THE BOUSFIELD LATTICE OF A TRIANGULATED CATEGORY AND STRATIFICATION

SRIKANTH B. IYENGAR AND HENNING KRAUSE

Abstract. For a tensor triangulated category which is well generated in the sense of Neeman, it is shown that the collection of Bousfield classes forms a set. This set has a natural structure of a complete lattice which is then studied, using the notions of stratification and support.

Contents

1. Introduction

In recent work with Benson [4, 5, 6], we introduced a notion of stratification of a triangulated category via the action of a graded commutative ring, and used it to classify the localizing subcategories of the stable module category of a finite group. In this paper, we relate these ideas to work on the Bousfield lattice of the stable homotopy category of spectra, initiated in [9], and through this, also to the theory of support being developed by Balmer [2]. Tensor triangulated categories, meaning triangulated categories admitting set-indexed coproducts and carrying a tensor product compatible with the triangulated structure, are an appropriate framework for our analysis. The stable homotopy category of spectra and the stable module category of a finite group are important examples, but there are many more.

Let $(\mathsf{T}, \otimes, \mathbb{1})$ be a tensor triangulated category. Mimicking [9], for each object X in T the class of objects annihilated by the functor $X \otimes -$ is said to be the *Bousfield class* of X. Objects X and Y are *Bousfield equivalent* if their Bousfield classes coincide. Our first result, Theorem 3.1, involves the notion of well generated triangulated categories, introduced by Neeman [27], and asserts:

In a well generated tensor triangulated category, the Bousfield classes form a set.

This result was proved by Ohkawa [28] for the stable homotopy category, but seems to be new even for compactly generated triangulated categories; it gives an

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affirmative answer to Question 5.9 in [13], which asked if the Bousfield classes in the derived category of any commutative ring form a set.

The set of Bousfield classes has a structure of a complete lattice, called the *Bousfield lattice*. It has been studied by topologist for the stable homotopy category of spectra, see for example [9, 29, 16], but not much is known for general tensor triangulated categories. In this work we combine a general analysis of the Bousfield lattice with an analysis for some specific classes of tensor triangulated categories.

In Sections 4 and 5, we concentrate on compactly generated tensor triangulated categories that are *stratified* by an action of a graded-commutative noetherian ring [5]. For instance, in Corollary 4.3, we give a complete description of the Bousfield lattice of such a tensor triangulated category in terms of the prime ideal spectrum of the ring. This connection was first made by Hovey, Palmieri, and Strickland [17] in their work on axiomatic stable homotopy theory.

A new ingredient in our work is a connection with support and cosupport for objects in T, introduced in [4] and [7], respectively. For instance, the stratification condition implies that the tensor product formula

$$\operatorname{supp}_R(X \otimes Y) = \operatorname{supp}_R(X) \cap \operatorname{supp}_R(Y)$$

holds for all X, Y in T; see [6]. Since $\operatorname{supp}_R(X) = \varnothing$ only when $X = 0$, the Bousfield class $\mathrm{A}(X)$ of an object X has thus the form

$$\mathrm{A}(X) = \{Y \in \mathsf{T} \mid \operatorname{supp}_R(Y) \cap \operatorname{supp}_R(X) = \varnothing\}\,,$$

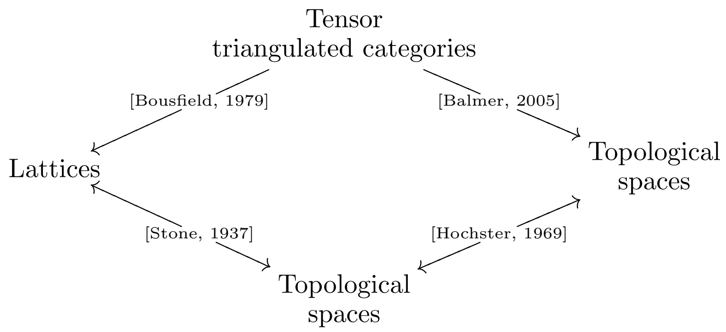
while the X-local objects can be described as follows:

$$\mathrm{A}(X)^{\perp} = \{Y \in \mathsf{T} \mid \operatorname{cosupp}_R(Y) \subseteq \operatorname{supp}_R(X)\}\,.$$

It follows that an object X in T is X-local precisely when $\operatorname{cosupp}_R(X) \subseteq \operatorname{supp}_R(X)$; this inclusion holds when X is compact, but not in general.

In Sections 6 and 7 of this paper, we employ concepts from lattice theory and introduce a notion of support which uses the Bousfield lattice. This is an attempt to develop a theory of support for tensor triangulated categories independent of any action of a ring. A typical example is the derived category of the category of quasi-coherent sheaves on a noetherian scheme. For other work in that direction, see Balmer and Favi [3], and Stevenson [31].

The support defined in terms of the Bousfield lattice is closely related to Balmer's theory of support [2]. The following diagram makes this more precise, and its commutativity will be a consequence of our analysis.



We hasten to add that one needs to restrict at each vertex to some appropriate class of objects to make sure that all arrows are well-defined. On the left, one composes the construction of the Bousfield lattice with Stone duality [32], which means that a distributive lattice is represented via its associated spectrum.

In [2], Balmer associates with any essentially small tensor triangulated category a spectrum of prime ideals together with a Zariski topology. In [10], it is shown that this space is spectral in the sense of Hochster [15]. There is a corresponding Hochster dual space and that completes the right half of the above diagram. There is also an objectwise interpretation of this diagram, assigning to any object of a tensor triangulated category its Bousfield class on the left and its support on the right. The commutativity of the above diagram then amounts to the following assertion which is a consequence of Theorem 7.6:

The Hochster dual of Balmer's spectrum of the category of compact objects in T *is homeomorphic to the spectrum associated with the sublattice of the Bousfield lattice of* T *generated by the compact objects. This homeomorphism identifies the support of a compact object* X*, which is a Zariski closed subset, with the quasi-compact open subset in the Stone topology corresponding to the Bousfield class of* X*.*

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2. Localizing subcategories

In this section we discuss a hierarchy of localizing subcategories. We work in the context of well generated triangulated categories in the sense of Neeman [27]. This includes the class of compactly generated categories. Note that localizing subcategories or Verdier quotient categories of a compactly generated triangulated category are usually not compactly generated; but they are often well generated. Thus the class of well generated triangulated categories seems to be the appropriate universe for studying triangulated categories.

Well generated triangulated categories. Let T be a *well generated triangulated category*. Thus T is a triangulated category with set-indexed coproducts and there is a set of perfect generators which are α-small for some regular cardinal α; see [27, 21] for details. In that case one calls T *α-well generated*.

Now suppose that T is α-well generated, and therefore β-well generated for every regular cardinal $\beta \geq \alpha$. We denote by T^α the full subcategory of *α-compact objects* in T. This is by definition the smallest triangulated subcategory of T which contains a set of α-small perfect generators of T and is closed under *α-coproducts*, that is, coproducts with less than α factors. Note that T^α does not depend on the choice of perfect generators of T [21, Lemma 5]. Moreover, $\mathsf{T} = \bigcup_{\beta \geqslant \alpha} \mathsf{T}^\beta$, where β runs through all regular cardinals.

If $\alpha = \aleph_0$, then T is called *compactly generated*. In that case $\mathsf{T}^{\aleph_0}$ coincides with the subcategory of compact objects and one writes T^c for $\mathsf{T}^{\aleph_0}$.

Denote by $\mathsf{Ab}_\alpha(\mathsf{T})$ the category of additive functors $(\mathsf{T}^\alpha)^{\mathrm{op}} \to \mathsf{Ab}$ into the category of abelian groups which send α-coproducts in T^α to products in Ab. This is an abelian category, and the *restricted Yoneda functor*

$$H_\alpha \colon \mathsf{T} \longrightarrow \mathsf{Ab}_\alpha(\mathsf{T}), \quad X \longmapsto \mathrm{Hom}_{\mathsf{T}}(-, X)|_{\mathsf{T}^\alpha}$$

is the universal cohomological and coproduct preserving functor into an abelian category which has set-indexed coproducts and exact α-filtered colimits [22, Proposition 6.10.1]. If $\alpha = \aleph_0$, then H_α is the universal functor into an abelian category satisfying Grothendieck's AB5 condition; see [20, Corollary 2.4], or [11, Proposition 7.3] for the stable homotopy category. For example, any module category is an AB5 category.

Localizing subcategories. A full subcategory of T is called *localizing* if it is a triangulated subcategory which is closed under set-indexed coproducts. A *localization functor* $L\colon \mathsf{T} \to \mathsf{T}$ is an exact functor that admits a natural transformation $\eta\colon \mathrm{Id}_{\mathsf{T}} \to L$ such that $L(\eta X)$ is an isomorphism and $L(\eta X) = \eta(LX)$ for all objects X in T.

Proposition 2.1. *Let T be an α-well generated triangulated category, where α denotes a fixed regular cardinal. Consider for a localizing subcategory S of T the following statements:*

(1) *There is an exact functor $F\colon \mathsf{T} \to \mathsf{T}$ preserving set-indexed coproducts such that $\mathsf{S} = \operatorname{Ker} F$.*
(2) *There is a cohomological functor $H\colon \mathsf{T} \to \mathsf{A}$ preserving set-indexed coproducts into an abelian category which has set-indexed coproducts and exact α-filtered colimits such that $\mathsf{S} = \operatorname{Ker} H$.*
(3) *There is a cohomological functor $H\colon \mathsf{T} \to \mathsf{A}$ preserving set-indexed coproducts into an abelian category which has set-indexed coproducts and exact β-filtered colimits for some regular cardinal β such that $\mathsf{S} = \operatorname{Ker} H$.*
(4) *There is an object X in T such that S equals the smallest localizing subcategory of T containing X.*
(5) *There is a localization functor $L\colon \mathsf{T} \to \mathsf{T}$ such that $\mathsf{S} = \operatorname{Ker} L$.*

Then the implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longleftrightarrow$ (4) $\Longrightarrow$ (5) *hold.*

Proof. (1) $\Rightarrow$ (2): Let H be the composite of F with the restricted Yoneda functor $\mathsf{T} \to \mathsf{Ab}_\alpha(\mathsf{T})$. Then $\operatorname{Ker} H = \mathsf{S}$.

(2) $\Rightarrow$ (3): Clear.

(3) $\Rightarrow$ (4): In [22, Theorem 7.5.1] it is shown that $\mathsf{S} = \operatorname{Ker} H$ is well generated. Now take for X the coproduct of a set of generators of S.

(4) $\Rightarrow$ (3): The Verdier quotient T/S is β-well generated for some regular cardinal β, by [27, Corollary 4.4.3] or [22, Theorem 7.2.1]. Let H be the composite of the canonical functor $\mathsf{T} \to \mathsf{T}/\mathsf{S}$ with the restricted Yoneda functor $\mathsf{T}/\mathsf{S} \to \mathsf{Ab}_\beta(\mathsf{T}/\mathsf{S})$. Then $\operatorname{Ker} H = \mathsf{S}$.

(4) $\Rightarrow$ (5): The Verdier quotient T/S has the property that $\operatorname{Hom}_{\mathsf{T}/\mathsf{S}}(X, Y)$ is a set for each pair of objects X, Y in T/S; see [27, Corollary 4.4.3] or [22, Theorem 7.2.1]. Thus the canonical functor $Q\colon \mathsf{T} \to \mathsf{T}/\mathsf{S}$ admits a right adjoint, by Brown representability. The composite of Q with its right adjoint is a localization functor with kernel S. □

Remark 2.2. The triangulated category S is well generated if and only if (3), equivalently (4), hold.

A *skeleton* of a category C is a full subcategory S such that each object in C is isomorphic to exactly one object in S; this is unique up to an isomorphism of categories. We write $|\mathsf{C}|$ for the cardinality of the collection of morphism of a skeleton of C.

Theorem 2.3. *Let T be an α-well generated triangulated category and consider the collection $\mathcal{H}$ of functors $H\colon \mathsf{T} \to \mathsf{A}$ such that*

(1) A *is abelian and has set-indexed coproducts and exact* α*-filtered colimits,*
(2) H *is cohomological and preserves set-indexed coproducts.*

Then the localizing subcategories of the form $\operatorname{Ker} H$ *for some* $H \in \mathcal{H}$ *form a set of cardinality at most* $2^{2^{|\mathsf{T}^\alpha|}}$.

Proof. Any functor $H \in \mathcal{H}$ can be extended to an exact functor $\bar{H}\colon \mathsf{Ab}_\alpha(\mathsf{T}) \to \mathsf{A}$ preserving set-indexed coproducts such that $H = \bar{H} H_\alpha$, by [22, Proposition 6.10.1]. Thus the kernel of $\bar{H}$ is a localizing subcategory of $\mathsf{Ab}_\alpha(\mathsf{T})$ and it determines $\operatorname{Ker} H$, since $H_\alpha(X) \neq 0$ for each non-zero object X in T. Recall that a full subcategory of an abelian category is *localizing* if it is closed under subobjects, quotient objects, extensions, and set-indexed coproducts.

The abelian category $\mathsf{Ab}_\alpha(\mathsf{T})$ is generated by the representable functors $H_\alpha(X)$ with $X \in \mathsf{T}^\alpha$, and it follows that each object of $\mathsf{Ab}_\alpha(\mathsf{T})$ is the α-filtered union of its subobjects of the form $H_\alpha(X)/U$ where $X \in \mathsf{T}^\alpha$ and $U \subseteq H_\alpha(X)$ is a subobject. Thus $\operatorname{Ker} \bar{H}$ is determined by the objects of the form $H_\alpha(X)/U$ which it contains. The number of pairs (U, X) where $X \in \mathsf{T}^\alpha$ and $U \subseteq H_\alpha(X)$ is bounded by $2^{|\mathsf{T}^\alpha|}$, since U is given by the family of subsets $U(C) \subseteq \operatorname{Hom}_{\mathsf{T}}(C, X)$ where C runs through the objects of T^α. It follows that the number of subcategories of the form $\operatorname{Ker} \bar{H}$ is bounded by the cardinal $2^{2^{|\mathsf{T}^\alpha|}}$. □

For the preceding result, the most interesting case is when $\alpha = \aleph_0$, which means that T is compactly generated. In that case $\mathsf{Ab}_\alpha(\mathsf{T})$ equals the category of T^c-modules, where the category of compact objects T^c is viewed as a ring with several objects. Given a localizing subcategory $\mathsf{C} \subseteq \mathsf{Ab}_\alpha(\mathsf{T})$, there exists a set of indecomposable injective modules Q_i such that a module X belongs to C if and only if $\operatorname{Hom}_{\mathsf{T}^c}(X, Q_i) = 0$ for all i; see [14, Chap. III]. The isomorphism classes of indecomposable injective T^c-modules form a set, which can be identified (via Brown representability) with the *Ziegler spectrum* $\operatorname{Sp}_{\mathrm{Zg}}(\mathsf{T})$ consisting of the isomorphism classes of indecomposable pure-injective objects in T; see [20, §1]. The map taking a localizing subcategory $\mathsf{S} \subseteq \mathsf{T}$ as in Theorem 2.3 to $\mathsf{S}^\perp \cap \operatorname{Sp}_{\mathrm{Zg}}(\mathsf{T})$ is injective, since $\mathsf{S} = {}^\perp(\mathsf{S}^\perp \cap \operatorname{Sp}_{\mathrm{Zg}}(\mathsf{T}))$. This provides another explicit method to bound the cardinality of the collection of these localizing subcategories.

3. Bousfield classes

In this section we recall the notion of a Bousfield class of an object in a tensor triangulated category, and prove that the collection of Bousfield classes form a set, provided the triangulated category is well generated.

Tensor triangulated categories. Let $(\mathsf{T}, \otimes, \mathbb{1})$ be a *tensor triangulated category.* Thus T is a triangulated category with a symmetric monoidal structure; $\otimes$ is its tensor product which is exact in each variable and $\mathbb{1}$ is the unit of the tensor product. We assume that T has set-indexed coproducts and that the tensor product preserves coproducts in each variable.

Recall that a subcategory $\mathsf{S} \subseteq \mathsf{T}$ is *tensor closed* if $X \in \mathsf{S}$ and $Y \in \mathsf{T}$ implies $X \otimes Y \in \mathsf{S}$. Given any object or class of objects X in T, we write $\operatorname{Loc}(X)$ for the smallest tensor closed localizing subcategory of T containing X.

For a subcategory $\mathsf{S} \subseteq \mathsf{T}$, we define its *orthogonal subcategories*

$$\mathsf{S}^\perp = \{Y \in \mathsf{T} \mid \operatorname{Hom}_{\mathsf{T}}(X, Y) = 0 \text{ for all } X \in \mathsf{S}\},$$
$${}^\perp\mathsf{S} = \{X \in \mathsf{T} \mid \operatorname{Hom}_{\mathsf{T}}(X, Y) = 0 \text{ for all } Y \in \mathsf{S}\}.$$

It is not hard to verify that $\mathsf{S} \subseteq {}^\perp(\mathsf{S}^\perp)$, and that equality holds when $\mathsf{S} = \operatorname{Ker} L$ for some localization functor $L\colon \mathsf{T} \to \mathsf{T}$.

Bousfield classes. Let X be an object in T. Following Bousfield [9], we define the *Bousfield class* of an object X to be the full subcategory of T with objects

$$\mathrm{A}(X) = \{Y \in \mathsf{T} \mid X \otimes Y = 0\}\,.$$

The objects in $\mathrm{A}(X)$ are said to be X-*acyclic*. Note that $\mathrm{A}(X)$ is a tensor closed localizing subcategory of T.

For the stable homotopy category of spectra, Ohkawa proved [28] that the collection of Bousfield classes forms a set of cardinality at most $2^{2^{\aleph_0}}$; see also [12]. More recently, Dwyer and Palmieri [13] investigated the Bousfield classes in the derived category of modules over some non-noetherian rings, and asked in Question 5.9 of op. cit., whether these Bousfield classes form a set. We answer their question in the affirmative, as follows.

Theorem 3.1. *For any α-well generated tensor triangulated category T, the collection of Bousfield classes forms a set of cardinality at most $2^{2^{|\mathsf{T}^\alpha|}}$.*

Proof. The Bousfield class of an object X is the kernel of the functor $X \otimes -$, which is exact and preserves coproducts. It thus follows from Proposition 2.1 and Theorem 2.3 that the collection of such localizing subcategories forms a set of cardinality at most $2^{2^{|\mathsf{T}^\alpha|}}$. □

The Bousfield lattice. A *lattice* is by definition a partially ordered set Λ with the property that for each pair of elements a, b in Λ there is a supremum, denoted $a \vee b$, and an infimum, denoted $a \wedge b$. A lattice Λ is *complete* if for any subset $A \subseteq \Lambda$ the supremum $\bigvee_{a\in A} a$ and the infimum $\bigwedge_{a \in A} a$ exist. In any partially ordered set the infimum can be expressed as a supremum and vice versa. For instance,

$$\bigwedge_{a\in A} a = \bigvee_{b \in B} b \qquad \text{where } B = \{b \in \Lambda \mid b \le a \text{ for all } a \in A\}\,.$$

Thus, Λ is complete if every subset in it has a supremum; equivalently, if every subset has an infimum. In any lattice, we write 0 for the unique minimal element and 1 for the unique maximal element, provided they exist.

Let T be a well generated tensor triangulated category and denote by $\mathbf{A}(\mathsf{T})$ the set of Bousfield classes in T. As in [9], there is a partial order $\le$ on $\mathbf{A}(\mathsf{T})$ given by:

$$\mathrm{A}(X) \le \mathrm{A}(Y) \quad \text{when} \quad \mathrm{A}(X) \supseteq \mathrm{A}(Y)\,.$$

For any set of objects X_i in T, one has

$$\bigvee_i \mathrm{A}(X_i) = \mathrm{A}(\coprod_i X_i)\,.$$

Thus $\mathbf{A}(\mathsf{T})$ is a complete lattice and we call it the *Bousfield lattice* of T.

We consider also the collection of tensor closed localizing subcategories of the form $\operatorname{Loc}(X)$, and denote it $\mathbf{Loc}(\mathsf{T})$. This time the partial order considered is the obvious one, namely, the one given by inclusion. Without additional hypotheses we do not know whether $\mathbf{Loc}(\mathsf{T})$ is a set or a proper class; see however Corollary 4.3.

In $\mathbf{Loc}(\mathsf{T})$ any subset has a supremum, given as before by

$$\bigvee_i \operatorname{Loc}(X_i) = \operatorname{Loc}(\coprod_i X_i)\,.$$

The infimum has the following explicit description.

Lemma 3.2. *Assume* T *is well generated. Then there is in* $\mathbf{Loc}(\mathsf{T})$ *an equality*

$$\bigwedge_i \operatorname{Loc}(X_i) = \bigcap_i \operatorname{Loc}(X_i)$$

for every set of objects $X_i \in \mathsf{T}$.

Proof. One needs to find an object $X \in \mathsf{T}$ such that $\bigcap_i \operatorname{Loc}(X_i) = \operatorname{Loc}(X)$. It follows from Proposition 2.1 that there are regular cardinals β_i and cohomological functors $H_i \colon \mathsf{T} \to \mathsf{A}_i$ such that A_i has exact β_i-filtered colimits and $\operatorname{Loc}(X_i) = \operatorname{Ker} H_i$ for all i. Put $\mathsf{A} = \prod_i \mathsf{A}_i$ and observe that A has β-exact filtered colimits for $\beta = \sum_i \beta_i$. Clearly, $\bigcap_i \operatorname{Loc}(X_i) = \operatorname{Ker} H$ for the functor $H \colon \mathsf{T} \to \mathsf{A}$ taking an object Y to $(H_i Y)$. Thus $\operatorname{Ker} H = \operatorname{Loc}(X)$ for some $X \in \mathsf{T}$ by Proposition 2.1. □

Remark 3.3. If $\mathbf{Loc}(\mathsf{T})$ is a set, then every tensor closed localizing subcategory $\mathsf{S} \subseteq \mathsf{T}$ belongs to $\mathbf{Loc}(\mathsf{T})$, since $\mathsf{S} = \bigvee_{X \in \mathsf{S}} \operatorname{Loc}(X)$.

4. Stratification

In this section we determine the structure of the Bousfield lattice, and of the lattice of localizing subcategories, for some classes of tensor triangulated categories. This involves the stratification property introduced and studied in [5, 6].

For the remainder of this section and the next, we assume $(\mathsf{T}, \otimes, \mathbb{1})$ is *compactly generated tensor triangulated category*, by which we mean that the following conditions are satisfied:

(1) T is a compactly generated triangulated category.
(2) The unit $\mathbb{1}$ is compact and all compact objects are strongly dualizable.
(3) The functor $\mathcal{H}om(-, Y)$ is exact for each object Y in T.

Here, $\mathcal{H}om(X, -)$ denotes for each object X in T the right adjoint of the tensor functor $X \otimes - \colon \mathsf{T} \to \mathsf{T}$, which exists by Brown representability. Thus

$$\operatorname{Hom}_{\mathsf{T}}(X \otimes Z, Y) \cong \operatorname{Hom}_{\mathsf{T}}(Z, \mathcal{H}om(X, Y))$$

for all objects Y, Z in T. We write $X^\vee = \mathcal{H}om(X, \mathbb{1})$, and the object X is *strongly dualizable* if the canonical morphism

$$X^\vee \otimes Y \to \mathcal{H}om(X, Y)$$

is an isomorphism for all Y in T.

Actions. Fix a graded-commutative noetherian ring R and a homomorphism $R \to \operatorname{End}^*_{\mathsf{T}}(\mathbb{1})$ into the graded endomorphism ring of $\mathbb{1}$. In this way R acts on T, that is, the graded abelian group

$$\operatorname{Hom}^*_{\mathsf{T}}(X, Y) = \bigoplus_{n \in \mathbb{Z}} \operatorname{Hom}_{\mathsf{T}}(X, \Sigma^n Y)$$

is an R-module, for all objects X, Y.

Let $\operatorname{Spec} R$ denote the set of graded prime ideals in R. In [4, §5] and [7, §4], we constructed for each $\mathfrak{p} \in \operatorname{Spec} R$ an adjoint pair of exact functors $\varGamma_{\mathfrak{p}} \colon \mathsf{T} \to \mathsf{T}$ and $\varLambda^{\mathfrak{p}} \colon \mathsf{T} \to \mathsf{T}$, and defined for each object X *support* and *cosupport* as follows:

$$\operatorname{supp}_R(X) = \{\mathfrak{p} \in \operatorname{Spec} R \mid \varGamma_{\mathfrak{p}} X \neq 0\}$$
$$\operatorname{cosupp}_R(X) = \{\mathfrak{p} \in \operatorname{Spec} R \mid \varLambda^{\mathfrak{p}} X \neq 0\}.$$

Note that $\varGamma_{\mathfrak{p}} X \cong \varGamma_{\mathfrak{p}} \mathbb{1} \otimes X$ and $\varLambda^{\mathfrak{p}} X \cong \mathcal{H}om(\varGamma_{\mathfrak{p}} \mathbb{1}, X)$.

Lemma 4.1. *Given objects* X, Y *in* T, *there are implications:*

$$\operatorname{Loc}(X) \subseteq \operatorname{Loc}(Y) \quad \Longrightarrow \quad \mathrm{A}(X) \leq \mathrm{A}(Y) \quad \Longrightarrow \quad \operatorname{supp}_R(X) \subseteq \operatorname{supp}_R(Y)\,.$$

Proof. The first implication follows from the fact that the functor $W \otimes -$ is exact and preserves set-indexed coproducts, for any object W in T.

For the second implication, observe that for any object X in T, a prime $\mathfrak{p}$ is in $\operatorname{supp}_R(X)$ if and only if $\varGamma_{\mathfrak{p}}\mathbb{1} \notin \mathrm{A}(X)$. □

Next we establish converses to the preceding lemma. We begin with a reformulation of some results from [5, 7]. Recall that

$$\operatorname{supp}_R(\mathsf{T}) = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p} \in \operatorname{supp}_R(X) \text{ for some } X \in \mathsf{T}\}\,.$$

This set coincides with $\operatorname{supp}_R(\mathbb{1})$, since $\varGamma_{\mathfrak{p}}X \cong \varGamma_{\mathfrak{p}}\mathbb{1} \otimes X$.

Function objects. The result below provides a formula for the cosupport of function objects. In the language of [6], condition (1) is the statement that T is *stratified* by R, as a tensor triangulated category.

Theorem 4.2. *The following conditions are equivalent:*

1. $\operatorname{Loc}(\varGamma_{\mathfrak{p}}\mathbb{1})$ *is a minimal non-zero element of* $\mathbf{Loc}(\mathsf{T})$*, for each* $\mathfrak{p} \in \operatorname{supp}_R(\mathsf{T})$.
2. $\operatorname{cosupp}_R(\mathcal{H}om(X,Y)) = \operatorname{supp}_R(X) \cap \operatorname{cosupp}_R(Y)$ *for all* X, Y *in* T.
3. $\operatorname{Loc}(X) = \{Y \in \mathsf{T} \mid \operatorname{supp}_R(Y) \subseteq \operatorname{supp}_R(X)\}$ *for all* X *in* T.
4. $\operatorname{Loc}(X) \subseteq \operatorname{Loc}(Y) \iff \operatorname{supp}_R(X) \subseteq \operatorname{supp}_R(Y)$*, for all* X, Y *in* T.

Proof. The equivalence of (1) and (2) is [7, Theorem 9.5], while that of (1), (3) and (4) follows from [5, Theorem 3.8]. □

Corollary 4.3. *Suppose that* T *is stratified as a tensor triangulated category by a graded-commutative noetherian ring* R*. Then the lattice of tensor closed localizing subcategories of* T *is isomorphic to the lattice of subsets of* $\operatorname{supp}_R(\mathsf{T})$ *via the map sending* $\operatorname{Loc}(X)$ *to* $\operatorname{supp}_R(X)$. □

Next we formulate the analogue of Theorem 4.2 for Bousfield classes.

Tensor products. The following result provides a formula for the support of tensor products. We say that the *tensor product formula* holds in T when the conditions below are satisfied.

Theorem 4.4. *The following conditions are equivalent:*

1. $\mathrm{A}(\varGamma_{\mathfrak{p}}\mathbb{1})$ *is a minimal non-zero element of* $\mathbf{A}(\mathsf{T})$*, for each* $\mathfrak{p} \in \operatorname{supp}_R(\mathsf{T})$.
2. $\operatorname{supp}_R(X \otimes Y) = \operatorname{supp}_R(X) \cap \operatorname{supp}_R(Y)$ *for all* X, Y *in* T.
3. $\mathrm{A}(X) = \{Y \in \mathsf{T} \mid \operatorname{supp}_R(Y) \cap \operatorname{supp}_R(X) = \varnothing\}$ *for all* X *in* T.
4. $\mathrm{A}(X) \leq \mathrm{A}(Y) \iff \operatorname{supp}_R(X) \subseteq \operatorname{supp}_R(Y)$*, for all* X, Y *in* T.

Moreover, these conditions hold when T *is stratified by* R.

Proof. (1) ⇒ (2): When (1) holds, for each $X \in \mathsf{T}$ and $\mathfrak{p} \in \operatorname{supp}_R(X)$, one has

$$\mathrm{A}(\varGamma_{\mathfrak{p}}X) = \mathrm{A}(\varGamma_{\mathfrak{p}}\mathbb{1})\,.$$

Indeed, the isomorphism $\varGamma_{\mathfrak{p}}X \cong \varGamma_{\mathfrak{p}}\mathbb{1} \otimes X$ implies $\mathrm{A}(\varGamma_{\mathfrak{p}}X) \leq \mathrm{A}(\varGamma_{\mathfrak{p}}\mathbb{1})$. The equality holds since $\mathrm{A}(\varGamma_{\mathfrak{p}}X) \neq \mathsf{T}$.

Now for any X and Y one has $\operatorname{supp}_R(X \otimes Y) \subseteq \operatorname{supp}_R(X) \cap \operatorname{supp}_R(Y)$. When $\mathfrak{p}$ is not in $\operatorname{supp}_R(X \otimes Y)$, one gets

$$\varGamma_{\mathfrak{p}}(X) \otimes Y \cong \varGamma_{\mathfrak{p}}(X \otimes Y) = 0\,,$$

so that $Y \in \mathrm{A}(\varGamma_{\mathfrak{p}}X)$; if in addition $\mathfrak{p}$ is in $\operatorname{supp}_R(X)$, then since $\mathrm{A}(\varGamma_{\mathfrak{p}}X) = \mathrm{A}(\varGamma_{\mathfrak{p}}\mathbb{1})$, it follows that $\mathfrak{p} \notin \operatorname{supp}_R(Y)$. Thus formula in (2) holds.

(2) ⇒ (3): This is immediate, for $X \otimes Y = 0$ if and only if $\operatorname{supp}_R(X \otimes Y) = \varnothing$, by [4, Theorem 5.2].

(3) $\Rightarrow$ (4): It is clear that when (3) holds and $\operatorname{supp}_R(X) \subseteq \operatorname{supp}_R(Y)$, one has $\mathrm{A}(X) \le \mathrm{A}(Y)$. The reverse implication holds always; see Lemma 4.1.

(4) $\Rightarrow$ (1): This is clear, for $\operatorname{supp}_R(\Gamma_{\mathfrak p}\mathbb{1}) = \{\mathfrak p\}$ for $\mathfrak p \in \operatorname{supp}_R(\mathsf T)$.

For the last conclusion, observe that part (4) of Theorem 4.2 in combination with Lemma 4.1 implies part (4) of the present theorem. □

Corollary 4.5. *Suppose that the tensor product formula holds in $\mathsf T$, for instance, when $\mathsf T$ is stratified by R as a tensor triangulated category. Then the Bousfield lattice of $\mathsf T$ is isomorphic to the lattice of subsets of $\operatorname{supp}_R(\mathsf T)$ via the map sending $\mathrm{A}(X)$ to $\operatorname{supp}_R(X)$. In particular, for all X, Y in $\mathsf T$,*

$$\mathrm{A}(X) \wedge \mathrm{A}(Y) = \mathrm{A}(X \otimes Y).$$

Proof. The inverse map sends $\mathcal U \subseteq \operatorname{supp}_R(\mathsf T)$ to $\mathrm{A}(\coprod_{\mathfrak p \in \mathcal U} \Gamma_{\mathfrak p}\mathbb{1})$. Indeed, note that

$$\operatorname{Loc}(X) = \bigvee_{\mathfrak p} \operatorname{Loc}(\Gamma_{\mathfrak p} X),$$

by [6, Theorem 7.2], where $\mathfrak p$ runs through all primes in $\operatorname{Spec} R$. This implies

$$\mathrm{A}(X) = \bigvee_{\mathfrak p \in \operatorname{supp}_R(X)} \mathrm{A}(\Gamma_{\mathfrak p} X) = \bigvee_{\mathfrak p \in \operatorname{supp}_R(X)} \mathrm{A}(\Gamma_{\mathfrak p}\mathbb{1}),$$

where the minimality of $\mathrm{A}(\Gamma_{\mathfrak p}\mathbb{1})$ is used for the second equality. It follows that the maps between $\mathbf{A}(\mathsf T)$ and subsets of $\operatorname{supp}_R(\mathsf T)$ are mutually inverse. □

Remark 4.6. In [17], condition (1) of Theorem 4.2 is formulated as Conjecture 6.1.2, while condition (1) of Theorem 4.4 is formulated as Conjecture 6.1.3 (with some additional assumptions on $\mathsf T$). A number of interesting consequences are proved in [17, Theorem 6.1.5], including a classification of the thick subcategories of $\mathsf T^{\mathsf c}$.

Example 4.7. Let A be a commutative noetherian ring. The derived category of the category of A-modules is a compactly generated tensor triangulated category with a natural A-action. It follows from Neeman's work [24], see also [6, Theorem 8.1], that this category is stratified by A, that is to say, the equivalent conditions in Theorem 4.2 hold.

Remark 4.8. In the context of Example 4.7, Dwyer and Palmieri [13, p. 429] ask if $\mathrm{A}(X) = \mathrm{A}(\bigoplus_n H^n X)$ for any complex X in the derived category of A. However, it is clear from Theorem 4.4 that this cannot hold in general, for there exist complexes X such $\operatorname{supp}_R(X) \neq \operatorname{supp}_R(\bigoplus_n H^n X)$; see [4, Example 9.4].

Example 4.9. Let k be a field and consider the category of k-linear maps $V \to W$. This is an abelian category, and tensor product over k induces on it a symmetric monoidal structure with tensor identity $\mathbb{1} = (k \xrightarrow{\mathrm{id}} k)$. The corresponding derived category $\mathsf D$ is then a compactly generated tensor triangulated category. Set $R = \operatorname{End}^*_{\mathsf D}(\mathbb{1}) = k$. Then the tensor product formula does not hold in $\mathsf D$.

Take for instance $X = (k \to 0)$ and $Y = (0 \to k)$. Then $X \otimes Y = 0$, and so

$$\operatorname{supp}_R(X \otimes Y) = \varnothing \neq \operatorname{Spec} R = \operatorname{supp}_R(X) \cap \operatorname{supp}_R(Y).$$

5. Local objects

Let $(\mathsf T, \otimes, \mathbb{1})$ be a compactly generated tensor triangulated category, as in Section 4. We follow [9] and say for a given object X that an object Y is *X-local* if the functor $\operatorname{Hom}_{\mathsf T}(-, Y)$ annihilates all X-acyclic objects. An object is *X-acyclic* if it is annihilated by $X \otimes -$. In this section we interpret this property in terms of support and cosupport of X and Y, respectively, when these notions are defined.

We begin with an elementary observation; recall that $X^\vee = \mathcal{H}om(X,\mathbb{1})$.

Proposition 5.1. *When $X \in \mathsf{T}$ is compact,* $\mathrm{A}(X) = \mathrm{A}(X^\vee)$ *and X is X-local.*

Proof. The object X is strongly dualizable and therefore a retract of $X \otimes X^\vee \otimes X$; see [23, Proposition III.1.2]. It follows that

$$\mathrm{A}(X) \le \mathrm{A}(X \otimes X^\vee \otimes X) \le \mathrm{A}(X^\vee)\,.$$

Since $(X^\vee)^\vee \cong X$, the inequalities above yield also $\mathrm{A}(X^\vee) \le \mathrm{A}(X)$.

Now when Y is X-acyclic, it is also $X^\vee$-acyclic, which yields the equality below:

$$\mathrm{Hom}_{\mathsf{T}}(Y,X) \cong \mathrm{Hom}_{\mathsf{T}}(Y,(X^\vee)^\vee) \cong \mathrm{Hom}_{\mathsf{T}}(X^\vee \otimes Y, \mathbb{1}) = 0$$

The first isomorphism holds as $(X^\vee)^\vee \cong X$, and the second one is by adjunction. Therefore X is X-local, as claimed. □

The preceding result does not, in general, extend to non-compact objects; see Example 5.5. A basic problem is to recognize when an object is X-local, and this has a satisfactory answer if T is stratified; this is explained next.

Cosupport. Henceforth we assume that T is endowed with an action of a graded-commutative noetherian ring R, as in Section 4. Given a subset $\mathcal{U} \subseteq \operatorname{Spec} R$, consider full subcategories

$$\mathsf{T}_{\mathcal{U}} = \{X \in \mathsf{T} \mid \operatorname{supp}_R(X) \subseteq \mathcal{U}\} \quad \text{and} \quad \mathsf{T}^{\mathcal{U}} = \{X \in \mathsf{T} \mid \operatorname{cosupp}_R(X) \subseteq \mathcal{U}\}\,.$$

These subcategories are related to each other:

Lemma 5.2. *Let $\mathcal{U} \subseteq \operatorname{Spec} R$ and $\mathcal{U}' = \operatorname{Spec} R \smallsetminus \mathcal{U}$. Then*

$$(\mathsf{T}_{\mathcal{U}})^\perp = \mathsf{T}^{\mathcal{U}'} \quad \textit{and} \quad \mathsf{T}_{\mathcal{U}} = {}^\perp(\mathsf{T}^{\mathcal{U}'})\,.$$

Proof. Fix an object Y in T. By definition, $\operatorname{cosupp}_R(Y) \subseteq \mathcal{U}'$ if and only if

$$\mathrm{Hom}_{\mathsf{T}}(\varGamma_{\mathfrak{p}}-, Y) \cong \mathrm{Hom}_{\mathsf{T}}(-, \varLambda^{\mathfrak{p}} Y) = 0$$

for all $\mathfrak{p} \in \mathcal{U}$. The local-global principle, [4, Theorem 3.6] implies

$$\mathsf{T}_{\mathcal{U}} = \mathrm{Loc}(\{\varGamma_{\mathfrak{p}} X \mid X \in \mathsf{T},\, \mathfrak{p} \in \mathcal{U}\})\,.$$

Thus $(\mathsf{T}_{\mathcal{U}})^\perp = \mathsf{T}^{\mathcal{U}'}$.

On the other hand, $\mathsf{T}_{\mathcal{U}} = \bigcap_{\mathfrak{p} \in \mathcal{U}'} \operatorname{Ker} \varGamma_{\mathfrak{p}}$. It follows from Proposition 2.1 that $\mathsf{T}_{\mathcal{U}}$ is the kernel of a localization functor. Thus $\mathsf{T}_{\mathcal{U}} = {}^\perp((\mathsf{T}_{\mathcal{U}})^\perp) = {}^\perp(\mathsf{T}^{\mathcal{U}'})$. □

Proposition 5.3. *The tensor product formula holds in T if and only if*

$$\mathrm{A}(X)^\perp = \{Y \in \mathsf{T} \mid \operatorname{cosupp}_R(Y) \subseteq \operatorname{supp}_R(X)\} \qquad \textit{for all } X \in \mathsf{T}\,.$$

Proof. Given the description of the Bousfield class $\mathrm{A}(X)$ in Theorem 4.4, this is an immediate consequence of Lemma 5.2. □

One consequence is that the Dichotomy Conjecture of Hovey and Palmieri [16, Conjecture 7.5] holds in our setting. The conjecture asserts that for each object X in T, there exists a non-zero compact object that is either X-acyclic or X-local.

Example 5.4. Suppose the tensor product formula holds in T.

Let $\mathfrak{p} \in \operatorname{supp}_R(\mathsf{T})$ be maximal and denote by $\mathbb{1}/\!\!/\mathfrak{p}$ the corresponding Koszul object [4, §5]. By its construction, the object is compact, and

$$\operatorname{supp}_R(\mathbb{1}/\!\!/\mathfrak{p}) = \{\mathfrak{p}\} = \operatorname{cosupp}_R(\mathbb{1}/\!\!/\mathfrak{p}),$$

by [6, Lemma 2.6] and [7, Lemma 4.12]. Given any object X in T, there are two possible cases. If $\mathfrak{p}$ is in $\operatorname{supp}_R(X)$, then the object $\mathbb{1}/\!\!/\mathfrak{p}$ is X-local, by Proposition 5.3. Otherwise, $\mathbb{1}/\!\!/\mathfrak{p}$ is X-acyclic, by Theorem 4.4.

When the tensor product formula holds in T (for example, when T is stratified by R), it follows from Proposition 5.3 that an object X is X-local if and only if $\operatorname{cosupp}_R(X)$ is contained in $\operatorname{supp}_R(X)$. This holds when X is compact, thanks to Proposition 5.1, but not always, as the following example shows.

Example 5.5. Let A be a commutative noetherian ring. Fix a prime ideal $\mathfrak{p}$ and let E be the injective hull of $A/\mathfrak{p}$. The derived category of A is then a noetherian R-linear category, where $R = A$, and there are equalities

$$\operatorname{cosupp}_R(E) = \{\mathfrak{q} \in \operatorname{Spec} R \mid \mathfrak{q} \subseteq \mathfrak{p}\} \quad \text{and} \quad \operatorname{supp}_R(E) = \{\mathfrak{p}\}.$$

For the computation of $\operatorname{cosupp}_R(E)$, see [7, Proposition 5.4]. Thus, E is not E-local. Recall that the derived category is stratified by A.

The preceding discussion raises the question: When is $\operatorname{cosupp}_R(X) \subseteq \operatorname{supp}_R(X)$? It follows from Proposition 5.1 that this is so when X is compact.

Proposition 5.6. *In any R-linear tensor triangulated category T, each compact object X satisfies* $\operatorname{cosupp}_R(X) \subseteq \operatorname{supp}_R(X)$.

Proof. Suppose $\mathfrak{p}$ is not in $\operatorname{supp}_R(X)$, and fix a compact object C. Then one has

$$X \otimes \varGamma_{\mathfrak{p}} C \cong X \otimes \varGamma_{\mathfrak{p}} \mathbb{1} \otimes C \cong \varGamma_{\mathfrak{p}} X \otimes C = 0\,.$$

Therefore $\varGamma_{\mathfrak{p}} C$ is X-acyclic. Using that X is X-local, by Proposition 5.1, this yields

$$\operatorname{Hom}_{\mathsf{T}}(C, \varLambda^{\mathfrak{p}} X) \cong \operatorname{Hom}_{\mathsf{T}}(\varGamma_{\mathfrak{p}} C, X) = 0\,.$$

Thus $\varLambda^{\mathfrak{p}} X = 0$, since C was arbitrary. It follows that $\mathfrak{p}$ is not in $\operatorname{cosupp}_R(X)$. ☐

The next example shows that the inclusion in the preceding result can be strict.

Example 5.7. Let G be a finite group, k a field whose characteristic divides the order of G, and set $R = H^*(G,k)$, the cohomology algebra of G.

Then $\mathsf{K}(\mathsf{Inj}\, kG)$, the homotopy category of complexes of injective kG modules is a compactly generated triangulated category, admitting a natural R-action. As explained in [7, Example 11.1], any non-zero compact object X in $\mathsf{K}(\mathsf{Inj}\, kG)$ satisfies

$$\operatorname{cosupp}_R(X) = \{H^{\geqslant 1}(G,k)\}\,.$$

On the other hand, for any any closed subset $\mathcal{V} \subseteq \operatorname{Spec} R$, there exists a compact object X with $\operatorname{supp}_R(X) = \mathcal{V}$, so the inclusion in Theorem 5.8 can be strict.

In contrast, in $\mathsf{StMod}\, kG$, which is also an R-linear compactly generated triangulated category, for any compact object (that is to say, for any finite dimensional kG-module) X one has $\operatorname{cosupp}_R(X) = \operatorname{supp}_R(X)$; see [7, Example 11.14].

A variation. To round off this discussion we prove a version of Proposition 5.6 for triangulated categories without using any tensor structure. Note that the definitions of support and cosupport in terms of the functors $\varGamma_{\mathfrak{p}}$ and $\varLambda^{\mathfrak{p}}$ do not require a tensor structure; see [4, §5] and [7, §4] for details.

Theorem 5.8. *Let T be a compactly generated R-linear triangulated category and X a compact object such that* $\operatorname{End}^*_{\mathsf{T}}(X)$ *is finitely generated over R. Then*

$$\operatorname{cosupp}_R(X) \subseteq \operatorname{supp}_R(X)\,.$$

Proof. Fix a prime $\mathfrak{p} \notin \operatorname{supp}_R(X)$ and a compact object C in T. It suffices to prove that $\operatorname{Hom}^*_{\mathsf{T}}(\varGamma_{\mathfrak{p}} C, X) = 0$. Then adjunction yields $\operatorname{Hom}^*_{\mathsf{T}}(C, \varLambda^{\mathfrak{p}} X) = 0$, and hence, since C was arbitrary, $\varLambda^{\mathfrak{p}} X = 0$, that is to say, $\mathfrak{p} \notin \operatorname{cosupp}_R(X)$.

Since X is compact and the R-module $\operatorname{End}^*_{\mathsf{T}}(X)$ is finitely generated, [4, Theorem 5.5] yields that $\operatorname{End}^*_{\mathsf{T}}(X)_{\mathfrak{p}} = 0$. Since the R-action on $\operatorname{Hom}^*_{\mathsf{T}}(\varGamma_{\mathfrak{p}}C, X)$ factors through $\operatorname{End}^*_{\mathsf{T}}(X)$, this then yields the second isomorphism below:

$$\operatorname{Hom}^*_{\mathsf{T}}(\varGamma_{\mathfrak{p}}C, X) \cong \operatorname{Hom}^*_{\mathsf{T}}(\varGamma_{\mathfrak{p}}C, X)_{\mathfrak{p}} \cong 0\,.$$

The first one holds as the R-module on its left is $\mathfrak{p}$-local, by [6, Proposition 2.3]. □

In the case when the R-linear category T is noetherian, in the sense of [6], the result above may be reformulated as follows.

Corollary 5.9. *Assume that for any compact object C in T the R-module $\operatorname{End}^*_{\mathsf{T}}(C)$ is finitely generated. Let $\mathcal{U} \subseteq \operatorname{Spec} R$ be a subset such that $\operatorname{Spec} R \smallsetminus \mathcal{U}$ is specialization closed, and S the localizing and colocalizing subcategory $\mathsf{T}_{\mathcal{U}} = \mathsf{T}^{\mathcal{U}}$. Then*

$$\mathsf{T}^c \cap {}^{\perp}\mathsf{S} \subseteq \mathsf{S}^{\perp}\,.$$

Proof. The equality $\mathsf{T}_{\mathcal{U}} = \mathsf{T}^{\mathcal{U}}$ is [7, Corollary 4.9].

Setting $\mathcal{V} = \operatorname{Spec} R \smallsetminus \mathcal{U}$, one gets ${}^{\perp}\mathsf{S} = \mathsf{T}_{\mathcal{V}}$ and $\mathsf{S}^{\perp} = \mathsf{T}^{\mathcal{V}}$, by Lemma 5.2. Now the assertion follows from Theorem 5.8. □

6. Stone duality and support

A basic idea in lattice theory, going back to Stone [32], is to *represent* a distributive lattice via its spectrum of prime ideals; see [8] for a modern treatment. In this section we consider a well generated tensor triangulated category T and identify a distributive sublattice of the Bousfield lattice of T. This enables us to introduce a notion of support for objects in T, with values in the associated topological space; it is an intrinsic notion of support that does not depend on an action of any ring.

Frames. A *frame* is a complete lattice in which the following *infinite distributivity* holds: for every element a and set of elements $\{b_i\}$ in Λ, there is an equality

$$a \wedge (\bigvee_i b_i) = \bigvee_i (a \wedge b_i)\,.$$

An element $p \neq 1$ in a frame Λ is called *prime* if $a \wedge b \leq p$ implies $a \leq p$ or $b \leq p$. We write $\operatorname{Sp}(\Lambda)$ for the prime elements in Λ, and for each $a \in \Lambda$ set

$$U(a) = \{p \in \operatorname{Sp}(\Lambda) \mid a \not\leq p\}\,.$$

It is not hard to verify that declaring sets of the form $U(a)$ to be open defines a topology on $\operatorname{Sp}(\Lambda)$; this is the *Stone topology*, and $\operatorname{Sp}(\Lambda)$ with this topology is called the *spectrum* of Λ. One says that Λ has *enough points* if $U(a) = U(b)$ implies $a = b$, for all $a, b \in \Lambda$. This means that the map sending $a \in \Lambda$ to $U(a)$ induces an isomorphism between Λ and the lattice of open subsets of $\operatorname{Sp}(\Lambda)$.

A morphism of frames $f\colon \Lambda \to \Gamma$ is a map such that for elements a, b and any set of elements $\{a_i\}$ in Λ, there are equalities

$$f(a \wedge b) = f(a) \wedge f(b) \quad \text{and} \quad f(\bigvee_i a_i) = \bigvee_i f(a_i)\,.$$

For example, primes in Λ correspond to *points*, that is, morphisms $\Lambda \to \mathbb{F}_2 = \{0, 1\}$. A morphism of frames $f\colon \Lambda \to \Gamma$ induces a continuous map $\operatorname{Sp}(f)\colon \operatorname{Sp}(\Gamma) \to \operatorname{Sp}(\Lambda)$ by taking a prime p to $\bigvee_{f(a)\leq p} a$. Alternatively, $\operatorname{Sp}(f)$ takes a point $q\colon \Gamma \to \mathbb{F}_2$ to the composite qf. Note that

$$U(f(a)) = \operatorname{Sp}(f)^{-1}(U(a))\,. \tag{6.1}$$

This yields a contravariant functor Sp into the category of topological spaces. Taking a space X to the lattice $\mathcal{O}(X)$ of open subsets provides a right adjoint to Sp. Thus there is a bijection

$$\operatorname{Hom}_{\mathsf{Top}}(X, \operatorname{Sp}(\Lambda)) \xrightarrow{\sim} \operatorname{Hom}_{\mathsf{Frm}}(\Lambda, \mathcal{O}(X)) \tag{6.2}$$

which sends f to the map $(a \mapsto f^{-1}U(a))$. Given a frame Λ, the adjunction morphism

$$\Lambda \longrightarrow \mathcal{O}(\operatorname{Sp}(\Lambda)), \quad a \mapsto U(a)\,, \tag{6.3}$$

is an isomorphism if and only if Λ has enough points. Given a space X, the adjunction morphism $X \to \operatorname{Sp}(\mathcal{O}(X))$ is a homeomorphism if and only if X is *sober*, that is, each non-empty irreducible closed subset has a unique generic point.

The following result summarizes the correspondence between lattices and spaces; it goes back to Stone [32] and is known as *Stone duality*.

Proposition 6.4. *Taking a frame to its spectrum and a topological space to its lattice of open subsets induces mutually inverse (contravariant) equivalences:*

$$\{\, \textit{frames with enough points} \,\} \xleftrightarrow{1-1} \{\, \textit{sober topological spaces} \,\} \qquad \square$$

Bousfield idempotents. Let now T be a well generated tensor triangulated category. An object X in T is said to be *Bousfield idempotent* if $\mathrm{A}(X) = \mathrm{A}(X \otimes X)$. This property only depends on $\mathrm{A}(X)$, since $\mathrm{A}(X) = \mathrm{A}(Y)$ implies

$$\mathrm{A}(X \otimes X) = \mathrm{A}(X \otimes Y) = \mathrm{A}(Y \otimes Y)\,.$$

Consider the set

$$\mathbf{D}(\mathsf{T}) = \{\mathrm{A}(X) \in \mathbf{A}(\mathsf{T}) \mid \mathrm{A}(X) = \mathrm{A}(X \otimes X)\}$$

with the partial order induced from $\mathbf{A}(\mathsf{T})$. The following result is due to Bousfield [9]; see also [16]. It shows that distributivity of the tensor product implies distributivity of $\mathbf{D}(\mathsf{T})$; hence our notation.

Proposition 6.5. *The partially ordered set $\mathbf{D}(\mathsf{T})$ is a frame. More precisely,*

$$\mathrm{A}(X) \wedge \mathrm{A}(Y) = \mathrm{A}(X \otimes Y) \quad \textit{and} \quad \bigvee_i \mathrm{A}(X_i) = \mathrm{A}(\coprod_i X_i)$$

hold in $\mathbf{D}(\mathsf{T})$ for all Bousfield idempotent objects X, Y and $\{X_i\}$ in T.

Proof. If U is a Bousfield idempotent object with $\mathrm{A}(U) \le \mathrm{A}(X) \wedge \mathrm{A}(Y)$, then

$$\mathrm{A}(U) = \mathrm{A}(U \otimes U) \le \mathrm{A}(U \otimes Y) \le \mathrm{A}(X \otimes Y)\,,$$

where the inequalities are easily verified. If X and Y are also Bousfield idempotent, then so is $X \otimes Y$, and hence one obtains that $\mathrm{A}(X) \wedge \mathrm{A}(Y) = \mathrm{A}(X \otimes Y)$.

Given a set of Bousfield idempotent objects X_i, we have in $\mathbf{A}(\mathsf{T})$

$$\begin{aligned} \mathrm{A}(\coprod_i X_i) &\ge \mathrm{A}\left((\coprod_i X_i) \otimes (\coprod_i X_i)\right) = \mathrm{A}\left(\coprod_{i,j} (X_i \otimes X_j)\right) = \bigvee_{i,j} \mathrm{A}(X_i \otimes X_j) \\ &\ge \bigvee_i \mathrm{A}(X_i \otimes X_i) = \bigvee_i \mathrm{A}(X_i) = \mathrm{A}(\coprod_i X_i). \end{aligned}$$

Thus $\coprod_i X_i$ is Bousfield idempotent, and this implies $\bigvee_i \mathrm{A}(X_i) = \mathrm{A}(\coprod_i X_i)$.

The infinite distributivity in $\mathbf{D}(\mathsf{T})$ follows from the fact that in T the tensor product distributes over set-indexed coproducts. $\square$

The distributive lattice $\mathbf{D}(\mathsf{T})$ provides the basis for an intrinsic notion of support.

Support. We set $\operatorname{Sp}(\mathsf{T}) = \operatorname{Sp}(\mathbf{D}(\mathsf{T}))$ and define for each object X in T its *support*

$$\operatorname{supp}_{\mathsf{T}}(X) = \{\mathrm{A}(P) \in \operatorname{Sp}(\mathsf{T}) \mid \mathrm{A}(X) \not\leq \mathrm{A}(P)\}\,.$$

By definition, this set is open when X is Bousfield idempotent.

Proposition 6.6. *The map* $\operatorname{supp}_{\mathsf{T}}(-)$ *has the following properties:*

(1) $\operatorname{supp}_{\mathsf{T}}(0) = \varnothing$ *and* $\operatorname{supp}_{\mathsf{T}}(\mathbb{1}) = \operatorname{Sp}(\mathsf{T})$.
(2) $\operatorname{supp}_{\mathsf{T}}(\coprod_i X_i) = \bigcup_i \operatorname{supp}_{\mathsf{T}}(X_i)$ *for every set of objects* $\{X_i\}$ *in* T.
(3) $\operatorname{supp}_{\mathsf{T}}(\Sigma X) = \operatorname{supp}_{\mathsf{T}}(X)$ *for every object* X *in* T.
(4) *For every exact triangle* $X' \to X \to X'' \to$ *in* T *one has*

$$\operatorname{supp}_{\mathsf{T}}(X) \subseteq \operatorname{supp}_{\mathsf{T}}(X') \cup \operatorname{supp}_{\mathsf{T}}(X'')\,.$$

(5) $\operatorname{supp}_{\mathsf{T}}(X \otimes Y) \subseteq \operatorname{supp}_{\mathsf{T}}(X) \cap \operatorname{supp}_{\mathsf{T}}(Y)$ *for all objects* X, Y *in* T*; equality holds when* X, Y *are Bousfield idempotent.*

Proof. The properties (1)–(4) and the inclusion in (5) follow from the fact that the map taking an object X to $\mathrm{A}(X)$ has the analogous properties and that the map taking $\mathrm{A}(X)$ to $\operatorname{supp}_{\mathsf{T}}(X)$ is order preserving. The second claim in (5) is clear from Proposition 6.5. □

For any objects X, Y in T, it follows from definitions that

$$\mathrm{A}(X) \leq \mathrm{A}(Y) \quad \Longrightarrow \quad \operatorname{supp}_{\mathsf{T}}(X) \subseteq \operatorname{supp}_{\mathsf{T}}(Y)\,.$$

The result below establishes a converse, under additional hypotheses on T. This provides a criterion for the existence of a reasonable notion of support.

Recall that T is a well generated tensor triangulated category.

Proposition 6.7. *The following conditions are equivalent:*

(1) $\operatorname{supp}_{\mathsf{T}}(X) \neq \varnothing$ *for every object* $X \neq 0$*, and*

$$\operatorname{supp}_{\mathsf{T}}(X \otimes Y) = \operatorname{supp}_{\mathsf{T}}(X) \cap \operatorname{supp}_{\mathsf{T}}(Y) \quad \textit{for all } X, Y \textit{ in } \mathsf{T}.$$

(2) $\operatorname{supp}_{\mathsf{T}}(X)$ *is an open subset of* $\operatorname{Sp}(\mathsf{T})$ *for every object* X*, and*

$$\mathrm{A}(X) \leq \mathrm{A}(Y) \iff \operatorname{supp}_{\mathsf{T}}(X) \subseteq \operatorname{supp}_{\mathsf{T}}(Y) \quad \textit{for all } X, Y \textit{ in } \mathsf{T}.$$

(3) *Every object of* T *is Bousfield idempotent and* $\mathbf{A}(\mathsf{T})$ *has enough points.*

Proof. (1) ⇒ (2): The assumption implies that for each object X in T

$$\mathrm{A}(X) = \{Y \in \mathsf{T} \mid \operatorname{supp}_{\mathsf{T}}(X) \cap \operatorname{supp}_{\mathsf{T}}(Y) = \varnothing\}.$$

From $\operatorname{supp}_{\mathsf{T}}(X \otimes X) = \operatorname{supp}_{\mathsf{T}}(X)$ then follows that $\mathrm{A}(X \otimes X) = \mathrm{A}(X)$. Thus X is Bousfield idempotent, and therefore $\operatorname{supp}_{\mathsf{T}}(X)$ is open.

If $\mathrm{A}(X) \not\leq \mathrm{A}(Y)$, then there exists a U such that $U \otimes X \neq 0$ and $U \otimes Y = 0$. Thus $\operatorname{supp}_{\mathsf{T}}(U) \cap \operatorname{supp}_{\mathsf{T}}(X) \neq \varnothing$ while $\operatorname{supp}_{\mathsf{T}}(U) \cap \operatorname{supp}_{\mathsf{T}}(Y) = \varnothing$, and hence $\operatorname{supp}_{\mathsf{T}}(X) \not\subseteq \operatorname{supp}_{\mathsf{T}}(Y)$. The other implication always holds.

(2) ⇒ (3): The frame $\mathbf{D}(\mathsf{T})$ has enough points, since $\operatorname{supp}_{\mathsf{T}}(X) = \operatorname{supp}_{\mathsf{T}}(Y)$ implies $\mathrm{A}(X) = \mathrm{A}(Y)$.

If an object X in T is such that $\operatorname{supp}_{\mathsf{T}}(X)$ is open, then there exists a Bousfield idempotent object Y such that $\operatorname{supp}_{\mathsf{T}}(X) = \operatorname{supp}_{\mathsf{T}}(Y)$; this implies $\mathrm{A}(X) = \mathrm{A}(Y)$, and hence X is also Bousfield idempotent.

(3) ⇒ (1): The equality $\operatorname{supp}_{\mathsf{T}}(X \otimes Y) = \operatorname{supp}_{\mathsf{T}}(X) \cap \operatorname{supp}_{\mathsf{T}}(Y)$ holds for all Bousfield idempotent objects X, Y by Proposition 6.6. If $X \neq 0$, then $\mathrm{A}(X) \neq \mathrm{A}(0)$, and therefore $\operatorname{supp}_{\mathsf{T}}(X) \neq \varnothing$ since the Bousfield lattice has enough points. □

Remark 6.8. In the stable homotopy category of spectra, there are objects that are not Bousfield idempotent [9, Lemma 2.5]; see [13, Theorem 6.1] for examples in the derived category of a ring. Also, a priori it is possible that $\mathrm{Sp}(\mathsf{T}) = \varnothing$.

On the other hand, any triangulated category T which is stratified by the action of a graded-commutative noetherian ring satisfies the equivalent conditions of Proposition 6.7; see Example 6.10 below.

Universality. The map $\mathrm{supp}_{\mathsf{T}}(-)$ enjoys the following universal property. Its statement is inspired by the universality of Balmer's support [2], and is an immediate consequence of Stone duality.

Proposition 6.9. *Suppose each object in T is Bousfield idempotent. Let U be a topological space and σ a map that assigns to each object $X \in \mathsf{T}$ an open subset $\sigma(X)$ of U with the following properties:*

1. *$\sigma(X \otimes Y) = \sigma(X) \cap \sigma(Y)$ for all X, Y in T,*
2. *$\sigma(\coprod_i X_i) = \bigcup_i \sigma(X_i)$ for every set of objects X_i in T,*
3. *$\mathrm{A}(X) = \mathrm{A}(Y) \implies \sigma(X) = \sigma(Y)$, for all X, Y in T.*

Then there exists a unique continuous map $f \colon U \to \mathrm{Sp}(\mathsf{T})$ such that

$$\sigma(X) = f^{-1}(\mathrm{supp}_{\mathsf{T}}(X)) \qquad \text{for all } X \in \mathsf{T}\,.$$

Proof. The map σ yields a frame morphism $\mathbf{D}(\mathsf{T}) \to \mathcal{O}(U)$ which then corresponds uniquely to a continuous map $f \colon U \to \mathrm{Sp}(\mathsf{T})$, by Stone duality (6.2). □

Example 6.10. Let T be a compactly generated tensor triangulated category stratified via the action of a graded-commutative noetherian ring R. Consider the set $\mathrm{supp}_R(\mathsf{T})$ endowed with the discrete topology. It follows from Theorem 4.4 that the map $\mathrm{supp}_R(-)$ satisfies all the properties listed in Proposition 6.9; it thus induces a continuous map $\mathrm{supp}_R(\mathsf{T}) \to \mathrm{Sp}(\mathsf{T})$. This map sends $\mathfrak{p}$ in $\mathrm{supp}_R(\mathsf{T})$ to $\mathrm{A}(\coprod_{\mathfrak{q}\neq\mathfrak{p}} \varGamma_{\mathfrak{q}}\mathbb{1})$ and is actually a homeomorphism, since $\mathrm{supp}_R(-)$ gives an isomorphism

$$\mathbf{D}(\mathsf{T}) = \mathbf{A}(\mathsf{T}) \xrightarrow{\sim} \mathbf{2}^{\mathrm{supp}_R(\mathsf{T})} = \mathcal{O}(\mathrm{supp}_R(\mathsf{T})),$$

by Corollary 4.5.

Functoriality. A functor $F \colon \mathsf{T} \to \mathsf{U}$ between tensor triangulated categories is called *tensor triangulated* if it is an exact functor that respects the monoidal structures; we do not assume that F preserves the tensor unit. We call such a functor *conservative* if $\mathrm{A}(X) = \mathrm{A}(Y)$ implies $\mathrm{A}(FX) = \mathrm{A}(FY)$ for all objects X, Y in T.

Proposition 6.11. *Let $F \colon \mathsf{T} \to \mathsf{U}$ be a tensor triangulated functor between well generated tensor triangulated categories. Assume F is conservative and preserves set-indexed coproducts. Then the map sending $\mathrm{A}(X)$ to $\mathrm{A}(FX)$ induces a morphism of frames $\mathbf{D}(\mathsf{T}) \to \mathbf{D}(\mathsf{U})$ and hence a continuous map $\mathrm{Sp}(F) \colon \mathrm{Sp}(\mathsf{U}) \to \mathrm{Sp}(\mathsf{T})$. For each object X in T, one has*

$$\mathrm{supp}_{\mathsf{U}}(FX) = \mathrm{Sp}(F)^{-1}(\mathrm{supp}_{\mathsf{T}}(X))\,.$$

Proof. The proof is straightforward. □

We do not have a general criterion for a functor to be conservative, but there is the following important example.

Proposition 6.12. *Let T be a well generated tensor triangulated category and S a tensor closed localizing subcategory such that S is well generated and $\mathsf{S}^{\perp}$ is tensor closed. Consider the induced tensor triangulated structure for the Verdier quotient T/S. Then the inclusion functor $\mathsf{S} \to \mathsf{T}$, the quotient functor $\mathsf{T} \to \mathsf{T}/\mathsf{S}$, and* *their right adjoints are tensor triangulated and conservative. These functors induce isomorphisms*

$$\mathbf{A}(\mathsf{T}) \xrightarrow{\sim} \mathbf{A}(\mathsf{S}) \times \mathbf{A}(\mathsf{T}/\mathsf{S}) \quad \textit{and} \quad \mathbf{D}(\mathsf{T}) \xrightarrow{\sim} \mathbf{D}(\mathsf{S}) \times \mathbf{D}(\mathsf{T}/\mathsf{S}).$$

The proof is based on the following lemma.

Lemma 6.13. *There exists an exact localization functor $L\colon \mathsf{T} \to \mathsf{T}$ with $\operatorname{Ker} L = \mathsf{S}$, and for each object X in T an exact triangle*

$$\varGamma X \to X \to LX \to \quad \textit{with} \quad \varGamma X \cong \varGamma \mathbb{1} \otimes X \quad \textit{and} \quad LX \cong L\mathbb{1} \otimes X$$

Proof. The assumption on S to be well generated implies that there exists an exact localization functor $L\colon \mathsf{T} \to \mathsf{T}$ with $\operatorname{Ker} L = \mathsf{S}$, by Proposition 2.1. This functor factors through the quotient functor $F\colon \mathsf{T} \to \mathsf{T}/\mathsf{S}$ via a functor $G\colon \mathsf{T}/\mathsf{S} \to \mathsf{T}$, which is a right adjoint of F and induces an equivalence $\mathsf{T}/\mathsf{S} \xrightarrow{\sim} \mathsf{S}^{\perp}$.

Completing for each object X in T the natural morphism $X \to LX$ yields a functorial exact triangle

$$\varGamma X \longrightarrow X \longrightarrow LX \longrightarrow$$

with $\varGamma X \in \mathsf{S}$. Now apply $- \otimes X$ to the localization triangle $\varGamma\mathbb{1} \to \mathbb{1} \to L\mathbb{1} \to$. Then $\varGamma\mathbb{1} \otimes X$ belongs to S, while $L\mathbb{1} \otimes X$ belongs to $\mathsf{S}^{\perp}$. Thus $LX \cong L\mathbb{1} \otimes X$ and $\varGamma X \cong \varGamma\mathbb{1} \otimes X$. Note that $\varGamma$ provides a right adjoint of the inclusion $\mathsf{S} \to \mathsf{T}$. □

Proof of Proposition 6.12. We keep the notation from Lemma 6.13 and its proof.

Given $X \in \mathsf{S}$ and $Y \in \mathsf{T}$, we have $X \otimes Y = 0$ iff $X \otimes \varGamma Y = 0$. Thus the inclusion is conservative. The identity $\mathrm{A}(\varGamma Y) = \mathrm{A}(Y) \cap \mathsf{S}$ implies that $\varGamma$ is conservative.

By a similar argument, the inclusion $\mathsf{S}^{\perp} \to \mathsf{T}$ and its left adjoint $L\colon \mathsf{T} \to \mathsf{S}^{\perp}$ are conservative. The composite $E\colon \mathsf{S}^{\perp} \to \mathsf{T} \to \mathsf{T}/\mathsf{S}$ is an equivalence of tensor triangulated categories, hence conservative. The composite of E with L is isomorphic to F, so the latter is conservative. The composite of E^{-1} with the inclusion $\mathsf{S}^{\perp} \to \mathsf{T}$ is isomorphic to G. Thus G is conservative.

The isomorphism $\mathbf{A}(\mathsf{T}) \to \mathbf{A}(\mathsf{S}) \times \mathbf{A}(\mathsf{T}/\mathsf{S})$ sends $\mathrm{A}(X)$ to $(\mathrm{A}(\varGamma X), \mathrm{A}(FX))$, and its inverse sends $(\mathrm{A}(U), \mathrm{A}(V))$ to $\mathrm{A}(U) \vee \mathrm{A}(GV)$. Note that X in T is Bousfield idempotent if and only if $\varGamma X$ and FX are Bousfield idempotent. □

Corollary 6.14. *Let $\mathsf{S}_1 = \operatorname{Loc}(X_1)$ and $\mathsf{S}_2 = \operatorname{Loc}(X_2)$ be tensor closed localizing subcategories of T such that $\mathsf{S}_1^{\perp}$ and $\mathsf{S}_2^{\perp}$ are tensor closed. Set $\mathsf{S} = \operatorname{Loc}(X_1 \amalg X_2)$. Then the Bousfield lattice $\mathbf{A}(\mathsf{T})$ admits the following decomposition:*

$$\mathbf{A}(\mathsf{T}) \cong \mathbf{A}(\mathsf{T}/\mathsf{S}) \times \mathbf{A}(\mathsf{S}_1/\mathsf{S}_1 \cap \mathsf{S}_2) \times \mathbf{A}(\mathsf{S}_2/\mathsf{S}_1 \cap \mathsf{S}_2) \times \mathbf{A}(\mathsf{S}_1 \cap \mathsf{S}_2)$$

Proof. From Proposition 6.12 one gets

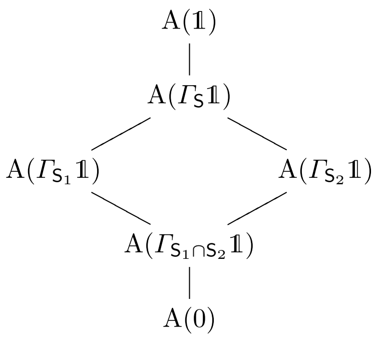
$$\begin{aligned} \mathbf{A}(\mathsf{T}) &\cong \mathbf{A}(\mathsf{T}/\mathsf{S}) \times \mathbf{A}(\mathsf{S}) \\ &\cong \mathbf{A}(\mathsf{T}/\mathsf{S}) \times \mathbf{A}(\mathsf{S}/\mathsf{S}_1 \cap \mathsf{S}_2) \times \mathbf{A}(\mathsf{S}_1 \cap \mathsf{S}_2) \\ &\cong \mathbf{A}(\mathsf{T}/\mathsf{S}) \times \mathbf{A}(\mathsf{S}_1/\mathsf{S}_1 \cap \mathsf{S}_2) \times \mathbf{A}(\mathsf{S}_2/\mathsf{S}_1 \cap \mathsf{S}_2) \times \mathbf{A}(\mathsf{S}_1 \cap \mathsf{S}_2). \end{aligned}$$

For the last isomorphism one uses the pair of standard isomorphisms

$$\mathsf{S}_1/\mathsf{S}_1 \cap \mathsf{S}_2 \xrightarrow{\sim} \mathsf{S}/\mathsf{S}_2 \xrightarrow{\sim} (\mathsf{S}/\mathsf{S}_1 \cap \mathsf{S}_2)/(\mathsf{S}_2/\mathsf{S}_1 \cap \mathsf{S}_2);$$

see [34, II.2.3]. □

The decomposition of $\mathbf{A}(\mathsf{T})$ is illustrated by the following Hasse type diagram. Given any localizing subcategory $\mathsf{S} \subseteq \mathsf{T}$, we write $\varGamma_{\mathsf{S}}$ for a right adjoint of the inclusion functor $\mathsf{S} \to \mathsf{T}$, assuming that it exists.



Example 6.15. Let T be a compactly generated tensor triangulated category with an action of a graded-commutative noetherian ring R. Denote for each $\mathfrak{p} \in \operatorname{Spec} R$ by $\varGamma_{\mathfrak{p}}\mathsf{T}$ the essential image of the functor $\varGamma_{\mathfrak{p}} \colon \mathsf{T} \to \mathsf{T}$. From the construction of $\varGamma_{\mathfrak{p}}$ in [4, §5] and Proposition 6.12, it follows that the functors $\varGamma_{\mathfrak{p}} \colon \mathsf{T} \to \varGamma_{\mathfrak{p}}\mathsf{T}$ and the inclusion $\varGamma_{\mathfrak{p}}\mathsf{T} \to \mathsf{T}$ are tensor triangulated and conservative. Thus these functors induce isomorphisms

$$\mathbf{A}(\mathsf{T}) \xrightarrow{\sim} \prod_{\mathfrak{p} \in \operatorname{Spec} R} \mathbf{A}(\varGamma_{\mathfrak{p}}\mathsf{T}) \quad \text{and} \quad \mathbf{D}(\mathsf{T}) \xrightarrow{\sim} \prod_{\mathfrak{p} \in \operatorname{Spec} R} \mathbf{D}(\varGamma_{\mathfrak{p}}\mathsf{T})\,.$$

The inverse maps send $(\mathrm{A}(X_{\mathfrak{p}}))_{\mathfrak{p}}$ to $\mathrm{A}(\coprod_{\mathfrak{p}} X_{\mathfrak{p}})$; see [6, Theorem 7.2]. When T is stratified by R, this yields the isomorphism $\mathbf{A}(\mathsf{T}) \xrightarrow{\sim} \mathbf{2}^{\operatorname{supp}_R(\mathsf{T})}$ from Corollary 4.5.

Example 6.16. Fix a separated noetherian scheme $(X, \mathcal{O}_X)$ and consider the derived category $\mathsf{D}(\operatorname{Qcoh} X)$ of the category of quasi-coherent $\mathcal{O}_X$-modules. This is a compactly generated tensor triangulated category. The Bousfield lattice of $\mathsf{D}(\operatorname{Qcoh} X)$ is isomorphic to the lattice of subsets of X. For the affine case, this assertion follows from Neeman's work [24]; see Example 4.7. The general case then follows, using Corollary 6.14; see also [1]. The model for this is Gabriel's analysis of the abelian category of quasi-coherent $\mathcal{O}_X$-modules [14, Chap. VI].

7. Support for compact objects

Let $(\mathsf{T}, \otimes, \mathbb{1})$ be a compactly generated tensor triangulated category, as in Section 4. The full subcategory T^c consisting of all compact objects is a skeletally small tensor triangulated category. Let $\mathbf{Th}(\mathsf{T}^c)$ denote the set of thick subcategories of T^c that are tensor closed. This set is partially ordered by inclusion and is a complete lattice, since for any set of elements C_i in $\mathbf{Th}(\mathsf{T}^c)$ there is an equality

$$\bigwedge_i \mathsf{C}_i = \bigcap_i \mathsf{C}_i\,.$$

In this section we discuss the support of objects in T^c, using the spectrum associated with $\mathbf{Th}(\mathsf{T}^c)$. Then we relate this support to the structure of the Bousfield lattice of the ambient category T. We begin by recalling pertinent facts about compactly generated lattices.

Compact generation. Fix a complete lattice Λ. An element $a \in \Lambda$ is *compact* if $a \le \bigvee_{i \in I} b_i$ implies $a \le \bigvee_{i \in J} b_i$ for some finite subset $J \subseteq I$. We write Λ^c for the partially ordered subset of compact elements in Λ. Note that $a, b \in \Lambda^c$ implies $a \vee b \in \Lambda^c$. The lattice Λ is *compactly generated* if every element in Λ is the supremum of compact elements.

Any compactly generated lattice is essentially determined by its subset of compact elements. To explain this, we need to introduce the ideal completion of a partially ordered set.

Let Γ be a partially ordered set having an infimum, and suppose that every finite subset has a supremum. A non-empty subset $I \subseteq \Gamma$ is an *ideal* of Γ if for all $a, b \in \Gamma$

(1) $a \le b$ and $b \in I$ imply $a \in I$, and
(2) $a, b \in I$ implies $a \vee b \in I$.

Given $a \in \Gamma$, let $I(a) = \{x \in \Gamma \mid x \le a\}$ denote the *principal ideal* generated by a. The set $\widehat{\Gamma}$ of all ideals of Γ is called the *ideal completion*[1] of Γ. This set is partially ordered by inclusion and in fact a compactly generated complete lattice. The map $\Gamma \to \widehat{\Gamma}$ sending $a \in \Gamma$ to $I(a)$ identifies Γ with $\widehat{\Gamma}^c$.

Lemma 7.1. *Let Λ be a compactly generated complete lattice. Then the map*

$$\Lambda \longrightarrow \widehat{\Lambda^c}, \quad a \mapsto I(a) \cap \Lambda^c = \{x \in \Lambda \mid x \le a \text{ and } x \text{ compact}\},$$

is a lattice isomorphism.

Proof. The inverse map sends an ideal $I \in \widehat{\Lambda^c}$ to its supremum in Λ. □

Coherent frames. A frame is called *coherent* if it is compactly generated and the compact elements form a sublattice containing 1 [19]. The correspondence from Proposition 6.4 between frames and topological spaces identifies the coherent frames with the spaces that are spectral. Following Hochster [15], a topological space is called *spectral* if it is T_0 and quasi-compact, the quasi-compact open subsets are closed under finite intersections and form an open basis, and every non-empty irreducible closed subset has a generic point.

The following characterization of the coherent frames is well-known; we sketch the proof for the convenience of the reader.

Proposition 7.2. *Let Λ be a complete lattice. The conditions below are equivalent:*

(1) *Λ is a coherent frame.*
(2) *Λ is a frame, has enough points, and the associated spectrum is spectral.*
(3) *Λ is the ideal completion of a distributive lattice having 0 and 1.*
(4) *Λ is compactly generated, distributive, and the compact elements form a sublattice containing 1.*

Proof. (1) ⇒ (2): Given elements $a, b \in \Lambda$ with $b \not\le a$, there exists $c \in \Lambda^c$ such that $c \not\le a$ and $c \le b$. The set $\{x \in \Lambda \mid a \le x,\, c \not\le x\}$ has a maximal element (using Zorn's lemma) which is prime. Thus Λ is a frame having enough points.

The sets $U(a)$ with $a \in \Lambda^c$ are precisely the quasi-compact open subsets of $\mathrm{Sp}(\Lambda)$; they are closed under finite intersections and $\mathrm{Sp}(\Lambda)$ is of this form since Λ^c is a sublattice of Λ. Given an irreducible subset $X \subseteq \mathrm{Sp}(\Lambda)$, the element $\bigwedge_{p \in X} p$ is prime and therefore a generic point of X. Thus $\mathrm{Sp}(\Lambda)$ is spectral.

(2) ⇒ (3): The lattice Λ is isomorphic to the lattice of open subsets of $\mathrm{Sp}(\Lambda)$, since Λ has enough points. The fact that $\mathrm{Sp}(\Lambda)$ is spectral means that Λ is isomorphic to the ideal completion of the lattice of quasi-compact open subsets $\Gamma = \mathcal{O}(\mathrm{Sp}(\Lambda))^c$. The required properties of Γ are easily checked.

(3) ⇒ (4): Suppose that $\Lambda = \widehat{\Gamma}$ for some distributive lattice Γ. The principal ideals are the compact elements in Λ. Therefore Λ is compactly generated. The

[1] Viewing a partially ordered set as a category, the ideal completion of Γ is nothing but the Ind-completion of Γ, that is, the category of functors $\Gamma^{\mathrm{op}} \to \mathsf{Sets}$ that are filtered colimits of representable functors.

maximal element $1 \in \Lambda$ is compact since Γ has a maximal element. The infimum of two compact elements is again compact, since $I(a) \wedge I(b) = I(a \wedge b)$. Distributivity of Λ follows from the distributivity of Γ, using that $I \vee J = \{a \vee b \mid a \in I,\, b \in J\}$.

(4) $\Rightarrow$ (1): Compact generation of Λ implies for every $a \in \Lambda$ and every directed subset $B \subseteq \Lambda$

$$a \wedge (\bigvee_{b \in B} b) = \bigvee_{b \in B} (a \wedge b).$$

Using that Λ is distributive, this identity extends to arbitrary subsets $B \subseteq \Lambda$, since

$$\bigvee_{b \in B} b = \bigvee_{\substack{B' \subseteq B \\ \text{finite}}} \bigvee_{b \in B'} b$$

and the finite subsets of B form a directed set. Thus Λ is a coherent frame. $\square$

Any compactly generated lattice is determined by its subset of compact elements, by Lemma 7.1. Combining this fact with Propositions 6.4 and 7.2 yields the following correspondence, which is another incarnation of Stone duality.

Corollary 7.3. *Taking a distributive lattice to the spectrum of its ideal completion induces a bijective correspondence:*

$$\left\{ \begin{matrix} \textit{distributive lattices} \\ \textit{with } 0 \textit{ and } 1 \end{matrix} \right\} \xleftrightarrow{\;1-1\;} \left\{ \begin{matrix} \textit{spectral} \\ \textit{topological spaces} \end{matrix} \right\}$$

The inverse takes a spectral space to its lattice of quasi-compact open subsets. $\square$

Hochster duality. Given a spectral topological space X, one defines its *Hochster dual* space X^* by taking the same points and turning the complements of quasi-compact open sets of X into basic open sets of X^*. Hochster proved that X^* is spectral and that $(X^*)^* = X$; see [15, Proposition 8].

Corollary 7.3 explains this duality: If a spectral space X corresponds to a distributive lattice Λ, then X^* corresponds to the opposite lattice Λ^{op}.

The lattice of thick subcategories. Let now T be a compactly generated tensor triangulated category, as in Section 4. We establish the lattice theoretic properties of $\mathbf{Th}(\mathsf{T}^c)$, and use them to define a notion of support for objects in T^c. The following map is our basic tool:

$$f\colon\ \mathbf{Th}(\mathsf{T}^c) \longrightarrow \mathbf{A}(\mathsf{T}), \qquad \mathsf{C} \mapsto \bigvee_{X \in \mathsf{C}} \mathrm{A}(X) = \mathrm{A}(\coprod_{X \in \mathsf{C}} X)\,.$$

Observe that $f(\mathsf{C}) = \mathsf{C}^{\perp}$ since the objects of C are strongly dualizing.

Lemma 7.4. *The map f is injective and its image is contained in $\mathbf{D}(\mathsf{T})$. Moreover,*

$$f(\mathsf{C} \cap \mathsf{D}) = f(\mathsf{C}) \wedge f(\mathsf{D}) \quad \textit{and} \quad f(\bigvee_i \mathsf{C}_i) = \bigvee_i f(\mathsf{C}_i)$$

for every set of elements $\mathsf{C}, \mathsf{D}, \mathsf{C}_i$ in $\mathbf{Th}(\mathsf{T}^c)$.

Proof. A compact object X is strongly dualizable, and therefore a retract of $X \otimes X^{\vee} \otimes X$; see [23, Proposition III.1.2]. Hence X is Bousfield idempotent. Using the properties of $\mathbf{D}(\mathsf{T})$ from Proposition 6.5, it follows that $f(\mathsf{C})$ belongs to $\mathbf{D}(\mathsf{T})$ for all C in $\mathbf{Th}(\mathsf{T}^c)$.

Next we compute $f(\mathsf{C}\cap\mathsf{D})$:

$$\begin{aligned} f(\mathsf{C}\cap\mathsf{D}) &\le f(\mathsf{C})\wedge f(\mathsf{D}) = \Big(\bigvee_{X\in\mathsf{C}} \mathrm{A}(X)\Big)\wedge\Big(\bigvee_{Y\in\mathsf{D}} \mathrm{A}(Y)\Big)\\ &= \bigvee_{(X,Y)\in\mathsf{C}\times\mathsf{D}} \big(\mathrm{A}(X)\wedge \mathrm{A}(Y)\big) = \bigvee_{(X,Y)\in\mathsf{C}\times\mathsf{D}} \mathrm{A}(X\otimes Y)\\ &\le \bigvee_{Z\in\mathsf{C}\cap\mathsf{D}} \mathrm{A}(Z) = f(\mathsf{C}\cap\mathsf{D}). \end{aligned}$$

The identity for $f(\bigvee_i \mathsf{C}_i)$ is clear.

It remains to verify the injectivity of f. Since C consists of compact objects

$$\operatorname{Loc}(\mathsf{C}) = {}^{\perp}(\operatorname{Loc}(\mathsf{C})^{\perp}) \quad\text{and}\quad \mathsf{C} = \operatorname{Loc}(\mathsf{C})\cap\mathsf{T}^c .$$

Indeed, both equalities are well-known; the first one can be easily deduced from Lemma 6.13; for the second, see [25, Lemma 2.2]. Since $f(\mathsf{C}) = \mathsf{C}^{\perp} = \operatorname{Loc}(\mathsf{C})^{\perp}$, it follows that $\mathsf{C} = {}^{\perp}f(\mathsf{C})\cap\mathsf{T}^c$, whence that f is injective. □

Let $\mathsf{C}\subseteq\mathsf{T}^c$ be a tensor closed thick subcategory and $\operatorname{Loc}(\mathsf{C})$ the localizing subcategory generated by C. Recall from Lemma 6.13 that there exists a localization functor $L_{\mathsf{C}}\colon\mathsf{T}\to\mathsf{T}$ with kernel $\operatorname{Loc}(\mathsf{C})$, and for each object X in T an exact triangle

$$\varGamma_{\mathsf{C}}X\to X\to L_{\mathsf{C}}X\to \quad\text{with}\quad \varGamma_{\mathsf{C}}X\cong\varGamma_{\mathsf{C}}\mathbb{1}\otimes X \quad\text{and}\quad L_{\mathsf{C}}X\cong L_{\mathsf{C}}\mathbb{1}\otimes X;$$

see also [17, Theorem 3.3.3]. In particular, $\operatorname{Loc}(\mathsf{C}) = \operatorname{Loc}(\varGamma_{\mathsf{C}}\mathbb{1})$. This leads to the following alternative description of the map $f\colon \mathbf{Th}(\mathsf{T}^c)\to\mathbf{A}(\mathsf{T})$.

Lemma 7.5. *Let $X\in\mathsf{T}$ such that $\operatorname{Loc}(\mathsf{C}) = \operatorname{Loc}(X)$. Then $f(\mathsf{C}) = \mathrm{A}(X)$. In particular, $f(\mathsf{C}) = \mathrm{A}(\varGamma_{\mathsf{C}}\mathbb{1})$.* □

A consequence is the fact that the elements of the form $f(\mathsf{C})$ have a complement in $\mathbf{A}(\mathsf{T})$. More precisely,

$$\mathrm{A}(\varGamma_{\mathsf{C}}\mathbb{1})\wedge\mathrm{A}(L_{\mathsf{C}}\mathbb{1}) = 0 \quad\text{and}\quad \mathrm{A}(\varGamma_{\mathsf{C}}\mathbb{1})\vee\mathrm{A}(L_{\mathsf{C}}\mathbb{1}) = 1.$$

The following result establishes the basic properties of the lattice $\mathbf{Th}(\mathsf{T}^c)$. The ambient category T is used in our proof, but it is not essential; see [10] for more general results. In what follows for any object X in T, we write $\operatorname{Th}(X)$ for the smallest tensor closed thick subcategory of T containing X.

Theorem 7.6. *Let T be a compactly generated tensor triangulated category. Then the lattice $\mathbf{Th}(\mathsf{T}^c)$ of tensor closed thick subcategories of T^c is a coherent frame. It is isomorphic to the sublattice of the Bousfield lattice $\mathbf{A}(\mathsf{T})$ consisting of the elements $\bigvee_{X\in\mathsf{C}}\mathrm{A}(X)$ with $\mathsf{C}\subseteq\mathsf{T}^c$.*

Proof. Lemma 7.4 provides the embedding of $\mathbf{Th}(\mathsf{T}^c)$ into $\mathbf{A}(\mathsf{T})$. In order to show that $\mathbf{Th}(\mathsf{T}^c)$ is a coherent frame, it suffices to verify that it is compactly generated, distributive, and that the compact elements form a sublattice; see Proposition 7.2.

For any object X in T^c, the category $\operatorname{Th}(X)$ is a compact element in $\mathbf{Th}(\mathsf{T}^c)$. This is clear, since $\operatorname{Th}(X)\le\bigvee_i\mathsf{C}_i$ if and only if $X\in\bigvee_i\mathsf{C}_i$, and keeping in mind the explicit construction of $\bigvee_i\mathsf{C}_i$ from $\bigcup_i\mathsf{C}_i$ by taking cones of morphisms, suspensions etc. Also, $\mathsf{C} = \bigvee_{X\in\mathsf{C}}\operatorname{Th}(X)$ for each C in $\mathbf{Th}(\mathsf{T}^c)$. Thus $\mathbf{Th}(\mathsf{T}^c)$ is compactly generated. The element $1\in\mathbf{Th}(\mathsf{T}^c)$ is compact since $\mathsf{T}^c = \operatorname{Th}(\mathbb{1})$. Given two objects X,Y in T^c, we have

$$\operatorname{Th}(X)\cap\operatorname{Th}(Y) = \operatorname{Th}(X\otimes Y),$$

since there are equalities

$$\begin{aligned} f(\mathrm{Th}(X)\cap \mathrm{Th}(Y)) &= f(\mathrm{Th}(X))\wedge f(\mathrm{Th}(Y)) \\ &= \mathrm{A}(X)\wedge \mathrm{A}(Y) \\ &= \mathrm{A}(X\otimes Y) \\ &= f(\mathrm{Th}(X\otimes Y)). \end{aligned}$$

Thus the compact elements form a sublattice. The distributivity of $\mathbf{Th}(\mathsf{T}^c)$ follows from the distributivity of $\mathbf{D}(\mathsf{T})$, using Lemma 7.4. □

Example 7.7. Let T be a compactly generated tensor triangulated category stratified by the action of a graded-commutative noetherian ring R. Suppose also that the graded endomorphism ring of each compact object is finitely generated over R. Consider the set $\mathrm{supp}_R(\mathsf{T})$ endowed with the Hochster dual of the Zariski topology; thus a subset of $\mathrm{supp}_R(\mathsf{T})$ is open if it is specialization closed. Then the map sending C to $\bigcup_{X\in\mathsf{C}}\mathrm{supp}_R(X)$ induces an isomorphism $\mathbf{Th}(\mathsf{T}^c)\xrightarrow{\sim}\mathcal{O}(\mathrm{supp}_R(\mathsf{T}))$; see [6, Theorem 6.1]. This isomorphism induces a homeomorphism $\mathrm{supp}_R(\mathsf{T})\xrightarrow{\sim}\mathrm{Sp}(\mathbf{Th}(\mathsf{T}^c))$; it sends $\mathfrak{p}$ in $\mathrm{supp}_R(\mathsf{T})$ to $\{X\in\mathsf{T}^c\mid \mathfrak{p}\notin\mathrm{supp}_R(X)\}$.

Support. We write $\mathrm{Sp}(\mathsf{T}^c)=\mathrm{Sp}(\mathbf{Th}(\mathsf{T}^c))$ and for each object X in T^c let

$$\begin{aligned} \mathrm{supp}_{\mathsf{T}^c}(X) &= \{\mathsf{P}\in\mathrm{Sp}(\mathsf{T}^c)\mid X\notin\mathsf{P}\} \\ &= \{\mathsf{P}\in\mathrm{Sp}(\mathsf{T}^c)\mid \mathrm{Th}(X)\not\subseteq\mathsf{P}\}. \end{aligned}$$

Note that $\mathrm{supp}_{\mathsf{T}^c}(X)$ is an open subset of $\mathrm{Sp}(\mathsf{T}^c)$ which is quasi-compact. For a tensor closed thick subcategory $\mathsf{C}\subseteq\mathsf{T}^c$, we write

$$\begin{aligned} \mathrm{supp}_{\mathsf{T}^c}(\mathsf{C}) &= \bigcup_{X\in\mathsf{C}}\mathrm{supp}_{\mathsf{T}^c}(X) \\ &= \{\mathsf{P}\in\mathrm{Sp}(\mathsf{T}^c)\mid \mathsf{C}\not\subseteq\mathsf{P}\}. \end{aligned}$$

Using this notation, Theorem 7.6 has the following consequence.

Corollary 7.8. *The spectrum* $\mathrm{Sp}(\mathsf{T}^c)$ *is a spectral topological space. The map that assigns to each object* $X\in\mathsf{T}^c$ *its support* $\mathrm{supp}_{\mathsf{T}^c}(X)$ *induces an inclusion preserving bijection between the set of tensor closed thick subcategories of* T^c *and the set of open subsets of* $\mathrm{Sp}(\mathsf{T}^c)$.

Proof. The spectrum of a coherent frame is spectral by Proposition 7.2. The map sending a tensor closed thick subcategory to an open subset is precisely the map (6.3) from Stone duality, which is bijective by Proposition 6.4. □

The following result connects the support in T^c with the one defined in T in terms of the Bousfield lattice of T.

Proposition 7.9. *The map* $\mathrm{supp}_{\mathsf{T}^c}(-)$ *has the following properties:*

(1) *The map* $f\colon \mathbf{Th}(\mathsf{T}^c)\to\mathbf{D}(\mathsf{T})$ *induces a continuous map* $\mathrm{Sp}(f)\colon \mathrm{Sp}(\mathsf{T})\to\mathrm{Sp}(\mathsf{T}^c)$ *such that for each object* X *in* T^c *there is an equality*

$$\mathrm{supp}_{\mathsf{T}}(X)=\mathrm{Sp}(f)^{-1}(\mathrm{supp}_{\mathsf{T}^c}(X))\,.$$

(2) $\mathrm{A}(X)\le\mathrm{A}(Y)\iff \mathrm{supp}_{\mathsf{T}^c}(X)\subseteq\mathrm{supp}_{\mathsf{T}^c}(Y)$*, for all* X,Y *in* T^c.

(3) $\mathrm{supp}_{\mathsf{T}^c}(X\otimes Y)=\mathrm{supp}_{\mathsf{T}^c}(X)\cap\mathrm{supp}_{\mathsf{T}^c}(Y)$*, for all* X,Y *in* T^c.

Proof. We apply Lemma 7.4 which lists the properties of f.

(1) The map f is a morphism of frames and yields therefore a continuous map between the associated spectra. In particular, the identity for $\mathrm{supp}_{\mathsf{T}}(X)$ follows from equation (6.1).

(2) Given objects X, Y in T^c, we have

$$\mathrm{A}(X) \leq \mathrm{A}(Y) \iff \mathrm{Th}(X) \subseteq \mathrm{Th}(Y) \iff \operatorname{supp}_{\mathsf{T}^c}(X) \subseteq \operatorname{supp}_{\mathsf{T}^c}(Y)\,.$$

The first equivalence follows from Lemma 7.4, while the second is a consequence of the fact that the frame $\mathbf{Th}(\mathsf{T}^c)$ has enough points, by Theorem 7.6.

(3) This follows from (2), since $\mathrm{A}(X) \wedge \mathrm{A}(Y) = \mathrm{A}(X \otimes Y)$, by Proposition 6.5. □

Corollary 7.10. *For any pair of tensor closed thick subcategories* C, D *of* T^c,

$$\mathsf{C} \cap \mathsf{D} = \mathrm{Th}(\{X \otimes Y \mid X \in \mathsf{C},\, Y \in \mathsf{D}\})\,.$$

Therefore a tensor closed thick subcategory $\mathsf{P} \subsetneq \mathsf{T}^c$ *is prime in* $\mathbf{Th}(\mathsf{T}^c)$ *if and only if* $X \otimes Y \in \mathsf{P}$ *implies* $X \in \mathsf{P}$ *or* $Y \in \mathsf{P}$, *for all objects* X, Y *in* T^c.

Proof. From the formula $\operatorname{supp}_{\mathsf{T}^c}(X \otimes Y) = \operatorname{supp}_{\mathsf{T}^c}(X) \cap \operatorname{supp}_{\mathsf{T}^c}(Y)$ it follows that $\mathsf{C} \cap \mathsf{D}$ and $\mathrm{Th}(\{X \otimes Y \mid X \in \mathsf{C},\, Y \in \mathsf{D}\}$ have the same support. Thus they coincide by Corollary 7.8. The second assertion is an immediate consequence. □

Remark 7.11. Let T be a compactly generated tensor triangulated category that is stratified via the action of a graded-commutative noetherian ring. Suppose also that the graded endomorphism ring of each compact object is finitely generated over R. The homeomorphisms

$$\operatorname{supp}_R(\mathsf{T}) \xrightarrow{\sim} \mathrm{Sp}(\mathsf{T}) = \mathrm{Sp}(\mathbf{D}(\mathsf{T})) \quad \text{and} \quad \operatorname{supp}_R(\mathsf{T}) \xrightarrow{\sim} \mathrm{Sp}(\mathsf{T}^c) = \mathrm{Sp}(\mathbf{Th}(\mathsf{T}^c))$$

from Examples 6.10 and 7.7 are compatible with $\mathrm{Sp}(f)$, in that, the following diagram is commutative.

$$\begin{array}{ccc} \operatorname{supp}_R(\mathsf{T}) & \xrightarrow{\mathrm{id}} & \operatorname{supp}_R(\mathsf{T}) \\ \downarrow\wr & & \downarrow\wr \\ \mathrm{Sp}(\mathsf{T}) & \xrightarrow{\mathrm{Sp}(f)} & \mathrm{Sp}(\mathsf{T}^c) \end{array}$$

In particular, the map $\mathrm{Sp}(f)$ is bijective. The topology on $\mathrm{Sp}(\mathsf{T})$ is discrete, while the one on $\mathrm{Sp}(\mathsf{T}^c)$ usually is not.

Stone versus Zariski topology. Fix a commutative ring A. Then $\operatorname{Spec} A$, the set of prime ideals of A with the Zariski topology, is the prototypical example of a spectral topological space [15]. One can think of the tensor triangulated category $\mathsf{D}^{\mathrm{per}}(A)$ of perfect complexes over A as a categorification of $\operatorname{Spec} A$, because the space $\mathrm{Sp}(\mathsf{D}^{\mathrm{per}}(A))$ endowed with the Stone topology is homeomorphic to the Hochster dual of $\operatorname{Spec} A$; see the Example 7.12 below, which shows that the formal notion of support for tensor triangulated categories is equivalent to the familiar notion from algebraic geometry.[2]

Example 7.12. Fix a quasi-compact and quasi-separated scheme $(X, \mathcal{O}_X)$; every noetherian scheme has these properties. The complexes of $\mathcal{O}_X$-modules with quasi-coherent cohomology form a compactly generated tensor triangulated category $\mathsf{T} = \mathsf{D}_{\mathrm{Qcoh}}(X)$, and its category of compact objects T^c identifies with the category $\mathsf{D}^{\mathrm{per}}(X)$ of perfect complexes [26]. In [33], Thomason classified the tensor closed thick subcategories of $\mathsf{D}^{\mathrm{per}}(X)$ using the following notion of support. For a complex $x \in \mathsf{D}^{\mathrm{per}}(X)$, write

$$\operatorname{supp}_X(x) = \{P \in X \mid x_P \neq 0\}\,.$$

[2]For a lattice theoretic analysis of this example, see [10]. In [30, p. 1442], Rota writes: 'To this day lattice theory has not made much of a dent in the sect of algebraic geometers; if it ever does, it will contribute new insights'.

Then the assignments

$$\mathsf{D}^{\mathrm{per}}(X) \supseteq \mathsf{C} \mapsto \bigcup_{x\in\mathsf{C}} \operatorname{supp}_X(x) \quad \text{and} \quad X \supseteq Y \mapsto \{x \in \mathsf{D}^{\mathrm{per}}(X) \mid \operatorname{supp}_X(x) \subseteq Y\}$$

induce bijections between

(1) the set of all tensor closed thick subcategories of $\mathsf{D}^{\mathrm{per}}(X)$, and
(2) the set of all subsets $Y \subseteq X$ of the form $Y = \bigcup_{i\in\Omega} Y_i$ with quasi-compact open complement $X \smallsetminus Y_i$ for all $i \in \Omega$.

The example above also illustrates the different topologies that are in use in describing the spectrum of a tensor triangulated category. Thus we return to the diagram from the introduction and can now explain its commutativity.

Fix a compactly generated tensor triangulated category T. It follows from Corollary 7.10 that the set $\operatorname{Sp}(\mathsf{T}^c)$ of prime elements of $\mathbf{Th}(\mathsf{T}^c)$ coincides with the spectrum of prime ideals of T^c defined by Balmer in [2]. Observe that $\mathbf{Th}(\mathsf{T}^c)$ has been identified with a sublattice of the Bousfield lattice in Theorem 7.6. Using the Stone topology, the support $\operatorname{supp}_{\mathsf{T}^c}(X)$ of a compact object X is a quasi-compact open subset of $\operatorname{Sp}(\mathsf{T}^c)$. Hochster duality [15] turns this into a closed set in the Zariski topology, which is used in [2].

Stratification revisited. Let T be a compactly generated tensor triangulated category. We propose a notion of stratification which does not involve the action of a graded-commutative noetherian ring; instead we use the spectrum $\operatorname{Sp}(\mathsf{T}^c)$. This provides the connection between the stratification from [6] and recent work of Balmer and Favi [3], and Stevenson [31].

Suppose that the space $\operatorname{Sp}(\mathsf{T}^c)$ satisfies the descending chain condition (dcc) on open subsets, that is, the Hochster dual is a noetherian space. Fix a prime P in $\operatorname{Sp}(\mathsf{T}^c)$. The following lemma implies that there are open subsets U, V such that $U \smallsetminus V = \{\mathsf{P}\}$.

Lemma 7.13. *Let X be a T_0-space satisfying the dcc on open subsets. Then there exists for each $x \in X$ a pair of open subsets U, V such that $U \smallsetminus V = \{x\}$.* □

Using the bijection from Corollary 7.8, we get tensor closed thick subcategories C, D of T^c such that for any Q in $\operatorname{Sp}(\mathsf{T}^c)$

$$\mathsf{C} \not\subseteq \mathsf{Q},\ \mathsf{D} \subseteq \mathsf{Q} \quad \Longleftrightarrow \quad \mathsf{Q} = \mathsf{P}.$$

Now define a functor $\varGamma_{\{\mathsf{P}\}} \colon \mathsf{T} \to \mathsf{T}$ by setting $\varGamma_{\{\mathsf{P}\}} = L_{\mathsf{D}}\varGamma_{\mathsf{C}}$. Here, L_{D} denotes the localization functor with kernel $\operatorname{Loc}(\mathsf{D})$, and $\varGamma_{\mathsf{C}}$ denotes the colocalization functor with essential image $\operatorname{Loc}(\mathsf{C})$; see Lemma 6.13. The functor $\varGamma_{\{\mathsf{P}\}}$ is studied in [3, §7]; it is the analogue of the local cohomology functor $\varGamma_{\mathfrak{p}}$ for a prime ideal $\mathfrak{p}$ of a graded-commutative noetherian ring R acting on T introduced in [4, §5]. The argument given in [4, Theorem 6.2] shows that the definition of $\varGamma_{\{\mathsf{P}\}}$ does not depend on the choice of C, D. Similarly, the analogue of [6, Lemma 2.4] gives

$$\varGamma_{\{\mathsf{P}\}}\varGamma_{\{\mathsf{Q}\}} = \begin{cases} \varGamma_{\{\mathsf{P}\}} & \text{if } \mathsf{P} = \mathsf{Q}, \\ 0 & \text{otherwise.} \end{cases} \tag{7.14}$$

Following [6, §3] and [31, §3], we say that the *local-global principle* holds for T if for each object X in T

$$\operatorname{Loc}(X) = \operatorname{Loc}(\{\varGamma_{\{\mathsf{P}\}}X \mid \mathsf{P} \in \operatorname{Sp}(\mathsf{T}^c)\})\,.$$

For example, the local-global principle holds when T has a model, by [31, Theorem 2.3.9], or when the dimension of $\operatorname{Sp}(\mathsf{T}^c)$ is finite [6, Corollary 3.5].

Following [6, §4], we say that T is *stratified* by $\operatorname{Sp}(\mathsf{T}^c)$ if

(1) $\mathrm{Sp}(\mathsf{T}^c)$, endowed with the Stone topology, satisfies the descending chain condition on open subsets,
(2) the local-global principle holds for T, and
(3) for each P in $\mathrm{Sp}(\mathsf{T}^c)$ there is no non-zero tensor closed localizing subcategory $\mathsf{S} \subseteq \mathsf{T}$ which is properly contained in $\varGamma_{\{\mathsf{P}\}}\mathsf{T}$.

A specific example of a category T which is stratified by $\mathrm{Sp}(\mathsf{T}^c)$ is the derived category $\mathsf{D}(\mathrm{Qcoh}\, X)$ of the category of quasi-coherent $\mathcal{O}_X$-modules for a separated noetherian scheme $(X, \mathcal{O}_X)$; see Example 6.16. Another example can be found in recent work of Stevenson [31] concerning certain singularity categories.

Let us end by pointing out an analogue of Example 6.15.

Proposition 7.15. *Suppose that the local-global principle holds for T. Then the functors $\varGamma_{\{\mathsf{P}\}}$ induce isomorphisms*

$$\mathbf{A}(\mathsf{T}) \xrightarrow{\sim} \prod_{\mathsf{P}\in\mathrm{Sp}(\mathsf{T}^c)} \mathbf{A}(\varGamma_{\{\mathsf{P}\}}\mathsf{T}) \quad \text{and} \quad \mathbf{D}(\mathsf{T}) \xrightarrow{\sim} \prod_{\mathsf{P}\in\mathrm{Sp}(\mathsf{T}^c)} \mathbf{D}(\varGamma_{\{\mathsf{P}\}}\mathsf{T})\,.$$

If T is stratified by $\mathrm{Sp}(\mathsf{T}^c)$, this yields an isomorphism $\mathbf{A}(\mathsf{T}) \xrightarrow{\sim} \mathbf{2}^{\mathrm{Sp}(\mathsf{T}^c)}$.

Proof. Sending $(\mathrm{A}(X_\mathsf{P}))_\mathsf{P}$ to $\mathrm{A}(\coprod_\mathsf{P} X_\mathsf{P})$ provides an inverse. This follows from the local-global principle and the identity (7.14). □

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Srikanth B. Iyengar, Department of Mathematics, University of Nebraska, Lincoln, NE 68588, U.S.A.

Henning Krause, Fakultät für Mathematik, Universität Bielefeld, 33501 Bielefeld, Germany.