On anomalies of E_8 gauge theory on String manifolds

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Abstract

In this note we revisit the subject of anomaly cancelation in string theory and M-theory on manifolds with String structure and give three observations. First, that on String manifolds there is no $E_8 \times E_8$ global anomaly in heterotic string theory. Second, that the description of the anomaly in the phase of the M-theory partition function of Diaconescu-Moore-Witten extends from the Spin case to the String case. Third, that the cubic refinement law of Diaconescu-Freed-Moore for the phase of the M-theory partition function extends to String manifolds.

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1 Introduction

The DMW anomaly [3] in the comparison of the partition functions of M-theory and type IIA string theory is given by the vanishing of the seventh integral Stiefel-Whitney class $W_7(X)$ of spacetime X. The cancelation of this anomaly in [9] leads to the emergence of elliptic cohomology and other generalized cohomology theories. The "String condition", i.e. the vanishing of half the Pontrjagin class $\frac{1}{2}p_1(X)$ is stronger than vanishing of W_7. There are also connections to generalized cohomology from the perspective of type IIB (target) string theory [10] [11]. It is natural then to ask how much the String condition plays in global aspects of string theory. It is the purpose of this note to provide a few observations that provide one step in shedding some light on this question.

Manifolds satisfying the String condition are called " String manifolds" and are characterized by having a String structure, i.e. with a lifting of the structure group of the tangent bundle from Spin $= O\langle 3\rangle$ to its 3-connected cover String $= O\langle 7\rangle$. Generic examples of String manifolds occur when the manifold is highly connected. The simplest case is perhaps the spheres, which occur in the compactification of eleven- and ten-dimensional supergravity (coupled to Yang-Mills) leading to gauged supergravity in lower dimensions (see [4]). While compact manifolds X with special holonomy G_2 and Spin(7) require a non-vanishing $p_1(X) \neq 0 \in H^4(X;\mathbb{Z})$ [8], topological generalized such structures can be satisfied for manifolds with vanishing p_1 [16]: S^7 admits a topological generalized G_2 structure and any manifold J of the form $S^1 \times N^7$, where N^7 is a Spin manifold, admits a topological generalized Spin(7) structure, since J (trivially) satisfies $p_1(J)^2 - 4p_2(J) = 0$ and $\chi(J) = 0$. Note that there is a difference between p_1 being zero and $\frac{1}{2}p_1$ being zero, due to the possible existence of 2-torsion. Furthermore, there exist *flat* manifolds of finite holonomy groups that have $p_1 \neq 0 \in H^4\mathbb{Z}$ [13].

The goal of this note is to point out the following:
(1) The global anomaly for heterotic string theory vanishes on String manifolds.
(2) The partition functions of M-theory and type IIA string theory match in the sense of [3] on String manifolds.
(3) The description of the phase of the M-theory partition function as a cubic refinement [2] extends to String manifolds.

One might wonder whether all what one needs to do to go from the Spin case to the string case is set the first Pontrjagin class of the tangent bundle to zero, which would essentially make trivial the task we set out to achieve. This turns out to be naive because we are considering global questions. In particular, we are not guaranteed that the desired obstructions vanish. More precisely, we are trying to extend bundles on String manifolds rather that on Spin manifolds, and the corresponding cobordism groups may introduce new invariants and obstructions in going from the Spin to the String case. That we show that this is not the case is not immediate and is in fact nontrivial. Mathematically, the results are possible due to the recent calculations of $M\text{String}_n\left(K(\mathbb{Z},4)\right)$ for $n \leq 14$ in the companion paper [7]. That paper provides the main technical mathematical results and this note provides the physical motivation and the corresponding consequences.

One might raise the following objection: If the anomaly vanishes for weaker condition then should it not vanish for the stronger one? The answer is not obvious as all without nontrivial calculations, and in fact the notions of "weaker" and "stronger" might be misleading in this context. We know that $\Omega_{11}^{\text{spin}}\left(K(\mathbb{Z},4)\right) = 0$ but we are not guaranteed that the corresponding cobordism group on String manifolds vanish. We show that this is the case, i.e. that $M\text{String}_{11}\left(K(\mathbb{Z},4)\right)$ is zero, and hence there are no new obstructions and the extension continues to be possible. A less nontrivial point is to check whether the string condition is preserved in forming the 11-dimensional mapping torus and its corresponding 12-dimensional bounding manifold. The global anomaly vanishes for S^{10} [17].

The objection would likewise be on both the cancelation of the DMW anomaly and the matching of the phases of the partition function. On the former there are indeed no subtleties, but the latter requires the analysis of subtle mod 2 invariants. In particular the question of extension of the $\mathbb{Z}_2$ invariant $f(a+b)-f(a)-f(b)$ of [3] becomes a question of extension on String manifolds of $M\mathrm{String}_{10}(Z)$ rather than $M\mathrm{Spin}(Z)$ for the Spin case, where Z is $K(\mathbb{Z},4)\times K(\mathbb{Z},4)$ or $K(\mathbb{Z},4)\wedge K(\mathbb{Z},4)$. Indeed one of the results is that $M\mathrm{String}_{10}(Z)=M\mathrm{Spin}_{10}(Z)$, so that there are no new invariants, hence obstructions, beyond the one coming from the Spin case.

2 Global Anomalies in $D=10$, $N=1$ Supergravity

We consider the following setting. An $E_8\times E_8$ bundle $V_1\oplus V_2$ on a ten-dimensional manifold M^{10}. We assume M^{10} to have a String structure, ie. that the spin bundle SM^{10} of M^{10} admits a lifting of the structure group from Spin(10) to its 7-connected cover String(10). The condition for such a lift is given by $\lambda(TM^{10})=0$, where $\lambda=\frac{1}{2}p_1$ is half the Pontrjagin class.

Due to the homotopy type of E_8, the $E_8\times E_8$ bundle on M^{10} is completely characterized by the degree four class

$$a(V_1\oplus V_2)=a(V_1)+a(V_2), \tag{2.1}$$

where $a=\frac{1}{2}p_1$. The anomaly cancelation condition is given by

$$\lambda(TM^{10})-a(V_1\oplus V_2)=0. \tag{2.2}$$

Assuming M^{10} to admit a String structure implies that

$$a(V_1\oplus V_2)=0. \tag{2.3}$$

Note that this does not necessarily imply that each factor is separately zero. We will thus take the condition to be

$$a(V_1)=-a(V_2), \tag{2.4}$$

so that V_1 has characteristic class a if and only if V_2 has corresponding class $-a$. Note that every class in $K(\mathbb{Z},4)$ represents the characteristic class of an E_8 bundle. We have the following fact from [18]

Lemma 2.1. *The effective action of heterotic string theory is invariant under diffeomorphisms $\varphi: M^{10}\to M^{10}$ that admit a lift to the spin bundle of M^{10} and to $V_1\oplus V_2$.*

Global anomalies are concerned with diffeomorphisms and/or gauge transformations that are not connected to the identity. The main question here is the string analog of the question raised in the spin case in [18]: *Is the effective action of $N=1$ supergravity with group $E_8\times E_8$ invariant under such φ?*
The study of global anomalies requires considering the mapping cylinder

$$X^{11}=\left(M^{10}\times S^1\right)_\varphi=\left(M^{10}\times I\right)/(x,0)\sim(\phi(x),1), \tag{2.5}$$

and asking whether the bundles extend to X^{11}.

Lemma 2.2. **i.** *If M^{10} is a string manifold then so is X^{11}.*
ii. *The bundle $V_1 \oplus V_2$ can be extended to an $E_8 \times E_8$ bundle over X^{11}.*
iii. *X^{11} bounds a 12-dimensional string manifold B^{12}.*

Proof. The first part follows from the multiplicative behavior of the Spin characteristic classes under Whitney sum. For part (ii), note that the String analog of lemma (2.1) holds. Then the action of ϕ on $V_1 \oplus V_2$ leads to the identification of the fiber of $V_1 \oplus V_2$ at $(\phi(x), 1)$ with the fiber at $(x, 0)$. For the third part we have from [6] that $M\mathrm{String}_{11}(\mathrm{pt}) = 0$ and thus any 11-dimensional string manifold bounds a 12-dimensional string manifold. □

We now have

Proposition 2.3. *The bundle $V_1 \oplus V_2$ extends from X^{11} to an $E_8 \times E_8$ bundle $W_1 \oplus W_2$ over B^{12}. Furthermore, we can have $a(W_1 \oplus W_2) = 0$.*

Proof. V_1 extends over B^{12} if and only if the cohomology class $a(V_1)$ extends to a cohomology class in $H^4(B^{12}; \mathbb{Z})$. Considering the pair (X^{11}, β). This vanishes if X^{11} bounds a String manifold B^{12} over which the class $\beta \in H^4(X^{11}; \mathbb{Z})$ can be extended. This means that the pair is an element of the cobordism group $\Omega_{11}(K(\mathbb{Z}, 4))$. It is shown in [7] that $M\mathrm{String}_{11}(K(\mathbb{Z}, 4))$ is zero. Therefore there is no obstruction and so B^{12} can always be chosen so that V_1 extends to an E_8 bundle W_1 over B^{12}.

Next we extend V_2. Corresponding to the inclusion $\iota : X^{11} \hookrightarrow B^{12}$, the tangent bundle of B^{12} decomposes as

$$TB^{12}|_{X^{11}} = TX^{11} \oplus \mathcal{N}, \tag{2.6}$$

where $\mathcal{N}$ is the normal bundle of X^{11} in B^{12}. Being a trivial real line bundle, $\mathcal{N}$ does not change the fact that the class λ of TX^{11} is zero, i.e.

$$\lambda(TX^{11}) = \lambda(TX^{11} \oplus \mathcal{O}) = 0. \tag{2.7}$$

extends to a trivial cohomology class in $H^4(B^{12}; \mathbb{Z})$. Since $a(V_1)$ extends to $a(W_1) \in H^4(B^{12}; \mathbb{Z})$ then $\beta = -a(W_1)$ is an element of $H^4(B^{12}; \mathbb{Z})$. Therefore, V_1 and V_2 both extend over B^{12}. Furthermore, the extension can be done in such a way that $a(W_1) = -a(W_2)$. □

Theorem 2.4. *There are no global anomalies for $E_8 \times E_8$ bundles $V_1 \oplus V_2$ on string manifolds M^{10}.*

Proof. Given that $(M^{10} \times S^1)_\varphi$ bounds a string manifold B^{12} over which $V_1 \oplus V_2$ can be extended, the change in the effective action S under φ will be as in the spin case [17] [18]

$$\Delta S = 2\pi i \left[\int_{B^{12}} \left(tr_1 F^2 + tr_2 F^2 - tr R^2 \right) \wedge I_8 - \int_{(M^{10} \times S^1)_\varphi} H_3 \wedge I_8 \right], \tag{2.8}$$

where

- F is the curvature of the E_8 bundle so that $tr_i F^2$ is the Chern class of the bundle W_i, $i = 1, 2$,
- R is the curvature of the tangent bundle TB^{12}, so that trR^2 is the Pontrjagin class,
- I_8 is the Green-Schwarz anomaly polynomial in characteristic classes of the gauge and tangent bundles, and hence satisfies $dI_8 = 0$.

Since $V_1 \oplus V_2$ extends to $W_1 \oplus W_2$ on B^{12} such that $a(W_1) + a(W_2) = 0 = \lambda(TB^{12})$ and hence (trivially) that $a(W_1 \oplus W_2) + \lambda(TB^{12}) = 0$, then the classes are trivial in cohomology and hence exact. Then H_3 can be defined to obey

$$dH_3 = tr_1 F^2 + tr_2 F^2 - trR^2 \tag{2.9}$$

not just on the mapping cylinder $(M^{10} \times S^1)_\varphi$ but also on the bounding manifold B^{12}. Inserting (2.9) in (2.8) we get

$$\Delta S = 2\pi i \left[\int_{B^{12}} dH_3 \wedge I_8 - \int_{(M^{10}\times S^1)_\varphi} H_3 \wedge I_8 \right]. \tag{2.10}$$

Now using Stokes' theorem for $(M^{10} \times S^1)_\varphi = \partial B^{12}$ the invariance result $\Delta S = 0$ follows. □

3 Refinement of the M-theory/IIA partition Function

The setting for the comparison of M-theory and type IIA string theory is as follows. M-theory is 'defined' on the eleven-dimensional total space Y^{11} of a circle bundle with base X^{10}, a ten-dimensional manifold on which type IIA string theory is defined. Both spaces X^{10} and Y^{11} are usually taken to be Spin manifolds and they have additional structures on them. On Y^{11} there is an E_8 bundle V. Due to the homotopy type of E_8 up to dimension 14, V over Y^{11} is completely characterized by a degree four class a as above. Various kinds of spinors are defined on both Y^{11} and X^{10}. In particular, for our purposes, there are elements λ of $\Gamma(SY^{11} \otimes V)$, i.e. sections of the spin bundle SY^{11} coupled to the E_8 vector bundle. On X^{10}, in addition to spinors, there are also the Ramond-Ramond (RR) fields of even degrees. In particular, there is a degree four field F_4. Such fields are images in cohomology of K-theory elements x, as they obey the quantization condition [14] [5] $F := \sum_{i=0}^{5} F_{2i} = \sqrt{\widehat{A}(X^{10})}\ \mathrm{ch}(x)$.

The comparison of the E_8 gauge theory description of M-theory to the K-theoretic description of type IIA string theory was performed in [3] at the level of the partition functions and is shown to agree upon dimensional reduction, i.e. integration over the S^1 fiber. The comparison involves those cohomology classes that lift to K-theory and the identification involve subtle torsion and mod 2 expressions.

Definition 3.1. The phase of the M-theory partition function on an eleven-dimensional spin manifold Y^{11} is [3] $\Phi_a = (-1)^f(a)$, where $f(a)$ is the mod 2 index of a Dirac operator coupled to V with characteristic class a.

Remarks
1. $f(a) = 0$ for $a = 0$, in which case $V = \bigoplus^{248} \mathcal{O}$, i.e. 248 copies of a trivial bundle.
2. The comparison to type IIA requires a corresponding mod 2 invariant in $KO(X^{10})$: For $x \in K(X^{10})$, $j(x)$ is the mod 2 index with values in the KO class $x \otimes \overline{x}$.
3. The refinement of the partition function to elliptic cohomology E in [9] introduces a mod 2 index with values in the $EO_2(X^{10}) = \mathbb{Z}_2[q]$ class $x \otimes \overline{x}$, where x is a class in $E(X^{10})$.
4. $f(a)$ cannot be a cubic function in $a \in H^4\mathbb{Z}$ since that would have dimension 12, which is greater than the dimensions of either X^{10} or Y^{11}.

The comparison between M-theory on Y^{11} and type IIA string theory on X^{10} proceeds from the embedding

$$SU(5) \times SU(5)/\mathbb{Z}_5 \subset E_8 \tag{3.1}$$

so that out of two $SU(5)$ vector bundles E and E' of Chern classes $c_2(E) = -a$ and $c_2(E') = -a'$ one builds an E_8 bundle whose characteristic class is $a + a'$. This requires a and a' to be elements of $H^4(X^{10}, \mathbb{Z})$ that lift to K-theory. The idea is then to compute $f(a+a') - f(a) - f(a')$. Using the decomposition (3.1), this is

$$\int_{X^{10}} c_2(E) c_3(E') \mod 2 = \int_{X^{10}} c_3(E) c_2(E') \mod 2. \tag{3.2}$$

Since $c_1(E) = 0$ then $c_3(E) = Sq^2(c_2(E)) \mod 2$, and similarly for E'. Then the main result on $f(a)$ is that it is a quadratic refinement of a bilinear form, i.e. f satisfies [3]

$$f(a + a') = f(a) + f(a') + \int_{X^{10}} a \cup Sq^2 a' . \tag{3.3}$$

This has an interpretation in terms of cobordism as follows [3].

Remarks
1. A result of Atiyah and Singer [1] states that the mod 2 index of the Dirac operator coupled to a vector bundle V on a Spin manifold X vanishes if X is the boundary of a Spin manifold over which V extends. $f(a)$ can be regarded as a $\mathbb{Z}_2$-valued function which vanishes when a extends to a Spin manifold B^{11} bounding X^{10} [3]. The class $a \in H^4(X^{10}, \mathbb{Z})$ can be extended to a bounding manifold B^{11} if and only if the map $\mu : X^{10} \to K(\mathbb{Z}, 4)$ can be extended to a map $\mu' : B^{11} \to K(\mathbb{Z}, 4)$, i.e. that (X^{10}, a) is zero as an element of the cobordism group $\Omega_{10}^{\text{spin}}(K(\mathbb{Z}, 4))$. Thus $f(a)$ can be viewed as an element of $\text{Hom}\left(\Omega_{10}^{\text{spin}}(K(\mathbb{Z}, 4)), \mathbb{Z}_2\right)$. More precisely, since $f(a) = 0$ for $a = 0$ then [3] $f \in \text{Hom}\left(\widetilde{\Omega}_{10}^{\text{spin}}(K(\mathbb{Z}, 4)), \mathbb{Z}_2\right)$.
2. A result of Stong [15] states that $\widetilde{\Omega}_{10}^{\text{spin}}(K(\mathbb{Z}, 4)) = \mathbb{Z}_2 \times Z_2$. Thus there are two invariants: one is linear [15]

$$v(a) = \int_{X^{10}} a \cup w_6 = \int_{X^{10}} a \cup Sq^2 w_4, \tag{3.4}$$

with $v(a + a') = v(a) + v(a')$, and another– the more relevant one– is nonlinear [12]

$$Q(a + a') = \int_{X^{10}} a \cup Sq^2 a' = \int_{X^{10}} (Sq^2 a) \cup a'. \tag{3.5}$$

$Q(a_1, a_2)$ vanishes if both a_1 and a_2 can be extended to B^{11} or if either a_1 or a_2 is zero. This leads to [3]: *$Q(a_1 + a_2)$ is a homomorphism from the bordism group $\Omega_{10}^{\text{spin}}(K(\mathbb{Z}, 4) \wedge K(\mathbb{Z}, 4)) = \mathbb{Z}_2$ to $\mathbb{Z}_2$. Thus, there is only one cobordism invariant Q.*

The first observation is straightforward

Observation 3.2. *There is no DMW anomaly in the M-theory partition function for String manifolds.*

Proof. The DMW anomaly for Spin manifolds is given by the vanishing of the seventh integral Stiefel-Whitney class $W_7 = 0$ [3]. Since $W_7 = Sq^3 \lambda$ then $\lambda = 0$ implies that $W_7 = 0$ and hence no anomaly. This fact has also been observed in [9]. □

The invariant in the case of String cobordism is still the Landweber-Stong invariant $Q(a_1, a_2)$ [7]. The analysis follows that of [3]. The main result in this section is then

Theorem 3.3. *The phases of M-theory and type IIA string theory coincide not just on Spin manifolds but also on String manifolds, i.e. $\Phi_M = \Phi_{IIA}$. Consequently, the corresponding partition functions match.*

4 The Cubic Refinement Law

In [2] the phase Φ of the M-theory partition function was interpreted as a cubic refinement of a trilinear form on the group $\check{H}^4(Y^{11})$ of degree four differential characters

$$\frac{\Phi_{(123)}\Phi_{(1)}\Phi_{(2)}\Phi_{(3)}}{\Phi_{(12)}\Phi_{(13)}\Phi_{(23)}\Phi} = (\check{a}_1\check{a}_2\check{a}_3)(Y^{11}) \in U(1) \ , \tag{4.1}$$

where we denoted $\Phi_{(ijk)} := \Phi([\check{C}] + \check{a}_i + \check{a}_j + \check{a}_k)$ and so on. The validity of this formula requires that $\check{C}$ and all $\check{a}_i$, $i = 1, 2, 3, 4$, extend simultaneously on the same Spin twelve-dimensional manifold Z^{12}. In the Spin case, the obstruction to extending $(Y^{11}, a_1, \cdots, a_k)$ is measured by $\Omega_{11}^{\rm spin}(\wedge^k K(\mathbb{Z}, 4))$. The group is zero for $k = 1$ by Stong's result. The result for $k = 2, 3, 4$ follows from an application of the Atiyah-Hirzebruch spectral sequence [2].

Now we would like to replace the String condition on Y^{11} with the String condition, i.e. we will assume that $\frac{1}{2}p_1(Y^{11}) = 0$. We know from [6] that $\Omega_{11}^{\rm String} = 0$. Furthermore, we know from [7] that $\Omega_{11}^{\rm String}(\wedge^k K(\mathbb{Z}, 4)) = 0$, for $k = 1, 2$. Extending the result to the case $k = 3, 4$ via the Atiyah-Hirzebruch spectral sequence we get

Proposition 4.1. *The cubic refinement law holds and is defined for String eleven-manifolds Y^{11}.*

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