On the branch point index of minimal surfaces

Stefan Hildebrandt and Anthony J. Tromba

Summary. Let Γ be a closed, sufficiently smooth Jordan curve in $\mathbb{R}^3$ and denote by $\mathcal{C}(\Gamma)$ the class of disk-type surfaces $X \in H^{1,2}(B,\mathbb{R}^3)$ which map ∂B continuously and monotonically onto Γ. Then any minimal surface $X \in \mathcal{C}(\Gamma)$ possesses only finitely many branch points in $\overline{B}$, and the *order* of any such point is well-defined, and also the *index* of an interior branch point is defined in a natural way if X is nonplanar. We show that also the index of boundary branch points can be defined if the curvature κ and the torsion τ of Γ are strictly nonzero. Secondly we derive upper bounds for the index of any branch point in terms of the total curvature of Γ or of its cut number.

In this paper, a *minimal surface*[1] is to be understood as a "disk-type minimal surface in $\mathbb{R}^3$", that is, as a nonconstant harmonic and conformally parametrized mapping $X : B \to \mathbb{R}^3$ of the unit disk $B := \{w = (u,v) \in \mathbb{R}^2 : |w| < 1\}$ into $\mathbb{R}^3$. In other words, a minimal surface X is a mapping of class $C^2(B,\mathbb{R}^3)$ which satisfies

$$\Delta X = 0 \text{ in } B \tag{1}$$

as well as

$$|X_u|^2 = |X_v|^2, \ \langle X_u, X_v\rangle = 0 \text{ in } B\ . \tag{2}$$

Using the Wirtinger operator $\frac{\partial}{\partial w} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right)$, we can write (2) in the complex form

$$\langle X_w, X_w\rangle = 0 \text{ in } B\ . \tag{3}$$

For any minimal surface X there is an *isotropic curve* $f : B \to \mathbb{C}^3$ such that $X = \text{Re} f$. Here "isotropic" means that f is a holomorphic curve satisfying $\langle f', f'\rangle = 0$. Since $f' = 2X_w$, the mapping $X_w : B \to \mathbb{C}^3$ is holomorphic.

A point $w_0 \in B$ is called *interior branch point* of X if $X_u(w_0) = 0$, which is equivalent to $X_w(w_0) = 0$, or to $f'(w_0) = 0$. Expanding $f' = 2X_w$ into a Taylor series about w_0 we obtain

$$X_w(w) = A(w - w_0)^n + \ldots \text{ for } |w - w_0| << 1 \tag{4}$$

with some $n \in \mathbb{N}$ and some $A \in \mathbb{C}^3 \setminus \{0\}$ satisfying $\langle A, A\rangle = 0$, since X is nonconstant and (3) holds true. Hence every interior branch point w_0 is isolated, and the normal

$$N(w) := |X_u(w) \wedge X_v(w)|^{-1}[X_u(w) \wedge X_v(w)]$$

[1] For details concerning the following results we refer to [1]

is defined for $0 < |w - w_0| << 1$. Setting $2A = \alpha - i\beta$ with $\alpha, \beta \in \mathbb{R}^3$, we obtain $|\alpha| = |\beta| > 0$ and $\langle \alpha, \beta \rangle = 0$. It follows that

$$\lim_{w \to w_0} N(w) = |\alpha \wedge \beta|^{-1}(\alpha \wedge \beta) \ ,$$

and so N can be extended to a continuous mapping $B \to \mathbb{R}^3$. By a well-known reasoning it follows that N can be understood as a harmonic mapping $N : B \to S^2$ satisfying

$$\langle N_w, N_w \rangle = 0 \ .$$

If we introduce Cartesian coordinates x^1, x^2, x^3 with respect to the orthonormal frame $\{e_1, e_2, N\}$ with $e_1 := \frac{\alpha}{|\alpha|}, e_2 := \frac{\beta}{|\beta|}$, we can write expansion (4) in the *normal form*

$$X_w(w) = (\varphi(w), \psi(w)) \tag{5}$$

with a holomorphic mapping $\varphi : B \to \mathbb{C}^2$ and a holomorphic function $\psi : B \to \mathbb{C}$. Here φ has the Taylor expansion

$$\begin{aligned} &\varphi(w) = A_1(w - w_0)^n + \dots \quad \text{for} \ \ |w - w_0| << 1 \\ &\text{with} \ \ A_1 \in \mathbb{C}^2 \setminus \{0\} \ \ \text{and} \ \ \langle A_1, A_1 \rangle = 0 \ , \end{aligned} \tag{6}$$

and ψ has the expansion

$$\psi(w) = R_m(w - w_0)^m + \dots \quad \text{for} \ \ |w - w_0| << 1 \ \ \text{with} \ R_m \in \mathbb{C} \ \text{and} \ m > n. \tag{7}$$

Clearly, X is planar if and only if $\psi(w) \equiv 0$ in B. If, however, X is nonplanar, then (7) holds with $R_m \neq 0$ for some $m > n$. In this case we call m the *index* of the interior branch point w_0 of X. Note that this definition of the index m differs from Nitsche's definition (cf. [2], p. 326). Nevertheless we can use the reasoning in [2] for estimating m defined above. For this purpose we recall Radó's lemma (cf. [3], pp. 793–794):

If $h \in C^0(\overline{B}) \cap C^2(B)$ is harmonic in B, $h(w) \not\equiv 0$ in B, and $\nabla^j h(w_0) = 0$ for $j = 0, 1, \dots, m$, then h has at least $2(m+1)$ different zeros on ∂B.

To apply this lemma to minimal surfaces which are bounded by a closed Jordan curve Γ in $\mathbb{R}^3$, we have to define the "Plateau class" $\mathcal{C}(\Gamma)$. This is the set of mappings $X \in H^{1,2}(B, \mathbb{R}^3)$ whose Sobolev trace, denoted by $X|_{\partial B}$, provides a continuous and monotonic mapping of ∂B onto Γ (cf. [1], Vol. I, Section 4.2). If X is a minimal surface of class $\mathcal{C}(\Gamma)$ then $X \in C^0(\overline{B}, \mathbb{R}^3)$, and $X|_{\partial B}$ yields a homeomorphism of ∂B onto Γ.

Consider now a minimal surface $X \in \mathcal{C}(\Gamma)$ with a nonplanar boundary contour Γ. Then X is nonplanar. If X has the interior branch point $w_0 \in B$ of index m, and if $X(w) = (X^1(w), X^2(w), X^3(w))$ is written in the normal form (5), we obtain

$$\begin{aligned} X^3(w) = X^3(w_0) \ &+ \text{Re} \ [c(w - w_0)^{m+1} \\ &+ O(|w - w_0|^{m+2})] \ \ \text{for} \ \ |w - w_0| << 1 \end{aligned} \tag{8}$$

with $c \in \mathbb{C} \setminus \{0\}$. Hence $h := X^3 - X^3(w_0)$ satisfies the assumptions of Radó's lemma, and therefore h has at least $2m+2$ zeros on ∂B. This means that the affine plane

$$\Pi := \{x = (x^1, x^2, x^3) \in \mathbb{R}^3 : x^3 = X^3(w_0)\}$$

intersects Γ in at least $2m+2$ different points, and we obtain:

If no affine plane in $\mathbb{R}^3$ intersects Γ in more than k different points, then the index m of w_0 is bounded by $2m+2 \leq k$.

Definition 1. *The* cut number $c(\Gamma)$ *of a closed Jordan curve Γ in $\mathbb{R}^3$ is the supremum of the number of intersection points of Γ with any affine plane in $\mathbb{R}^3$, i.e.*

$$c(\Gamma) := \sup\{\sharp(\Gamma \cap \Pi) : \Pi = \text{ affine plane}\}$$

If the cardinality of $\Gamma \cap \Pi$ is infinite for some Π, we interprete $\sharp(\Gamma \cap \Pi)$ and $c(\Gamma)$ as ∞. It is easy to see that $4 \leq c(\Gamma) \leq \infty$. Furthermore, for any nonplanar, real analytic, and regular closed Jordan curve Γ, the cut number $c(\Gamma)$ is finite.

By the above result we have:

Theorem 1. *If Γ is nonplanar then the index m of any interior branch point of a minimal surface $X \in \mathcal{C}(\Gamma)$ is bounded by $2m+2 \leq c(\Gamma)$, or equivalently by*

$$m \leq \frac{1}{2}c(\Gamma) - 1 \ . \tag{9}$$

Now we turn to boundary branch points. If Γ is a regular, real analytic contour, then there exists a real analytic extension $\tilde{X}$ of X as a minimal surface $\tilde{X} : B_0 \to \mathbb{R}^3$ defined on a disk B_0 with $B \subset\subset B_0$. Then a *boundary branch point* of X is a point $w_0 \in \partial B$ such that the tangential derivative $X_\theta(w_0)$ of X at w_0 vanishes. The relation $X_\theta(w_0) = 0$ is equivalent to $\tilde{X}_w(w_0) = 0$, and so it makes sense to define:

The index m of a boundary branch point w_0 of X is the index of the interior branch point w_0 of a real analytic extension $\tilde{X}$ of X.

If Γ is not real analytic, this reasoning no longer works. So we may ask: What can be done if Γ is merely smooth, say, a regular curve of class $C^{q,\alpha}$ with $q \geq 2$ and $\alpha \in (0,1)$? It well known that in this case, any minimal surface $X \in \mathcal{C}(\Gamma)$ is of class $C^{q,\alpha}(\overline{B}, \mathbb{R}^3)$, and if $X_\theta(w_0) = 0$ then

$$X_w(w) = A(w - w_0)^n + o(|w - w_0|^n) \ \text{ for } \ w \in \overline{B}, \ \ w \to w_0 \ , \tag{10}$$

with some $A \in \mathbb{C}^3 \setminus \{0\}$ satisfying $\langle A, A \rangle = 0$, and some $n \in \mathbb{N}$. If $X_\theta(w_0) = 0$ and if (10) holds, one calls w_0 a *boundary branch point* of the minimal surface $X \in \mathcal{C}(\Gamma)$. We conclude that every branch point $w_0 \in \overline{B}$ is isolated, and therefore X possesses at most finitely many interior and boundary branch points.

By the same reasoning as above we show that $N_0 = \lim_{w \to w_0} N(w)$ exists, and so $N(w)$ can be continued continuously to a mapping $N : \overline{B} \to S^2$ which is harmonic in B. Using $\alpha, \beta \in \mathbb{R}^3$ with $2A = \alpha - i\beta, |\alpha| = |\beta| > 0$, $\langle \alpha, \beta \rangle = 0$, we can once again write X_w in the normal form

$$X_w(w) = (\varphi(w), \psi(w)), \varphi : \overline{B} \to \mathbb{C}^2, \psi : \overline{B} \to \mathbb{C}$$

with

$$\begin{aligned} &\varphi(w) = A_1(w - w_0)^n + o(|w - w_0|^n) \ \text{ as } \ w \to w_0, A_1 \in \mathbb{C}^2 \setminus \{0\}, \\ &\psi(w) = o(|w - w_0|^n) \ \text{ as } \ w \to w_0 \ . \end{aligned} \tag{11}$$

If we assume $X \in C^{q+2}(\overline{B}, \mathbb{R}^3)$ for some integer q with $q > n$ (which follows from $\Gamma \in C^{q+2,\alpha}$), then Taylor's formula implies that for $w \to w_0$ we have the asymptotic expansions

$$\begin{aligned} \varphi(w) \ &= \ A_1(w - w_0)^n + \ldots + A_{m-n+1}(w - w_0)^m + O(|w - w_0|^{m+1}), \\ & \qquad\qquad\qquad\qquad\qquad\qquad A_j \in \mathbb{C}^2, A_1 \neq 0 \ , \\ \psi(w) \ &= \ R_m(w - w_0)^m + O(|w - w_0|^{m+1}), R_m \in \mathbb{C}, w \to w_0 \ , \end{aligned} \tag{12}$$

provided that $D^\mu_w \psi(w_0) = 0$ for $\mu = 1, \ldots, m-1$ and m is an integer with $n + 1 \leq m \leq q$ (note that $D^\mu_w \psi(w_0) = 0$ for $\mu = 0, 1, \ldots, n$). If $R_m \neq 0$ for some $m \in \{n+1, n+2, \ldots, q\}$, it makes sense to call m the *index* of the boundary branch point w_0 of X, as this definition agrees with that given in the case of a real analytic nonplanar contour Γ. However, it is not clear to us whether or not such an $R_m \neq 0$ always exists, even if Γ is a nonplanar contour of class C^∞; but the following result guarantees the existence of an index $m \in \mathbb{N}$ with $m \geq n + 1$ in a fairly general situation.

Theorem 2. *Suppose that Γ is a regular contour of class $C^{3n+6,\alpha}, n \in \mathbb{N}, \alpha \in (0, 1)$, and assume that both the curvature $\boldsymbol{\kappa}$ and the torsion $\boldsymbol{\tau}$ of Γ never vanish. Secondly, let $X \in \mathcal{C}(\Gamma)$ be a minimal surface with a boundary branch point $w_0 \in \partial B$ of the order $n \in \mathbb{N}$. Then the pair $\{X, w_0\}$ possesses an index $m \in \mathbb{N}$ which is estimated by*

$$n + 1 \leq m \leq 3n + 2 \ . \tag{13}$$

Before we prove this result, we want to draw some conclusions. To this end we recall the Gauss-Bonnet formula for a (branched) minimal surface $X \in \mathcal{C}(\Gamma)$ with $\Gamma \in C^{2,\alpha}, \alpha \in (0, 1)$; cf. [1], Vol. 2, Section 7.11: Let $w_1, \ldots, w_J$ and $\zeta_1, \ldots, \zeta_L$ be the interior branch points and the boundary branch points of X, and K be the Gauss curvature of X, $dA = |X_u \wedge X_v|$ the area element, and κ_g the geodesic curvature of Γ on X. Then

$$1 + \sum_{j=1}^{J} n(w_j) + \frac{1}{2} \sum_{\ell=1}^{L} n(\zeta_\ell) + \frac{1}{2\pi} \int_X |K| dA = \frac{1}{2\pi} \int_\Gamma \kappa_g ds \ , \tag{14}$$

where $n(w_j)$ and $n(\zeta_k)$ denote the orders of w_j and ζ_k respectively. Since $|\kappa_g| \leq \boldsymbol{\kappa}$, we have

$$\int_\Gamma \kappa_g ds \leq \int_\Gamma \kappa \ ds = \text{ total curvature of } \Gamma. \tag{15}$$

Set

$$\boldsymbol{r}(\Gamma) := \frac{1}{2\pi} \int_\Gamma \boldsymbol{\kappa} \ ds - 1 \quad (\geq 0) \ , \tag{16}$$

and note that the order of any boundary branch point ζ_ℓ is even because of the monotonicity of the boundary values $X|_{\partial B}$. Then it follows from (14)-(16) that

$$J, \ \ L, \ \ n(w_j) \leq \boldsymbol{r}(\Gamma) \tag{17}$$

and

$$n(\zeta_\ell) \leq 2\boldsymbol{r}(\Gamma) \tag{18}$$

Combining these estimates with Theorem 2, we arrive at the following result:

Theorem 3. *Suppose that the curvature κ and the torsion τ of $\Gamma \in C^{q+6,\alpha}, 0 < \alpha < 1$, with $q \geq 6r(\Gamma)$ are nowhere zero, and let $\zeta_1, \ldots, \zeta_L$ be the boundary branch points of a minimal surface $X \in \mathcal{C}(\Gamma)$. Then $L \leq r(\Gamma)$, and any ζ_ℓ has an order $n(\zeta_\ell)$ and an index $m(\zeta_\ell)$ such that $n(\zeta_\ell) \leq 2r(\Gamma)$ and*

$$n(\zeta_\ell) + 1 \ \leq \ m(\zeta_\ell) \ \leq \ 3n(\zeta_\ell) + 2 \ \leq \ 6r(\Gamma) + 2 \tag{19}$$

holds true.

Now we turn to the *proof of Theorem 2.* By a rotation about the origin of B we can achieve that the boundary branch point $w_0 \in \partial B$ is moved into the point $w = 1$. Thus, without loss of generality, we can assume that $w_0 = 1$. For technical reasons it will be more convenient to work with minimal surfaces which are parametrized on the upper halfplane

$$\mathcal{H} := \{z \in \mathbb{C} : \text{Im } z > 0\}$$

instead of the unit disk B. Therefore we consider the conformal mapping $w \mapsto z = g(w)$ with

$$g(w) := -i\frac{w-i}{w+1} \qquad w \in \overline{B} \setminus \{-1\} \ ,$$

which maps B onto $\mathcal{H}$ and $S^1 \setminus \{-1\}$ such that $g(1) = 0$. The inverse $z \mapsto w = g^{-1}(z)$ is given by

$$g^{-1}(z) := \frac{1+iz}{1-iz}, \quad z = x + iy \in \overline{\mathcal{H}} \ .$$

Transforming $X(w)$ to the new parameter z, we obtain the minimal surface Y, given by

$$Y(z) = X(g^{-1}(z)), \quad z \in \overline{\mathcal{H}} \ , \tag{20}$$

which has the branch point $z = 0$ of the even order $n = 2\nu, \nu \in \mathbb{N}$. Analogous to (11) and (12) we have the representation

$$Y_z(z) = (\tilde{\varphi}(z), \tilde{\psi}(z)), \tilde{\varphi} : \overline{\mathcal{H}} \to \mathbb{C}^2, \tilde{\psi} : \overline{\mathcal{H}} \to \mathbb{C} \ ,$$

$$\left.\begin{aligned} &\tilde{\varphi}(z) = \tilde{A}_1 z^n + \ldots + \tilde{A}_{m-n+1} z^m + O(|z|^{m+1}), \\ &\tilde{\psi}(z) = \tilde{R}_m z^m + O(|z|^{m+1}), \end{aligned}\right\} \begin{aligned} &\text{for } \ z \in \overline{\mathcal{H}} \setminus \{0\}, \\ &z \to 0 \ , \end{aligned} \tag{21}$$

with $m \geq n+1$, $\tilde{A}_1 \in \mathbb{C}^2 \setminus \{0\}, \langle \tilde{A}_1, \tilde{A}_1 \rangle = 0$.

We want to show that there is an integer $m \leq 3n+2$ such that

$$Y_z^3(z) = \tilde{R}_m z^m + O(|z|^{m+1}) \ \text{ for } \ |z| << 1 \ \text{ and } \ \tilde{R}_m \neq 0 \ . \tag{22}$$

We argue by contradiction. Suppose not, then we would have

$$Y_z^3(z) = O(|z|^{3n+3}) \ \text{ for } \ |z| << 1 \ . \tag{23}$$

Let $\gamma(s) = (\gamma_1(s), \gamma_2(s), \gamma_3(s)), |s| << 1$, be the local representation of Γ with respect to its arc-length parameter s such that $\gamma(0) = Y(0)$ and $\gamma'(0) = e_1 = (1, 0, 0)$. By (21) we have

$$Y_x(x) := Y_x(x, 0) = (n+1)a \ e_1 x^n + O(x^{n+1}), \ \ n = 2\nu \ , \tag{24}$$

with some $a > 0$. Then s and x are related by $s = \sigma(x)$ with

$$\sigma'(x) = |Y_x(x)| = (n+1)a \ x^n + O(x^{n+1}) \ ,$$

whence

$$\sigma(x) = a \ x^{n+1} + O(x^{n+2}) \ \text{ as } \ x \to 0 \ . \tag{25}$$

Since $Y(x) = \gamma(\sigma(x))$ for $|x| << 1$, the third component Y^3 of Y is given by

$$Y^3(x) = \gamma_3(\sigma(x)) = \gamma_3(a \ x^{n+1} + O(x^{n+2})) \ \text{ for } \ x \to 0 \ . \tag{26}$$

Because of (23), we have $Y^3_x(x) = O(x^{3n+3})$ as $x \to 0$, which implies

$$Y^3(x) = O(x^{3n+4}) \ \text{ as } \ x \to 0 \ . \tag{27}$$

On the other hand,

$$\gamma(s) = \gamma'(0)s + O(s^2) \ \text{ as } \ s \to 0 \ ,$$

and so (26) yields

$$Y^3(x) = \gamma_3'(0)a\, x^{n+1} + O(|x|^{n+2}) \ \text{ as } \ x \to 0 \ .$$

Since $a > 0$, it follows $\gamma_3'(0) = 0$ on account of (27). Thus,

$$\gamma_3(s) = \frac{1}{2}\gamma_3''(0)s^2 + O(s^3) \ \text{ as } \ s \to 0 \ ,$$

and by virtue of (26),

$$Y^3(x) = \frac{1}{2}\gamma_3''(0)a^2x^{2n+2} + O(x^{2n+3}) \ \text{ as } \ x \to 0 \ .$$

Then $a > 0$ and (27) imply $\gamma_3''(0) = 0$, whence

$$\gamma_3(s) = \frac{1}{6}\gamma_3'''(0)s^3 + O(s^4) \ \text{ as } \ x \to 0 \ .$$

Applying once again (26), we arrive at

$$Y^3(x) = \frac{1}{6}\gamma_3'''(0)a^3x^{3n+3} + O(x^{3n+4}) \ \text{ as } \ x \to 0$$

and finally $a > 0$ and (27) yield $\gamma_3'''(0) = 0$. Thus we have found:

$$\gamma_3'(0) = 0, \quad \gamma_3''(0) = 0, \quad \gamma_3''(0) = 0 \ ,$$

and so the three vectors $\gamma'(0), \gamma''(0), \gamma'''(0)$ are linearly dependent. This will contradict our assumption $\boldsymbol{\kappa}(s) \neq 0$ and $\boldsymbol{\tau}(s) \neq 0$ for any s. To see this we introduce the Frenet triple $\boldsymbol{T}(s), \boldsymbol{N}(s), \boldsymbol{B}(s)$ of the curve Γ satisfying $\boldsymbol{T} = \gamma', \boldsymbol{T}' = \gamma'', \boldsymbol{T}'' = \gamma'''$ and

$$\begin{aligned} \boldsymbol{T}' &= \quad\ \ \boldsymbol{\kappa}\boldsymbol{N} \\ \boldsymbol{N}' &= -\boldsymbol{\kappa}\boldsymbol{T} \qquad\quad +\boldsymbol{\tau}\boldsymbol{B} \\ \boldsymbol{B}' &= \qquad -\boldsymbol{\tau}\boldsymbol{N} \quad . \end{aligned}$$

Then $\boldsymbol{T}_3(0) = 0, \boldsymbol{T}'_3(0) = 0, \boldsymbol{T}''_3(0) = 0$, and from $\boldsymbol{T}' = \boldsymbol{\kappa}\boldsymbol{N}$ and $\boldsymbol{\kappa}(s) \neq 0$ it follows that $\boldsymbol{N}_3(0) = 0$. Since

$$\boldsymbol{N}' = \left(\frac{1}{\boldsymbol{\kappa}}\right)'\boldsymbol{T}' + \frac{1}{\boldsymbol{\kappa}}\boldsymbol{T}''$$

we obtain $\boldsymbol{N}_3'(0) = 0$ whence $\boldsymbol{\tau}(0)\boldsymbol{B}_3(0) = 0$. Because of $\boldsymbol{\tau}(s) \neq 0$ it follows that $\boldsymbol{B}_3(0) = 0$, and so $\boldsymbol{T}(0), \boldsymbol{N}(0), \boldsymbol{B}(0)$ are linearly dependent. This is a contradiction since $(\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B})$ is an orthonormal frame, hence the assumption (23) is impossible. This completes the proof of Theorem 2. □

We mention that the numbers n and m play an essential role in Tromba's branch point theory, cf. [4]. A branch point w_0 is called *exceptional* if

$$m + 1 \equiv 0 \mod(n+1) \;;$$

otherwise it is said to be *nonexceptional*. For interior branch points this theory is completed, whereas at the boundary the theory is to be perfected. The most notable result for boundary branch points in case of a smooth, but not necessarily real analytic boundary contour Γ is due to D. Wienholtz [5].

The authors should like to thank the Hausdorff Research Institute for Mathematics of Bonn University for the support of their work.

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Stefan Hildebrandt
Mathematisches Institut
Universität Bonn
Beringstrasse 1
D-53115 Bonn
Germany

Anthony J. Tromba
Department of Mathematics
University of California
at Santa Cruz
Santa Cruz, CA 95064
USA