looking at the question from a 27-dimensional perspective, a Kaluza-Klein mechanism comes to mind. We do not attempt to discuss this problem fully here but merely provide some possibilities that are compatible with the structures that we have. In dimensional reduction from ten and eleven dimensions to lower dimensions, holonomy plays an important role as it gives some handle on the differential forms involved, as well as on supersymmetry.

From the cohomology of $\mathbb{O}P^2$, the possible topological terms generated from this internal space come from $X_i \in H^i(\mathbb{O}P^2)$ for $i = 8, 16$, so that their linear combination generates a candidate degree sixteen term

$$\rho_{16} := aX_{16} + bX_8^2, \tag{4.1}$$

where X_8 and X_{16} are eight- and sixteen-forms, respectively, and a and b are some parameters.

Remarks

1. Since the degree 16 generator is built out of the degree 8 generator, namely the first is proportional to u^2 and the second is u, then equation (4.1) is redundant as X_{16} is really built out of X_8^2. Thus equation (4.1) should be replaced by

$$\rho_{16} = bX_8^2. \tag{4.2}$$

2. In terms of the generator u of $H^8(\mathbb{O}P^2;\mathbb{Z})$, the expression at the integral level should be

$$\rho_{16} = \alpha u^2, \tag{4.3}$$

with $\alpha \in \mathbb{Q}$.

3. The term ρ_{16} would be thought of as a degree 16 analog of the one loop term I_8 in M-theory and type IIA string theory from [31]. It would appear as a topological term in the action, rationally as

$$S_{(27)}^{\text{top}} = \int_{M^{27}} L_{(27)}^{\text{top}} = \int_{M^{27}} \rho_{16} \wedge L_{(11)^{\text{top}}}, \tag{4.4}$$

where $L_{(11)}^{\text{top}}$ is the topological Lagrangian in eleven dimensions given by

$$L_{(11)}^{\text{top}} = \frac{1}{6} G_4 \wedge G_4 \wedge C_3 - I_8 \wedge C_3. \tag{4.5}$$

Then we have

$$
\begin{aligned}
S^{\rm top}_{(27)} &= \int_{Y^{11}} L^{\rm top}_{(11)} \int_{\mathbb{O}P^2} \rho_{16} \\
&= \alpha \int_{Y^{11}} L^{\rm top}_{(11)} \\
&= \alpha\, S^{\rm top}_{(11)}\ .
\end{aligned} \tag{4.6}
$$

In section 2.3 we discussed the question of whether the higher-dimensional 'theory' in our case is supersymmetric. In any case holonomy would give us a handle on whatever differential forms end up appearing. The holonomy group of $\mathbb{O}P^2$ is Spin(9) and there is in fact a Spin(9)-invariant 8-form that generalizes the Kähler 2-form for $\mathbb{C}\mathrm{P}^2$ and the fundamental or Cayley 4-form on $\mathbb{H}P^2$ [23]. The Spin(9) representation $\Lambda^8(\Delta_9) = \Lambda^8(\mathbb{R}^{16})$ contains a unique 8-form which is invariant under the action of Spin(9). Note that $\mathbb{O}P^2$ does not admit an almost complex structure [20] nor an almost quaternionic structure [14].

The Cayley 8-form. The explicit expression for the 8-form is given in terms of the cross product of vectors $V_i = (0, e_i)$, $e_i \in \mathbb{O}$, $i = 1, \cdots, 8$, in the tangent plane $\mathbb{O} \oplus \mathbb{O}$ to $\mathbb{O}P^2$ by [23] [21]

$$
\omega_8(V_1, V_2, \cdots, V_8) = \frac{1}{8!} \sum_{\sigma \in \Sigma_8} \varepsilon(\sigma) \left[\left(V_{\sigma(1)} \times V_{\sigma(2)}\right)\left(V_{\sigma(3)} \times V_{\sigma(4)}\right)\right] \left[\left(V_{\sigma(5)} \times V_{\sigma(6)}\right)\left(V_{\sigma(7)} \times V_{\sigma(8)}\right)\right]. \tag{4.7}
$$

Note that ω_8 is nonzero, real, takes the value 1 on $\mathbb{O}P^2$, and reduces to a product of two fundamental Cayley calibration 4-forms ϕ upon restriction to $\mathbb{O}P^1$ [21]

$$
\omega_8(e_1, e_2, \cdots, e_8) = \frac{1}{35} \sum_{P_8^4} \phi(e_1, e_2, e_3, e_4) \cdot \phi(e_5, e_6, e_7, e_8). \tag{4.8}
$$

In fact there is another expression for the Cayley 8-form which corresponds to the integral generator of the cohomology ring of $\mathbb{O}P^2$. This is described as follows [1]. Let u_i and v_i, $i = 1, \cdots, 8$, be 1-forms on $T\mathbb{O}P^2$ satisfying

$$
\begin{aligned}
v_i(e_j, 0) &= \delta_{ij} & \qquad v_i(0, e_j) &= 0 \\
u_i(e_j, 0) &= 0 & \qquad u_i(0, e_j) &= \delta_{ij}\ .
\end{aligned} \tag{4.9}
$$

Various 2-forms can be formed, such as $\omega_{IJ} = v_I \wedge v_J$ and $\eta_{KL} = u_K \wedge u_L$, where I, J, K, L are various combinations of pairs of i and j. The Spin(9)-invariant 8-form ω_8 is the sum of eight 8-forms $\omega_8 = \sum_{i=1}^{8} \omega_8^i$, where ω_8^i are built out of wedge products of the v_i, u_j, ω_{IJ} and η_{KL}. More precisely,

$$
\omega_8^1 = -14(v_1 \wedge \cdots \wedge v_8 - u_1 \wedge \cdots \wedge u_8), \tag{4.10}
$$

and ω_8^m, $m = 2, \cdots, 8$ are quartic expressions in ω_{IJ} and η_{KL}. The action of the Lie algebra $\mathfrak{spin}(9)$ on any 8-form φ is

$$
(\alpha\varphi)(X_1, \cdots, X_8) = \sum_{i=1}^{8} \varphi(X_1, \cdots, \alpha X_i, \cdots, X_8), \tag{4.11}
$$

for $\alpha \in \mathfrak{spin}(9)$ and $X_1, \cdots, X_8 \in T\mathbb{O}P^2$. The 8-form ω_8 satisfies $\alpha\omega_8 = 0$, so that it is Spin(9)-invariant. The advantage of this approach is that the identification with the cohomology generator is possible and transparent, even though it take some work to write down the form itself. Set

$$
\omega_{16} = v_1 \wedge v_2 \wedge \cdots \wedge v_8 \wedge u_1 \wedge u_2 \wedge \cdots \wedge u_8, \tag{4.12}
$$

the analog of the epsilon symbol whose integral is the volume form of $\mathbb{O}P^2$. The wedge product of ω_8 with itself gives

$$\omega_8 \wedge \omega_8 = 1848\ \omega_{16}\ . \tag{4.13}$$

Set $\mathcal{J}_8 = \frac{60}{\pi^4}\omega_8$. We will need the volume of $\mathbb{O}P^2$. For sake of this calculation we can take $\mathbb{O}P^2$ to be

$$\left\{\text{lines in } \mathbb{O}^3 \cong \mathbb{R}^{24}\right\} = \frac{S^{23}}{S^7}\ . \tag{4.14}$$

Now using the fact that the volume of the sphere S^{d-1} of unit radius and geodesic length 2π is $2\pi^{\frac{d}{2}}/\Gamma(\frac{d}{2})$, we get

$$\mathrm{vol}(\mathbb{O}P^2) = \frac{2\pi^{11}}{\Gamma(11)}\frac{\Gamma(3)}{2\pi^3} = \frac{3!\ \pi^8}{11!}\ , \tag{4.15}$$

with normalization of geodesic length π. Now evaluating the wedge product of the 8-form with itself over $\mathbb{O}P^2$, and using (4.13), gives

$$\begin{aligned}\int_{\mathbb{O}P^2} \mathcal{J}_8 \wedge \mathcal{J}_8 &= \int_{\mathbb{O}P^2} \frac{2^4 \cdot 3^2 \cdot 5^2}{\pi^8}\omega_8 \wedge \omega_8 \\ &= \frac{(2^4 \cdot 3^2 \cdot 5^2)(2^3 \cdot 3 \cdot 7 \cdot 11)}{\pi^8} \int_{\mathbb{O}P^2} \omega_{16} = 1\ ,\end{aligned} \tag{4.16}$$

since $1848 = 2^3 \cdot 3 \cdot 7 \cdot 11$. Let $H^*_{DR}(\mathbb{O}P^2)$ denote the de Rham cohomology ring of $\mathbb{O}P^2$. Let

$$r : H^*_{DR}(\mathbb{O}P^2) \longrightarrow H^*(\mathbb{O}P^2;\mathbb{R}) \tag{4.17}$$

be the de Rham isomorphism, and

$$\jmath : H^*(\mathbb{O}P^2;\mathbb{Z}) \longrightarrow H^*(\mathbb{O}P^2;\mathbb{R}) \tag{4.18}$$

be the homomorphism induced by the natural homomorphism from $\mathbb{Z}$ to $\mathbb{R}$. Finally, the structure of the cohomology ring $H^*(\mathbb{O}P^2;\mathbb{Z}) = \mathbb{Z}[u]/u^3$, which implies that the generator of degree 16 is the square of the generator of degree 8, gives

$$r([\mathcal{J}_8]) = \pm\jmath(u)\ . \tag{4.19}$$

Therefore the class $[\mathcal{J}_8]$ of the closed differential form $\mathcal{J}_8$ corresponds to the integral generator u of $H^8(\mathbb{O}P^2;\mathbb{Z})$.

Note that the 8-form has the following properties:
(1) The 8-form defines a unique parallel form on $\mathbb{O}P^2$.
(2) Since the signature of $\mathbb{O}P^2$ is positive (see (3.6)), then the 8-form is self-dual.

Remarks
1. At the rational level we can thus use ω_8 to build a Spin(9)-invariant degree sixteen expression

$$\rho_{16}^{\mathbb{R}} = \omega_8 \wedge \omega_8 \tag{4.20}$$

that we integrate and insert as part of the action as $\int_{\mathbb{O}P^2} \rho_{16}^{\mathbb{R}}$.
2. Assume that there are fields $\mathcal{F}_8$ and $\mathcal{F}_{16}$ in the 27-dimensional 'theory' with potentials $\mathcal{C}_7$ and $\mathcal{C}_{15}$. In the dimensional reduction on $\mathbb{O}P^2$ to eleven dimensions, a natural Spin(9)-invariant ansatz for the fields may be taken, at the rational level, to be

$$\mathcal{F}_8 = \omega_8, \qquad\qquad \mathcal{F}_{16} = \omega_8 \wedge \omega_8\ , \tag{4.21}$$

and similar expressions at the integral level in terms of $\mathcal{J}_8$. Note that since ω_{16} is essentially the volume form, then such an ansatz is the analog of the Freund-Rubin ansatz [34] in the reduction of eleven-dimensional supergravity to lower dimensions.

The integration of ρ_{16} over $\mathbb{O}P^2$ in the second step of equation (4.6) requires the existence of a fundamental class $[\mathbb{O}P^2]$ for the Cayley plane. The Cayley 8-form $\mathcal{J}_8$ allows for such an evaluation at the rational and integral level. The next question is about torsion. The existence of such a fundamental class at that level is neither automatic nor obvious. In order to state the following result we recall some notation. Let $\beta : H^i(Y^{11}; \mathbb{Z}_3) \to H^{i+1}(Y^{11}; \mathbb{Z})$ be the Bockstein homomorphism corresponding to the reduction modulo 3, $r_3 : \mathbb{Z} \to \mathbb{Z}_3$, i.e. associated to the short exact sequence

$$0 \longrightarrow \mathbb{Z}_3 \longrightarrow \mathbb{Z}_9 \longrightarrow \mathbb{Z}_3 \longrightarrow 0, \tag{4.22}$$

and $P_3^1 : H^j(Y^{11}; \mathbb{Z}_3) \to H^{j+4}(Y^{11}; \mathbb{Z}_3)$ be the Steenrod reduced power operation at $p = 3$. Then we have

Theorem 4.1. *A fundamental class exists provided that $\beta P_3^1 x_4 = 0$, where x_4 is the mod 3 class on Y^{11} pulled back from BF_4 via the classifying map.*

Proof. Consider the fiber bundle $E \to Y^{11}$ with fiber $\mathbb{O}P^2$ and structure group F_4. There is a universal bundle of this type. $\mathbb{O}P^2$ bundles over Y^{11} are pullbacks of the universal bundle

$$\mathbb{O}P^2 = F_4/\text{Spin}(9) \longrightarrow B\text{Spin}(9) \longrightarrow BF_4 \tag{4.23}$$

by the classifying map $f : Y^{11} \to BF_4$. Since BF_4 is path-connected and $\mathbb{O}P^2$ is connected then we can apply the Serre spectral sequence to the fibration (4.23). We consider two cases for the coefficients of the cohomology: $\mathbb{Z}_p$(or any field in general), p a prime, and $\mathbb{Z}$ coefficients.

Coefficients in $\mathbb{Z}_p$: The important primes are $p = 2, 3$ as these are the torsion primes of F_4. For $p = 2$ the inclusion map $i : \text{Spin}(9) \hookrightarrow F_4$ induces a map on the classifying spaces so that $H^*(B\text{Spin}(9); \mathbb{Z}_p)$ is a free $H^*(BF_4; \mathbb{Z}_p)$-module on generators $1, x, x^2$ with $x \in H^8(B\text{Spin}(9); \mathbb{Z}_p)$ the universal Leray-Hirsch generator that maps to $x \in H^8(\mathbb{O}P^2; \mathbb{Z}_p)$. Here we use the fact [69] that the Serre spectral sequence for a fiber bundle $F \to E \to B$ collapses if and only if the corresponding Poincaré series $\mathcal{P}(-) := \sum_{n \geq 0} t^n \dim_{\mathbb{Z}_p} H^n(-; \mathbb{Z}_p)$ satisfies $\mathcal{P}(E) = \mathcal{P}(F)\mathcal{P}(B)$. In our case the Serre spectral sequence of (4.23) collapses [55]. This follows from the equality of the corresponding Poincaré polynomials

$$\begin{aligned} \frac{\mathcal{P}(B\text{Spin}(9))}{\mathcal{P}(BF_4)} &= \frac{(1-t^4)^{-1}(1-t^6)^{-1}(1-t^7)^{-1}(1-t^8)^{-1}(1-t^{16})^{-1}}{(1-t^4)^{-1}(1-t^6)^{-1}(1-t^7)^{-1}(1-t^{16})^{-1}(1-t^{24})^{-1}} \\ &= \frac{1-t^{24}}{1-t^8} = 1 + t^8 + t^{16}, \end{aligned} \tag{4.24}$$

which is just the Poincaré polynomial $\mathcal{P}(\mathbb{O}P^2)$ of the Cayley plane. This implies that the Leray-Hirsch theorem holds, i.e. that the map $H^*(\mathbb{O}P^2) \otimes H^*(BF_4) \to H^*(B\text{Spin}(9))$ is an isomorphism of $H^*(BF_4)$-modules. This implies in particular that $H^*(B\text{Spin}(9))$ is a free BF_4-module on $1, x, x^2$, where x is either w_8 or $w_8 + w_4^2$. The Wu formula with $w_1 = w_2 = 0$ for both cases gives that $Sq^1 x = Sq^2 x = Sq^3 x = Sq^5 x = 0$ so that

$$Sq\, x = x + Sq^4 x + Sq^6 x + Sq^7 x + x^2. \tag{4.25}$$

The elements x_4, $Sq^2 x_4$, $Sq^3 x_4 \in H^* BF_4$ are mapped to the elements $w_4, w_6 = Sq^2 w_4, w_7 = Sq^3 w_4 \in H^* B\text{Spin}(9)$. The Leray-Hirsch theorem holds for the universal bundle, and consequently for all $\mathbb{O}P^2$ bundles [55].

For $p = 3$ the argument is similar except that now the generators in degrees 4 and 8 are related as $p_1 = \overline{p}_1$ and $p_2 = \overline{p}_2 + \overline{p}_1^2$, respectively [98]. Here p_i are the Pontrjagin classes (see the appendix).

Coefficients in $\mathbb{Z}$: We would like to find the differentials for

$$H^*(B\mathrm{Spin}(9); \mathbb{Z}) \Longleftarrow H^*\left(BF_4, H^*(\mathbb{O}P^2; \mathbb{Z})\right). \tag{4.26}$$

The class u maps under the differential to a $\mathbb{Z}_3$ class of degree 9 which we will call α. The lowest degree class on the fiber is x_8, so the differentials begin with d_9. The differential is d_9 on x_8 so that the class is $\beta P_3^1 x_4$, where x_4 is the mod 3 class on Y^{11} coming from BF_4

$$Y^{11} \longrightarrow BF_4 \longrightarrow K(\mathbb{Z}_3, 9). \tag{4.27}$$

We thus have a 3-torsion class of $\mathbb{O}P^2$ bundles. The obstruction in $H^9(Y^{11}; \mathbb{Z})$ coming from $H^9(BF_4; \mathbb{Z})$ is zero if and only if there exists a degree 16 class, say ρ_{16}, that restricts on each fiber to the fundamental class. □

Thus the vanishing of d_9 provides us with a fundamental class which we use to integrate over $\mathbb{O}P^2$.

Remark. The Pontrjagin classes p_2 and p_4 of $\mathbb{O}P^2$ are divisible by three. There is always a class in M^{27} that restricts on the fiber to three times the generator of the cohomology of $\mathbb{O}P^2$.

4.2 Torsion classes and effect on the M-theory partition function

In subtle situations the fields in the physical theory can be torsion classes in cohomology. We consider terms in the action coming from BF_4 or from the fiber $\mathbb{O}P^2$. We will show that torsion classes from BF_4 are compatible with the description in [28] of the phase of the M-theory partition function.

4.2.1 Classes from BF_4

1. $\mathbb{Z}_2$ coefficents: The cohomology ring of BF_4 with coefficients in $\mathbb{Z}_2$ is given from (3.65) by the polynomial ring

$$H^*(BF_4; \mathbb{Z}_2) = \mathbb{Z}_2\left[x_4, Sq^2 x_4, Sq^3 x_4, x_{16}, Sq^8 x_{16}\right], \tag{4.28}$$

where x_4 and x_{16} are polynomial generators of degree 4 and 16, respectively. From the structure of the cohomology ring (4.28) we see that we can pull back classes from BF_4 and that these are in fact compatible with the fields of M-theory. In particular, there is a degree four class x_4, as in all Lie groups of dimension greater than or equal to three, which could be matched with the field strength G_4 in M-theory. In fact, since *any* degree four class can be the characteristic class a_{E_8} of an E_8 bundle, then a class pulled back from F_4 can certainly be at the same time a class of some E_8 bundle. Hence an F_4 class is possible in the shifted quantization condition

$$[G_4] - \frac{\lambda}{2} = a_{E_8} \in H^4(Y^{11}; \mathbb{Z}), \tag{4.29}$$

discovered in [101].

The higher degree classes are also relevant. We also have the degree six and the seven generators $Sq^2 x_4$ and $Sq^3 x_4$, respectively, which, when nonzero, would appear in the phase of the partition function. The comparison of M-theory on Y^{11} with type IIA string theory on a ten-manifold X^{10} involves the bilinear form [28]

$$\mu(a, b) = \int_{X^{10}} a \cup Sq^2 b, \tag{4.30}$$

for $a, b \in H^4(X^{10};\mathbb{Z})$. This can be viewed [28] as a torsion pairing

$$\begin{aligned} T : H^4_{\rm tor}(X^{10};\mathbb{Z}) \times H^7_{\rm tor}(X^{10};\mathbb{Z}) &\longrightarrow U(1) \\ (\, a \, , \; Sq^3 b \,) &\longmapsto \int_{X^{10}} a \cup Sq^2 c, \end{aligned} \tag{4.31}$$

where $Sq^3 b = \beta(Sq^2 c) = Sq^1 Sq^2 c = Sq^3 c$. In our case a and b can be $f^* x_4$. Thus we have

Proposition 4.2. *$\mathbb{Z}_2$ classes from BF_4 are compatible with the M-theory partition function, i.e. they produce no anomalies and they do not change the value of the partition function.*

2. $\mathbb{Z}_3$ coefficents: If we restrict to low degrees, say ≤ 16, then we have the truncated polynomial

$$H^*(BF_4;\mathbb{Z}_3) \cong \mathbb{Z}_3[x_4, x_8] + \Lambda(x_9). \tag{4.32}$$

Now the main observation is that the class x_9, being $\beta P^1_3 x_4$, is the same as the class required to be cancelled in theorem 4.1. If we kill this class then we are left with only the degree four and the degree eight classes x_4 and x_8. Since x_8 is $P^1_3 x_4$, then this $\mathbb{Z}_3[x_4, P^1_3 x_4]$ is also compatible with the mod 3 description of the anomalies in M-theory described in [89]. Therefore,

Proposition 4.3. *$\mathbb{Z}_3$ classes from BF_4 are compatible with the partition function of M-theory once the anomaly in theorem 4.1 is cancelled.*

4.2.2 Classes from $\mathbb{O}P^2$

Recall that we have introduced fields $\mathcal{F}_8$ and $\mathcal{F}_{16}$ with corresponding potentials $\mathcal{C}_7$ and $\mathcal{C}_{15}$, respectively (see (4.21)). Assuming that the 27-dimensional 'theory' indeed has such fields, we consider some consequences in this section. We emphasize that we do not have enough knowledge about the dynamics (if and when it exists) in 27 dimensions so we will concentrate on the topology. We will concentrate on the first field, because of the cohomology of $\mathbb{O}P^2$, i.e. that the second would probably be a 'composite' of the first.

Imposing conventional Dirac quantization on the field $\mathcal{C}_7$ gives that these fields are classified topologically by a class $x \in H^8(\mathbb{O}P^2;\mathbb{Z})$, so that x is represented in de Rham cohomology by $\frac{\mathcal{F}_8}{2\pi}$,

$$x = \left[\frac{\mathcal{F}_8}{2\pi}\right] \; . \tag{4.33}$$

In analogy to the case in string theory [103] and M-theory [101] [102], we consider the construction of the partition function corresponding to $\mathcal{C}_7$. This is done in terms of a theta function on $T = H^8\left(\mathbb{O}P^2; U(1)\right)$. However, since $\mathbb{O}P^2$ has no torsion in cohomology, then T will be the torus

$$T = H^8(\mathbb{O}P^2;\mathbb{R})/H^8(\mathbb{O}P^2;\mathbb{Z}) \; . \tag{4.34}$$

Furthermore, the construction requires a function

$$\Omega : H^8(\mathbb{O}P^2;\mathbb{Z}) \longrightarrow \mathbb{Z}_2 \; , \tag{4.35}$$

obeying the law

$$\Omega(x+y) = \Omega(x)\Omega(y)(-1)^{x\cdot y} \; , \tag{4.36}$$

where $x \cdot y$ is the intersection pairing $\int_{\mathbb{O}P^2} x \cup y$ on $\mathbb{O}P^2$. The function Ω enters into the determination of the line bundle $\mathcal{L}$ on T. The partition function of the $\mathcal{C}_7$ field will then be a holomorphic section of $\mathcal{L}$.

The signature of $\mathbb{O}P^2$, which has dimension 16, is by definition the signature of the quadratic form

$$\begin{aligned} H^8(\mathbb{O}P^2;\mathbb{Q}) &\longrightarrow \mathbb{Q} \\ v &\longmapsto \langle v^2, [\mathbb{O}P^2]\rangle, \end{aligned} \tag{4.37}$$

and is given in (3.6), whose value is 1.

The intersection form. For a manifold M^{2n} of dimension $2n$, the universal coefficient theorem implies that

$$H_n(M^{2n};\mathbb{R}) \cong H_n(M^{2n}) \otimes \mathbb{R} \cong (H_n(M^{2n})/T_n) \otimes \mathbb{R}\ . \tag{4.38}$$

Torsion elements do not affect the intersection number: if α_n, β_n are torsion elements so that $r\alpha_n, s\beta_n \in H_n(M^{2n};\mathbb{R})$, then

$$\langle r\alpha_n, s\beta_n\rangle = rs\langle \alpha_n, \beta_n\rangle, \tag{4.39}$$

so that the intersection forms over $\mathbb{R}$ and $\mathbb{Z}$ have the same matrix. Then $H_n(M^{2n};\mathbb{R})$ has a basis in which the intersection form has integer coefficients. Since the cup product is anti-commutative then the intersection form is symmetric for even n and antisymmetric for odd n. The intersection form of $\mathbb{O}P^2$ is not even. This can be seen in two ways. First that the signature of $\mathbb{O}P^2$ given in (3.6), which is the signature of the intersection matrix of the middle cohomology of $\mathbb{O}P^2$, is not zero. Second, the Steenrod operation Sq^4k does not decompose in the similar way that Sq^{4k+2} does. In the latter case, the Adem relation $Sq^{4k+2} = Sq^2 Sq^{4k} + Sq^1 Sq^{4k} Sq^1$ implies that $x_{4k+2}^2 = Sq^{4k+2} x_{4k+2} = 0$.

Now we look at mod 2 and integral bilinear forms. We have

Proposition 4.4. *1. The bilinear form*

$$\begin{aligned} H^8(\mathbb{O}P^2;\mathbb{Z}_2) \times H^8(\mathbb{O}P^2;\mathbb{Z}_2) &\longrightarrow \mathbb{Z}_2 \\ (\ a_8\ ,\ a_8\ ) &\longmapsto \int_{\mathbb{O}P^2} a_8 \cup a_8 \end{aligned} \tag{4.40}$$

is given by $\int_{\mathbb{O}P^2} a_8 \cup w_8$.

2. The bilinear form over $\mathbb{Z}$

$$H^8(\mathbb{O}P^2;\mathbb{Z}) \times H^8(\mathbb{O}P^2;\mathbb{Z}) \longrightarrow \mathbb{Z} \tag{4.41}$$

is an odd $\mathbb{Z}$*-form.*

Proof. Consider the first part. Since $w_8^2 = p_4$ mod 2 and $w_{16} = 3u^2 = e$ mod 2, then the total Stiefel-Whitney class of $\mathbb{O}P^2$ is $w = 1 + u + u^2$, with coefficients of u reduced mod 2 [20] (see equation (2.27). The fact that the first seven Stiefel-Whitney classes of $\mathbb{O}P^2$ vanish implies that the Wu class $\nu(\mathbb{O}P^2)$ reduces to the element $w_8(\mathbb{O}P^2) \in H^8(\mathbb{O}P^2;\mathbb{Z}_2)$ [51]. Consequently, the Stiefel-Whitney class $w_8(\mathbb{O}P^2)$ is characterized by the condition [35]

$$y_8 \cup y_8 = y_8 \cup w_8(\mathbb{O}P^2) \text{ for any } y_8 \in H^8(\mathbb{O}P^2;\mathbb{Z}_2). \tag{4.42}$$

Next consider the second part. In [35] it was shown that, for a compact manifold M^{16} admitting a Spin(9)-structure, the quadratic form

$$H^8(M^{16};\mathbb{Z})/\text{Tor} \times H^8(M^{16};\mathbb{Z})/\text{Tor} \longrightarrow H^{16}(M^{16};\mathbb{Z}) \tag{4.43}$$

is an even $\mathbb{Z}$-form if and only if $w_8(M^{16}) = 0$. Since $\mathbb{O}P^2$ has no torsion in cohomology, $H^{16}(\mathbb{O}P^2;\mathbb{Z}) = \mathbb{Z}$, and $w_8(\mathbb{O}P^2)$ is nonzero, then the result follows immediately. □

In fact, we know that the value of the intersection form is given by the signature, which is 1, from (3.6).

4.3 Further terms and compatibility with other theories

4.3.1 Kinetic terms

We have not so far included any kinetic terms in the discussion. The main reason is that we do not know the nature of the resulting 'theory' and whether it will have such terms. If we take the proposal in [48], there are difficulties with the Einstein-Hilbert, i.e. the gravitational kinetic, term because the obvious choice does not give the correct term in bosonic string theory in twenty-six dimensions upon dimensional reduction, but is off by a factor of 125/121. This is also linked with difficulties of finding coset symmetries [54] [64]. Thus we exclude the gravitational terms from the discussion. We go back to some of this in section 4.3.3. To some limited extent, we do consider the kinetic term for the M-theory C-field provided this field lifts and provided that such a term does in fact appear.

Assuming a kinetic term for G_4, then the EOM would be rationally

$$d *_{27} G_4 = \frac{1}{2} G_4 \wedge G_4 \wedge Z_{16} + I_8 \wedge Z_{16}, \tag{4.44}$$

where $*_{27}$ is the Hodge duality operator in 27 dimensions. The right hand side is a degree 24 differential form, whose class is of the form

$$\Theta_{24}^{\mathbb{R}} := \left[\frac{1}{2} G_4 \wedge G_4 + I_8\right] \wedge Z_{16}. \tag{4.45}$$

As we have argued earlier, a term such as Z_{16} can only be a composite, i.e. a square of degree eight expressions, due to the cohomology of $\mathbb{O}P^2$. We are interested in the integral lift of that degree 24 expression. The term in brackets in (4.45) has an integral lift given by the class Θ_8, defined in [27], as $[\Theta_8(a)]_{\mathbb{R}} = \frac{1}{2} a_{\mathbb{R}}(a_{\mathbb{R}} - \lambda_{\mathbb{R}}) + 30\widehat{A}_8$. The integral lift of Z_{16} is just u^2 where u is the generator of $H^8(\mathbb{O}P^2; \mathbb{Z})$. Thus we have

Proposition 4.5. *The integral lift of $\Theta_{24}^{\mathbb{R}}$ is given by*

$$[\Theta_{24}] = [\Theta_8] \cup u^2 \tag{4.46}$$

The study of this class, and further refinements thereof, could be useful.

Remark. Having $*_{27} G_4$ and $[\Theta_{24}]$ signals the appearance of 21-branes in the 27-dimensional theory. Requirement of decoupling of this brane from the membrane, so that a well-defined partition function can be constructed, gives that the class $[\Theta_{24}]$ be trivial in cohomology, so that the fields are cohomologically trivial on the brane. One obvious way to ensure this is to require triviality of $[\Theta_8]$. If we do not require this then we can find some other way to do this. We do not just set u to zero. But we can do something when reducing coefficients. Let P_5^1 be the Steenrod reduced power operation $P_5^1 : H^k(\mathbb{O}P^2; \mathbb{Z}_5) \longrightarrow H^{k+8}(\mathbb{O}P^2; \mathbb{Z}_5)$. Let $\overline{u}$ be the generator u with coefficients reduced mod 5. In this case, for $k = 8$, the action of P_5^1 is given by multiplication with $5L_2$, where L_2 is the degree 8 term in the L-genus [43].

$$P_5^1 \overline{u} = \frac{1}{9}(7p_2 - p_1^2)\overline{u} = -2p_2\overline{u} = -2\overline{u}^2. \tag{4.47}$$

This implies the following.
1. We can make $[\Theta_{24}]$ zero by imposing the condition $P_5^1 \overline{u} = 0$. This is analogous to the mod 3 case in [89].
2. For each homeomorphism $\phi : \mathbb{O}P^2 \to \mathbb{O}P^2$, $\phi^* \overline{u} = \overline{u}$ [20]. Hence $\overline{u}$ is invariant under continuous deformations of $\mathbb{O}P^2$.

4.3.2 Compatibility with ten-dimensional superstring theories

We have looked at the proposed 'theory' in twenty-seven dimensions in relation to M-theory in eleven dimensions. The question will now be whether the structures we discussed are compatible with other known theories. Given that the 27-dimensional 'theory' is proposed in such a way that it is by construction compatible with M-theory (as we know it) then, since all five superstring theories in ten dimensions are obtained from M-theory via dimensional reduction and/or dualities, the 27-dimensional construction is compatible with these superstring theories. We will actually reduce the $F_4 - \mathbb{O}P^2$-bundle to ten dimensions along the M-theory circle and check this explicitly.

We consider the $\mathbb{O}P^2$ bundle M^{27} with structure group F_4. The transition functions on Y^{11}, with patches U_i and U_j, will be

$$g_{ij} : U_i \cap U_i \longrightarrow \mathrm{Diff}(\mathbb{O}P^2), \tag{4.48}$$

are $\mathrm{Diff}(\mathbb{O}P^2)$-valued ($\mathrm{Diff}^+$ if orientation-preserving). If we take Y^{11} to be the product $X^{10} \times S^1$ and view the circle as the interval $[0,1]$ with the ends glued together then we can form the diagram

$$\begin{array}{ccc} \mathbb{O}P^2 & \xrightarrow{=} & \mathbb{O}P^2 \\ \downarrow & & \downarrow \\ M^{27} & \longrightarrow & \pi^* M^{27} \\ \downarrow & & \downarrow \\ X^{10} \times S^1 & \xleftarrow[\pi]{} & X^{10} \times [0,1] \; . \end{array} \tag{4.49}$$

The bundle $\pi^* M^{27}$ is isomorphic to a bundle $\xi^{26} \times [0,1]$ over $X^{10} \times [0,1]$. Gluing at $[0,1]$ we get a map from X^{10} to $\mathrm{Aut}(\xi^{26})$, the automorphism group of the bundle ξ. Therefore,

Proposition 4.6. *From a bundle M^{27} over $X^{10} \times S^1$ we get*

1. *a bundle $\xi^{26} \to X^{10}$ with fiber $\mathbb{O}P^2$ and structure group F_4, and*
2. *a gauge group element of ξ^{26}, i.e. a map $X^{10} \to \mathrm{Aut}(\xi^{26})$.*

If the bundle is trivial then the automorphims of ξ^{26} will be the automorphisms of the fiber, i.e. F_4. A map from X^{10} to F_4 might then be regarded as a classifying map for based loop bundles, since $B\Omega F_4 = F_4$. Thus, in this special case, we have an F_4 bundle and an ΩF_4 bundle over X^{10}. This is analogous to the case of E_8 [88].

The diffeomorphism group above is very large and is not easy to work with. Instead we will invoke a condition that is familiar from Kaluza-Klein theory, namely to assume that the original bundle comes from a principal F_4-bundle

$$\begin{array}{ccc} F_4 & \longrightarrow & P \\ & & \downarrow \\ & & Y^{11}, \end{array} \tag{4.50}$$

so that we effectively consider the reduction of the structure group $\mathrm{Diff}^+(\mathbb{O}P^2)$ to the subgroup F_4, the isometry group of the $\mathbb{O}P^2$ fiber. This is analogous to the case when Y^{11} itself is taken as the total space of a circle bundle over X^{10}. A priori the structure group is $\mathrm{Diff}^+(S^1)$, in which the transition functions are valued. Restricting to $U(1) \subset \text{Diff}^+(S^1)$, we get a principal circle bundle $U(1) \to Y^{11} \to X^{10}$. In fact, in this case, the reduction is always possible and no condition is required. Now we are presented with a situation which is analogous to having an E_8 bundle [101] in eleven dimensions that we are asking to reduce to ten dimensions. The result, analogously to the E_8 case [3] [68], is

$$\begin{array}{ccc} F_4 & \longrightarrow & P \\ & & \downarrow \\ S^1 & \longrightarrow & Y^{11} \\ & & \downarrow \\ & & X^{10} \end{array} \qquad \Longrightarrow \qquad \begin{array}{ccc} LF_4 & \longrightarrow & Q \\ & & \downarrow \\ & & X^{10} \;. \end{array} \tag{4.51}$$

The homotopy type of F_4 is identical to the homotopy type of E_8 in degrees less than eleven, and so rationally $F_4 \sim S^3$, $\Omega F_4 \sim S^4$, so that $LF_4 \sim S^3 \times S^4$. Thus, at the rational level, we expect a degree three and a degree four class from the LF_4 bundle. At the integral level, since $F_4 \sim K(\mathbb{Z}, 3)$, then

$$LF_4 \sim K(\mathbb{Z}, 3) \times K(\mathbb{Z}, 4), \qquad \deg < 11. \tag{4.52}$$

This can be shown as follows. We have LF_4 bundles which are classified by maps to BLF_4. The sequence $\Omega X \to LX \to X$ for $X = BF_4$ gives

$$F_4 \longrightarrow LBF_4 \longrightarrow BF_4 \;. \tag{4.53}$$

Since F_4 is connected, then LBF_4 and BLF_4 are homotopy equivalent. We can then replace LBF_4 with BLF_4 in (4.54). Since 2 and 3 are the only torsion primes for F_4, then for $p \geq 5$ the sequence

$$F_4 \longrightarrow BLF_4 \underset{s}{\overset{\text{ev}}{\rightleftarrows}} BF_4 \tag{4.54}$$

splits on mod p cohomology, so that

$$H^*(BLF_4; \mathbb{Z}_p) \cong H^*(BF_4; \mathbb{Z}_p) \otimes H^*(F_4; \mathbb{Z}_p), \qquad p \geq 5 \;, \tag{4.55}$$

as algebras. At the torsion primes we use the Serre spectral sequence corresponding to the sequence (4.54). From (4.28) we see for $p = 2$ that in degrees ≤ 15,

$$H^*(BF_4; \mathbb{Z}_2) = \mathbb{Z}_2 \left[x_4, Sq^2 x_4, Sq^3 x_4\right] \;. \tag{4.56}$$

The differential d acting on x_4 is zero because of the section s in (4.54). From (4.56), for $p = 2$, and from (4.32), for $p = 3$, we see that all the generators are connected by cohomology operations, Sq^i and P^j, respectively. Thus, since $x_{i>4} = \mathcal{O} x_4$, for some cohomology operation $\mathcal{O}$, then all the differentials are zero. Thus the spectral sequence collapses and the fibration is a product.

The LF_4 bundle over X^{10} is therefore a $K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 3)$ bundle. The first factor, $K(\mathbb{Z}, 2)$ gives the NS field H_3 and the second factor, $K(\mathbb{Z}, 3)$ gives the RR field F_4 in ten dimensions. Hence at the topological level, compatibility of F_4 with ten-dimensional type IIA is reduced to that of E_8, which follows from [28] [3] [68]. The compatibility with type IIB, and hence with F-theory, also follows from T-duality as for the E_8 case [32]. Therefore, we can give the following statement.

Proposition 4.7. *Consider the $\mathbb{O}P^2$ bundle over Y^{11} with structure group reduced to F_4 as above. Then*

1. *The reduction of the F_4 bundle on the circle in Y^{11} leads to an LF_4 bundle over X^{10}.*

2. *At the topological level, the $\mathbb{O}P^2$ bundle, with the above assumptions, is compatible with type II string theory.*

4.3.3 Compatibility with the bosonic string

The question is whether the 27-dimensional structure is compatible with the bosonic string theory in twenty-six dimensions, on X^{26}. We have addressed some aspects of this in section 2.3 in relation to fermions and supersymmetry, and so we consider other aspects in this section. The form fields we have introduced, including G_4 from M-theory, are all of dimensions that are multiples of 4. Since the bosonic string spectrum does not involve G_4 and the action does not obviously get the topological terms that we introduced, then the relation between M^{27} and X^{26}, if a dimensional reduction, could be a one-dimensional orbifold, [4] i.e. $S^1/\mathbb{Z}_2$, where we assume a $\mathbb{Z}_2$ parity on all form fields of degrees of the form $4k$ in such a way that they disappear in the same way that G_4 gets killed in going from M-theory to the heterotic theory and also from the bosonic theory in [48] to twenty-six dimensions. Thus, the forms coming from the $\mathbb{O}P^2$ bundles can be made compatible with bosonic string theory.

One difficulty with the proposal in [48] was raised in [64], which is that the action does not support a coset symmetry that would include the bosonic string theory. This was also observed in [54]. The question is whether our proposal can evade these objections. In [64] the reduction was on tori, but ours is a coset space with large and sparse homotopy cells. In [54] the analysis was based on assumptions, such as Lorentz symmetry, that we do not know whether they hold for the higher-dimensional case, and the search was made based on the classification of simple Lie algebras. It is possible that the higher structures will not be entirely described by such classical notions (although of course we used some of these notions in our own discussion). Furthermore, in both [64] and [54] gravity was involved. The Einstein-Hilbert term in twenty-seven dimensions does not give the correct term in twenty-six dimensions [48], and this is related to the lack of coset symmetry structure [64] mentioned above. We have not included the gravitational kinetic terms in our discussion, mainly for this reason, but also because there is a possibility that the theory will not be of the the usual form. This was also raised in [64]. It is possible that the theory will be nonlocal or topological. We cannot answer this in any definitive way here.

Thus, given the discussion about supersymmetry at the end of section 2.3 and the above discussion, it would be desirable to find a compatibility diagram of the schematic form

$$\begin{array}{ccc} M^{27} & \xrightarrow{\;?\;} & X^{26} \\ {\scriptstyle \mathbb{O}P^2 \text{ reduction}} \big\downarrow & & \big\downarrow {\scriptstyle \text{Lattice reduction}} \\ Y^{11} & \xrightarrow[\text{reduction}]{S^1 \text{ or } S^1/\mathbb{Z}_2} & M^{10} \; . \end{array} \tag{4.57}$$

This requires further investigation but we have not immediately seen an obstruction for this to hold.

5 Connection to Cobordism and Elliptic Homology

5.1 Cobordism and boundary theories

In this section we consider the question of extension of the theories in eleven and twenty-seven dimensions to bounding theories in twelve and twenty-eight dimensions, respectively, assuming the spaces to be String

[4] Alternatively, the relation between the twenty-seven - and the twenty-six-dimensional theories could be more involved such as in the case of heterotic/type II duality.

and taking into account the F_4 bundles. As mentioned in the introduction, our discussion will make contact with a version of elliptic cohomology constructed by Kreck and Stolz [60]. In that paper the emphasis was on the Spin case corresponding geometrically to quaternionic projective plane $\mathbb{H}P^2$ bundles, but the authors assert the existence of a $BO\langle 8\rangle$ version corresponding to octonionic projective plane $\mathbb{O}P^2$ bundles. Let us denote this theory by $E^{\langle 8\rangle}$ or, equivalently, by $E^{\mathbb{O}}$. Before making the connection we first recall some basic facts.

Recall the definition of cobordism in our case. Two eleven-manifolds Y^{11} and M^{11} are cobordant if there exists a twelve-manifold Z^{12} with $\partial Z^{12} = Y^{11} \coprod (-M^{11})$, where $\coprod$ denotes disjoint union and $(-M^{11})$ is M^{11} with the reverse orientation. In fact cobordism is an equivalence relation $Y^{11} \sim M^{11}$, and the set of equivalence classes Ω forms an abelian group under the disjoint union operation. Furthermore, in the general case, the Cartesian product $X \times Y$ makes $\Omega = \sum_{n=0}^{\infty} \Omega_n$ into a graded super-commutative ring, with $1 = [\mathrm{pt}]$ as unit. Depending on the structure on the tangent bundle of the space, one gets corresponding cobordism groups. For example, the hierarchy of structure: un-oriented, oriented, spin, string, corresponds to the tangent structure groups O, SO, Spin, and String, which in turn correspond to the cobordism groups with such structures: Ω^{O}, Ω^{SO}, Ω^{Spin}, and Ω^{String}, respectively. Note that we have used the covariant notation for bordism instead of cobordism. [5] Next, the (co)bordism groups can be 'evaluated' for a space X as follows. The bordism groups $\Omega_n^G(X)$, for G any of the above groups, for a space X consist of equivalence classes of maps $f: M^n \to X$, where $f_0 \sim f_1$ if these are obtained by restricting some map $g: Z^{m+1} \to X$ to $\partial Z^{m+1} = M_0^m \coprod (-M_1^m)$.

Note that we can also consider the String condition, discussed above in section 3, from an eleven-dimensional point of view. One point that we utilize is that $\Omega_{11}^{\mathrm{spin}}(\mathrm{pt})$, the Spin cobordism group in eleven dimensions, is zero. This means that any eleven-dimensional Spin manifold bounds a twelve-dimensional one. It is also the case that the $BO\langle 8\rangle$ cobordism group $\Omega_{11}^{\langle 8\rangle}(\mathrm{pt})$ is zero [36], so that the extension from an eleven-dimensional String manifold to the corresponding boundary is unobstructed. Thus, if the space Y^{11} in which M-theory is defined admits a String structure then this always bounds a twelve-dimensional String manifold Z^{12}.

Generalized cohomology theories can, in fact, be obtained as quotients of cobordism (see [61] for some exposition on this for physicists) by classic results [25]. For instance, Spin cobordism $\Omega_*^{\mathrm{spin}} = \Omega_*^{\langle 4\rangle}$ is closely related to real K-theory KO, a fact we used in section 3. For a space X, $KO^*(\mathrm{pt})$ can be made into an Ω_*^{spin}-module and there is an isomorphism of $KO^*(X)$ with $\Omega_*^{\mathrm{spin}}(X) \otimes_{\Omega_*^{\mathrm{spin}}} KO^*(\mathrm{pt})$. As we have seen, this is related to the mod 2 index of the Dirac operator with values in real bundles in ten dimensions which appears in the mod 2 part of the partition function [28]. There is an analogous construction for elliptic cohomology, where there the starting point is $\Omega_*^{\langle 8\rangle}$. This fact is related to the elliptic refinement of the mod 2 index which then has values in a real version of elliptic cohomology [61].

5.2 Cobordism of $BO\langle 8\rangle$-manifolds with fiber $\mathbb{O}P^2$

Now we go back to our main discussion of relating the cobordisms of the eleven- and twenty-seven-dimensional theories together with the F_4-$\mathbb{O}P^2$ bundles. Thus we are led to the study of the cobordism groups $\Omega_i^{\langle 8\rangle}(BF_4)$ for $i = 11$ and 27. We will also be interested in relating these two groups.

We have an 11-dimensional base manifold Y^{11}, assumed to admit a String(11) structure, with an $\mathbb{O}P^2$ bundle such that the total space is M^{27} and the structure group is F_4. Let $\mathcal{I} \in \Omega_{27}^{\langle 8\rangle}$ be the ideal generated

[5] This is analogous to the distinction between cohomology –contravariant functor– and homology – covariant functor.

by elements of the form $[M^{27}] - [\mathbb{O}P^2][Y^{11}]$ where, as before, $M^{27} \to Y^{11}$ is a fibration with fiber $\mathbb{O}P^2$ and structure group F_4. We have

Proposition 5.1. *Let Y^{11} be a compact manifold with a String structure on which M-theory is taken, and let M^{27} be the String manifold on which the 27-dimensional theory is taken, realizing the Euler triplets geometrically. Then such 27-manifolds M^{27} are in the ideal $\mathcal{I}$ of $\Omega_{27}^{\langle 8\rangle}$ generated by $\mathbb{O}P^2$ bundles.*

Our setting is given in the following diagram

$$\begin{array}{ccc}
\mathbb{O}P^2 \longrightarrow & M^{27} & \\
 & {\scriptstyle \pi}\downarrow & \searrow{\scriptstyle f'} \\
 & Y^{11} & \xrightarrow{f} N\,.
\end{array} \tag{5.1}$$

First we ignore the structure group and consider N to be a point. As in Section 5.1, let $\Omega_*^{\langle 8\rangle}$ be the cobordism ring of manifolds with $w_1 = w_2 = \frac{1}{2}p_1 = 0$. This ring has only 2-torsion and 3-torsion, with the 3-torsion being a $\mathbb{Z}_3$ summand in dimensions 3, 10, and 13 (this is known only up to roughly dimension 16).

Note that cobordim groups $\Omega_*^{\langle n\rangle}$ arise as homotopy groups of the Thom spectra $MO\langle n\rangle$, in the sense that the former groups are the homotopy groups of the spectra (this is general for any type of cobordism). Hence the Thom spectrum for the String cobordism ring is $MO\langle 8\rangle$, and $\Omega_*^{\langle 8\rangle} = \pi_*\left(MO\langle 8\rangle\right)$. We can actually gain information about $\Omega_*^{\langle 8\rangle}$ by looking at topological modular forms. This is due to the following fact. Let $MO\langle 8\rangle \to tmf$ be any multiplicative map whose underlying genus is the Witten genus. Then the induced map on the homotopy groups $\pi_* MO\langle 8\rangle \to \pi_* tmf$ is surjective [47]. The low-dimensional homotopy groups of tmf are [47] [6]

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
π_k tmf	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	0	$\mathbb{Z}\oplus\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/6$	0	$\mathbb{Z}$	$\mathbb{Z}/3$	$\mathbb{Z}/2$	$\mathbb{Z}/2$

The 2-primary components ${}_{(2)}\Omega_*^{\langle 8\rangle}$ of $\Omega_*^{\langle 8\rangle}$ are given by [36] (see also [55] [97])

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
${}_{(2)}\Omega_k^{\langle 8\rangle}$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/8$	0	0	$\mathbb{Z}/2$	0	$\mathbb{Z}\oplus\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$(\mathbb{Z})^2$

By comparing the two tables, we can indeed see the 'missing' $\mathbb{Z}/3$ factors.

Note that in dimension 11, the result of [36] implies that $\Omega_{11}^{\langle 8\rangle} = 0$ since the 2-primary part is zero and there is no torsion in that dimension. There does not seem to be a computation for dimensions as high as 27. This implies that the map

$$\varrho : \Omega_{11}^{\langle 8\rangle}(\mathrm{pt}) \longrightarrow \Omega_{27}^{\langle 8\rangle}(\mathrm{pt}) \tag{5.2}$$

[6]Here we prefer to use the notation for cyclic groups used in homotopy theory, e.g. $\mathbb{Z}/2$ in place of $\mathbb{Z}_2$. We hope this will be clear.

is a map whose domain is 0, and is thus not interesting.

We next allow the structure group F_4 so that there is a map from Y^{11} to its classifying space BF_4. Thus we are considering $N = BF_4$ and the classifying map to be f in (5.1). In this case, instead of the map ϱ we will consider the map

$$\varrho' : \Omega_{11}^{\langle 8 \rangle}(BF_4) \longrightarrow \Omega_{27}^{\langle 8 \rangle}(BF_4) \quad (5.3)$$

$$[Y^{11}, f] \longmapsto [M^{27}, f'] \ , \quad (5.4)$$

which maps bordism classes of 11-manifolds, together with a map f to BF_4, to bordism classes of 27-manifolds together with a map f' to BF_4. Now both the domain and the range are in general non-empty unless certain condition are applied.

Remarks

1. The classifying space BF_4 has at least interesting degree four cohomology. However, we have seen that for the String condition to be multiplicative on $\mathbb{O}P^2$ bundles then we must kill x_4 coming from BF_4. This would then mean that we should in this case consider $BF\langle x_4 \rangle$ instead of BF_4.

2. Killing x_4 as above would lead to the rational homotopy type

$$BF_4\langle x_4 \rangle \sim S^{12} \times \text{higher spheres} \ , \quad (5.5)$$

so that the first homotopy is in dimension 12. This then would mean that should consider $\Omega_{11}^{\langle 8 \rangle}(BF_4\langle x_4 \rangle)$, which is zero, by dimension.

3. If we use $BF_4\langle x_4 \rangle$ instead of BF_4, then this might cause some problems for the description of the fields of M-theory in terms of $\mathbb{O}P^2$ bundles, since there we used the Lie group F_4 on the nose. In other words, unlike the case for compact E_8 in eleven dimensions, F_4 appears not merely topologically, but via representation theory. However, compare to the arguments in [86] for the E_8 model of the C-field in M-theory. It should be checked that the representations coming from the Lie 2-group $F_4\langle x_4 \rangle$ respect the discussion in section 2.

We can actually say more about the extensions of the F_4 bundle. We have

Proposition 5.2. *The F_4 bundle on a String manifold Y^{11} can be extended to Z^{12} where $\partial Z^{12} = Y^{11}$.*

Proof. We look for cobordism obstructions. Extending the bundle would be obstructed by $\Omega_{11}^{\langle n \rangle}(BF_4)$. Since the homotopy type of F_4 is $(3, 11, 15, 23)$ then that of BF_4 is $(4, 12, 16, 24)$ so that up to dimension 11 the classifying space BF_4 has the homotopy type of $K(Z, 4)$, much the same as E_8 does (and in fact all exceptional Lie groups except E_6) in that range. Now we reduce the problem to checking whether $\Omega_{11}^{\langle n \rangle}\langle (K(\mathbb{Z}, 4)) \rangle$ is zero. This is indeed so by calculations of Stong [96], for $n = 4$, and Hill [42], for $n = 8$. □

Let $T_{27}^{\langle 8 \rangle}(BF_4)$ be the subgroup of $\Omega_{27}^{\langle 8 \rangle}(BF_4)$ consisting of bordism classes $[M^{27}, f \circ \pi]$, i.e. the classes that factor through the base Y^{11}. It could happen that some of the classes $[Y^{11}, f]$ of the bordism group of the base are zero. Let $\widetilde{T}_{27}^{\langle 8 \rangle}(BF_4)$ be the subgroup whose elements satisfy the additional assumption that $[Y^{11}, f] = 0$ in $\Omega_{11}^{\langle 8 \rangle}(BF_4)$. Corresponding to the diagram (3.48) there is a classifying map

$$\psi : \Omega_{11}^{\langle 8 \rangle}(BF_4) \longrightarrow \Omega_{27}^{\langle 8 \rangle}(\text{pt}) \quad (5.6)$$

which takes the class $[Y^{11}, f]$ to the class $[M^{27} = f^*E]$. The image $T_{27}^{\langle 8 \rangle} = \text{im}\, \psi$ of this map is the set of total spaces of $\mathbb{O}P^2$ bundles in $\Omega_{27}^{\langle 8 \rangle}$. If we forget the classifying map f then instead of (5.6) we can map

$$\lambda : \Omega_{11}^{\langle 8 \rangle}(BF_4) \longrightarrow \Omega_{11}^{\langle 8 \rangle}(\text{pt}) \ , \quad (5.7)$$

where now the class $[Y^{11}, f]$ lands in the class $[Y^{11}]$ by simply forgetting f. Obviously, the kernel of λ makes up the classes $[Y^{11}, f]$ which map to $[Y^{11}]$ that are zero in $\Omega_{11}^{\langle 8\rangle}$. Such classes $[Y^{11}, f]$ map under ψ to total spaces of $\mathbb{O}P^2$ bundles with zero-bordant bases in $\Omega_{11}^{\langle 8\rangle}$. It is clear that $\psi(\ker\lambda)$ is the subgroup $\widetilde{T}_{27}^{\langle 8\rangle}$. That is, we have

$$
\begin{aligned}
T_{27}^{\langle 8\rangle} &:= \operatorname{im}\psi = \Big\{\text{total spaces of } \mathbb{O}P^2 \text{ bundles in } \Omega_{27}^{\langle 8\rangle}(\mathrm{pt})\Big\} && (5.8)\\
\widetilde{T}_{27}^{\langle 8\rangle} &:= \psi(\ker\lambda)\\
&= \Big\{\text{total spaces of } \mathbb{O}P^2 \text{ bundles with zero bordant base in } \Omega_{27}^{\langle 8\rangle}(\mathrm{pt})\Big\}. && (5.9)
\end{aligned}
$$

Note that, as mentioned above, the 2-primary part of $\Omega_n^{\langle 8\rangle}$ for $n \leq 16$ is calculated in [36]. For $n = 11$ this is zero. This implies that the kernel of λ is all of $\Omega_{11}^{\langle 8\rangle}(BF_4)$, i.e. all cobordism classes of total spaces have zero bordant bases. Then we have

Proposition 5.3. *$T_{27}^{\langle 8\rangle}$ and $\widetilde{T}_{27}^{\langle 8\rangle}$ coincide for base String manifolds of dimension eleven.*

There are two cases to consider in order to determine whether or not the above spaces are trivial:
1. If $\Omega_{27}^{\langle 8\rangle}$ turns out to be zero, then the map ψ will be trivial in that degree.
2. If it turns out that $\Omega_{27}^{\langle 8\rangle} \neq 0$, then the map ψ is not trivial. It would then mean that $T_{27}^{\langle 8\rangle} = \widetilde{T}_{27}^{\langle 8\rangle} \neq \emptyset$. However, looking carefully at the map ψ we notice that its domain is zero. This is because the homotopy type of F_4 is $K(\mathbb{Z}, 3)$ up to dimension ten, so that the homotopy type of BF_4 is $K(\mathbb{Z}, 4)$ up to dimension eleven. This means that $\Omega_{11}^{\langle 8\rangle}(BF_4) = \Omega_{11}^{\langle 8\rangle}((K\mathbb{Z}, 4)) = 0$. This then implies that the map ψ is trivial. In modding out by the corresponding equivalence to form

$$E_{27}^{\mathbb{O}} = E_{27}^{\langle 8\rangle} = \Omega_{27}^{\langle 8\rangle}/T_{27}^{\langle 8\rangle}\ , \tag{5.10}$$

we simply get

Proposition 5.4. *The homology theory is just the bordism ring $E_{27}^{\mathbb{O}} = \Omega_{27}^{\langle 8\rangle}$.*

Remarks
1. Proposition 5.4 implies that in dimension 27 we do not get anything smaller or simpler than bordism.
2. The two spaces (5.9) and (5.9) have been characterized in the quaternionic case, i.e. when the fiber is $\mathbb{H}P^2$ with structure group $PSp(3)$, as

$$
\begin{aligned}
T_{27}^{\langle 4\rangle} &= \ker(\alpha) && (5.11)\\
\widetilde{T}_{27}^{\langle 4\rangle} &= \ker(\Phi_{\mathrm{och}}), && (5.12)
\end{aligned}
$$

i.e. as the kernels of the Atiyah invariant in [92] and the Ochanine genus in [60], respectively. We see that in our case, $\alpha(M^{27}) = 0$, but $\Phi_{\mathrm{och}}(M^{27})$ is not necessarily zero. This provides another justification for the calculations leading to theorem 3.21. In fact, we can use the nontriviality of the Ochanine genus to check whether or not the homology theory is empty. Since, using Theorem 3.21, we can find a 27-dimensional manifold M^{27} with $\Phi_{\mathrm{och}}(M^{27}) \neq 0$, the Spin cobordism group is nonzero $\Omega_{27}^{\langle 4\rangle} \neq 0$. Consequently, we have the following result for the corresponding String cobordism group.

Theorem 5.5. $\Omega_{27}^{\langle 8\rangle} \neq 0$.

Remark. Alternatively, the theorem can proved using information about tmf. Since the orientation map from $M\mathrm{String} = MO\langle 8\rangle$ to tmf is surjective [6] then it is enough to know that the homotopy group of tmf in dimension 27 is nonzero. Indeed, [7] at least $\pi_{27}(tmf) \supset \mathbb{Z}/3$, so that $\Omega_{27} = \pi_{27}(M\mathrm{String}) \neq 0$.

In [97], the Witten genus was proposed as a candidate for the replacement of α in the octonionic case, so that

$$T_{27}^{\langle 8\rangle}(\mathrm{pt}) = \ker(\alpha^{\mathbb{O}}) := \ker(\Phi_W). \tag{5.13}$$

Indeed, we have shown in Proposition 3.15 that the Witten genus is zero for our 27-dimensional manifolds, which are $\mathbb{O}P^2$ bundles. The extension of the the 'new Atiyah invariant' $\alpha^{\mathbb{O}}$ would be to a 'new Ochanine genus'

$$\Phi_{\mathrm{och}}^{\mathbb{O}} : \Omega_*^{\langle 8\rangle} \longrightarrow \mathbb{Q}[E_4, E_6][[q]], \tag{5.14}$$

i.e. to the power series ring over rationalized coefficients of level 1 elliptic cohomology, such that the constant term is the Witten genus. We have seen in theorem 3.3 that the Witten genus of $\mathbb{O}P^2$ is zero, so that in the current context, the constant term is zero. We do not know what the higher terms are, and so they can conceivably be nonzero. The 'new Ochanine genus' is expected to be related to $K3$-cohomology. Such a theory has not yet been explicitly constructed but it should exist.

Define the functor $X \to \Omega_*^{\langle 8\rangle}(X)/\mathcal{I}$, where $\mathcal{I}$ is the ideal introduced in the beginning of this section. The question is whether this is a generalized (co)homology theory. The desired homology theory $E_n^{\mathbb{O}}$ is formed by dividing $\Omega_*^{\langle 8\rangle}$ by $\widetilde{T}_n$ and inverting the primes 2 and 3 [97]. However, there is one extra condition required, which is the invertibility of the element $v = \mathbb{O}P^2$. By taking the limit in

$$E_n^{\mathbb{O}}(X)[\mathbb{O}P^2]^{-1} = \lim_j E_{n+16j}^{\mathbb{O}}(X) \tag{5.15}$$

over the sequence of homomorphisms given by multiplying by $\mathbb{O}P^2$ the resulting theory is

$$e\ell\ell_*^{\mathbb{O}}(X) = E_*^{\mathbb{O}}(X)[\mathbb{O}P^2]^{-1} = \bigoplus_{k\geq 0} \Omega_{*+16k}(X)/\sim, \tag{5.16}$$

where the equivalence relation $\sim$ is generated by identifying $[Y, f] \in \Omega_*^{\langle 8\rangle}(X)$ with $[M, f\circ\pi] \in \Omega_{*+16k}^{\langle 8\rangle}(X)$ for an $\mathbb{O}P^2$ bundle $\pi : M \to Y$, with structure group $\mathrm{Isom}\,\mathbb{O}P^2 = F_4$, i.e. the total space of an $\mathbb{O}P^2$ bundle is identified with its base. A full construction of this theory is not yet achieved by homotopy theorists but it is believed that this should be possible in principle. We mentioned towards the end of Section 5.1 that $KO^*(\mathrm{pt})$ can be made into an Ω_*^{spin}-module and the existence of an isomorphism relating $KO^*(X)$ and $KO^*(\mathrm{pt})$. The octonionic version of Kreck-Stolz theory is arrived at by replacing $KO^*(\mathrm{pt})$ by $e\ell\ell_n^{\mathbb{O}}(\mathrm{pt})$, i.e.

$$\Omega_*^{\langle 8\rangle}(X) \otimes_{\Omega_*^{\langle 8\rangle}} e\ell\ell_*^{\mathbb{O}}(\mathrm{pt}) \longrightarrow \mathrm{e}\ell\ell_*^{\mathbb{O}}(\mathrm{X}) \tag{5.17}$$

is an isomorphism away from the primes 2 and 3 [97].

Remarks.

1. The model for elliptic homology in fact involves indefinitely higher cobordism groups in increments of 16,

$$e\ell\ell_{11}^{\mathbb{O}}(Y^{11}) = \bigoplus_{k\geq 0} \Omega_{11+16k}/\sim, \tag{5.18}$$

where $\sim$ is an equivalence that provides a correlation between topology in M-theory and topology in dimensions $27, 43, \cdots, 11+16k, \cdots, \infty$. We have two points to make:

[7] I thank Mike Hill for pointing out the $\mathbb{Z}/3$ summand in this homotopy group.

- The first bundle with total space an $\mathbb{O}P^2$ bundle over Y^{11} is related to Ramond's Euler multiplet.
- As the pattern continues in higher and higher dimensions, one is tempted to seek physical interpretations for such theories as well. While this direction is tantalizing, we do not pursue it in this paper.

2. There is another homology theory that one can form, namely by identifying the image of ψ with the trivial bundle as in [60]. The construction is analogous. The advantage here is that we do not kill $\O P^2$, as dividing by T has the effect of killing the fiber.

We have seen connections between eleven-dimensional M-theory and the putative theory in twenty-seven dimensions. If the latter theory in twenty-seven dimensions is fundamental, then it should ultimately be studied also without restricting to the relation to M-theory. This is analogous to the case of M-theory itself in relation to ten-dimensional type IIA string theory. Since M-theory is, as far as we know, a fundamental theory, then it should be (and it is being) studied without necessarily assuming a circle bundle for the eleven-dimensional manifold. In other words, what about 27-dimensional manifolds that are not the total space of $\mathbb{O}P^2$ bundles over eleven-manifolds? Hence

Proposal. To study the bosonic theory as a fundamental theory in twenty-seven dimensions we should also consider modding out by the equivalence relation (the ideal).

For example, extension problems can be studied in this way.

5.3 Families

It is desirable to consider the $\mathbb{O}P^2$ bundle as a family problem of objects on the fiber of M^{27} parametrized by points in the base Y^{11}. The family of these 16-dimensional String manifolds will define an element of the cobordism group

$$MO\langle 8\rangle^{-16}(Y^{11}). \tag{5.19}$$

Remarks 1. We have seen in section 3.3.2 that the total space of an $\mathbb{O}P^2$ bundle is not necessarily String even if Y^{11} is String. However, we do get a family of String manifolds provided we kill the degree four class pulled back from BF_4 (see Prop. 3.14).
2. Unfortunately, genera are multiplicative on fiber bundles so that the vanishing of $\Phi_W(\mathbb{O}P^2)$ will force the Witten genus of M^{27} to be zero as well. Also taking higher and higher bundles – so as to get fibers of dimensions higher than 16– as in (5.18) will not help in making the Witten genus nonzero. tmf is the home of the parametrized version of the Witten genus, but we do not see modular forms in this picture. This is to be contrasted with the $\mathbb{H}P^2$ case where the Witten genus is $E_4/288$.
3. Nevertheless, the elliptic genus $\Phi_{\rm ell}$ of $\mathbb{O}P^2$ is not zero, so the total space will not automatically have a zero elliptic genus. However, elliptic genera are defined for Spin manifolds of dimension divisible by 4. Our base space Y^{11} is eleven-dimensional and so will automatically have zero elliptic genus. This also applies for the Witten genus. One way out of this is instead to consider the bounding twelve-dimensional theory, i.e. the extension of the topological terms from $Y^{11} = \partial Z^{12}$ to Z^{12} as in [101]. If we also take a 28-dimensional

coboundary for M^{27}, i.e. $\partial W^{28} = M^{27}$, we would then have

$$
\begin{array}{ccccc}
\mathbb{O}P^2 \longrightarrow & M^{27} & \hookrightarrow & W^{28} & \\
& \downarrow \pi & & \downarrow \pi & \searrow \\
& Y^{11} & \hookrightarrow & Z^{12} & \xrightarrow{f} BF_4\ .
\end{array}
\tag{5.20}
$$

Such an extension would involve cobordism obstructions. The manifolds extend nicely, as $\Omega_{11}^{\langle n \rangle} = 0$ for both $n = 4$ (Spin) and $n = 8$ (String). The bundles also extend as shown in Proposition 5.2. It is tempting to propose that the theories should be defined on the $(12 + 16m)$-dimensional spaces, and then restriction to the boundaries would be a special instance.

We have provided evidence for some relations between M-theory and an octonionic version of Kreck-Stolz elliptic homology. Strictly speaking, both theories are conjectural, and we hope that this contribution motivates more active research both on completing the mathematical construction of this elliptic homology theory (part of which is outlined in [97]) as well as making more use of the connection to M-theory. In doing so, we even hope that M-theory itself would in turn give more insights into the homotopy theory.

In closing we hope that further investigation will help shed more light on the mysterious appearance of the exceptional groups E_8 and F_4 and to give a better understanding of their role in M-theory.

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6 Appendix: Some Properties of $\mathbb{O}P^2$

In this appendix we summarize the topological properties of the Cayley plane $\mathbb{O}P^2$ which are useful for proving some of the results in the text.

1. **Betti numbers:** The only nonzero Betti numbers are $b_0 = 1$, $b_8 = 1$, $b_{16} = 1$.

2. **Integral cohomology:** The cohomology ring is

$$
H^*(\mathbb{O}P^2; \mathbb{Z}) = \mathbb{Z}[u]/u^3, \tag{6.1}
$$

   where $u \in H^8(\mathbb{O}P^2; \mathbb{Z})$ is the canonical 8-dimensional generator coming from S^8. Thus $H^0\mathbb{Z} = H^8\mathbb{Z} = H^{16}\mathbb{Z} = \mathbb{Z}$ and $H^i = 0$ otherwise. Note that there is no torsion in cohomology. Consider the last Hopf map $S^7 \longrightarrow S^{15} \xrightarrow{f} S^8$. The spheres S^7 and S^8 are oriented, so that generators $a \in H^7(S^7;\mathbb{Z}) = \mathbb{Z}$ and $b \in H^8(S^8;\mathbb{Z})$ can be specified. The mapping cone $\mathcal{C}(f)$ is $\mathbb{O}P^2$. The exactness of the cohomology long exact sequence corresponding to f gives the isomorphisms

$$\begin{aligned} \imath &: H^{15}(S^{15};\mathbb{Z}) \xrightarrow{\cong} H^{16}(\mathbb{O}P^2;\mathbb{Z}) \\ \jmath^* &: H^8(\mathbb{O}P^2;\mathbb{Z}) \xrightarrow{\cong} H^8(S^8;\mathbb{Z}) \ . \end{aligned} \tag{6.2}$$

Let $a' = \imath(a) \in H^{16}(\mathbb{O}P^2;\mathbb{Z})$, and let $u \in H^8(\mathbb{O}P^2;\mathbb{Z})$ be the unique element such that $\jmath^*(u) = b$. Since $H^{16}(\mathbb{O}P^2;\mathbb{Z}) = \mathbb{Z}$ then there exists a unique integer $H(f)$, the *Hopf invariant*, such that $u \cup u = H(f)a'$. It is a classical result that this is equal to one. Therefore $a' = u^2$. This justifies the above claim about the cohomology of $\mathbb{O}P^2$.

3. **Euler class:** Let u be a generator of $H^8(\mathbb{O}P^2;\mathbb{Z})$. The Euler class of $\mathbb{O}P^2$ is $e = \pm 3u^2$.

4. **Pontrjagin classes**: The total tangential Pontrjagin class is given by [20]

$$p(T\mathbb{O}P^2) = 1 + 6u + 39u^2, \tag{6.3}$$

so that the nonzero Pontrjagin classes are $p_2 = 6u$, $p_4 = 39u^2$. Choosing that orientation which is defined by u^2, the non-vanishing Pontrjagin numbers are $p_2^2[\mathbb{O}P^2] = 36$, $p_4[\mathbb{O}P^2] = 39$.

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