FOURIER ANALYSIS, LINEAR PROGRAMMING, AND DENSITIES OF DISTANCE AVOIDING SETS IN $\mathbb{R}^n$

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Abstract. In this paper we derive new upper bounds for the densities of measurable sets in $\mathbb{R}^n$ which avoid a finite set of prescribed distances. The new bounds come from the solution of a linear programming problem. We apply this method to obtain new upper bounds for measurable sets which avoid the unit distance in dimensions $2, \ldots, 24$. This gives new lower bounds for the measurable chromatic number in dimensions $3, \ldots, 24$. We apply it to get a new, short proof of a recent result of Bukh which in turn generalizes theorems of Furstenberg, Katznelson, Weiss and Bourgain and Falconer about sets avoiding many distances.

1. Introduction

Let $d_1, \ldots, d_N$ be positive real numbers. We say that a subset A of the n-dimensional Euclidean space $\mathbb{R}^n$ *avoids the distances* $d_1, \ldots, d_N$ if the distance between any two points in A is never $d_1, \ldots, d_N$. We define the *upper density* of a Lebesgue measurable set $A \subseteq \mathbb{R}^n$ as

$$\overline{\delta}(A) = \limsup_{T \to \infty} \frac{\operatorname{vol}(A \cap [-T,T]^n)}{\operatorname{vol}[-T,T]^n}.$$

In this expression $[-T,T]^n$ denotes the regular cube in $\mathbb{R}^n$ with side $2T$ centered at the origin. We denote the *extreme density* which a measurable set in $\mathbb{R}^n$ that avoids the distances $d_1, \ldots, d_N$ can have by

$$m_{d_1,\ldots,d_N}(\mathbb{R}^n) = \sup\{\, \overline{\delta}(A) \;:\; A \subseteq \mathbb{R}^n \text{ is measurable and avoids distances } d_1, \ldots, d_N \,\}.$$

In this paper we derive upper bounds for this extreme density from the solution of a linear programming problem.

To formulate our main theorem we consider the function Ω_n given by

$$\Omega_n(t) = \Gamma\Big(\frac{n}{2}\Big)\Big(\frac{2}{t}\Big)^{\frac{1}{2}(n-2)} J_{\frac{1}{2}(n-2)}(t), \text{ for } t > 0, \quad \Omega_n(0) = 1, \tag{1}$$

where $J_{\frac{1}{2}(n-2)}$ is the *Bessel function of the first kind* with *parameter* $(n-2)/2$. To fix ideas we plotted the graph of the function Ω_4 in Figure 1.

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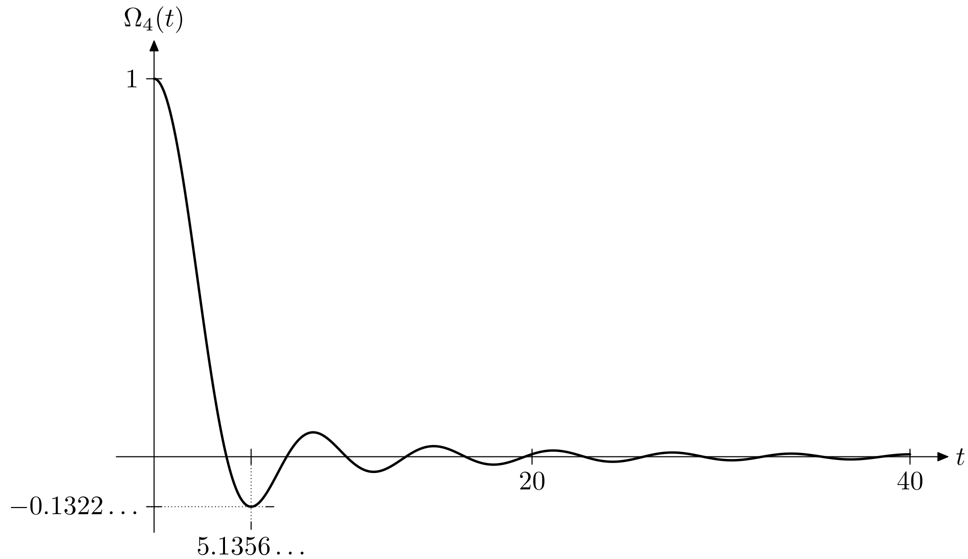

FIGURE 1.1. Graph of the function $\Omega_4(t) = \frac{2}{t}J_1(t)$

Theorem 1.1. *Let $d_1, \ldots, d_N$ be positive real numbers. Let $A \subseteq \mathbb{R}^n$ be a measurable set which avoids the distances $d_1, \ldots, d_N$. Suppose there are real numbers z_0, $z_1, \ldots, z_N$ which sum up to at least one and which satisfy*

$$z_0 + z_1\Omega_n(td_1) + z_2\Omega_n(td_2) + \cdots + z_N\Omega_n(td_N) \geq 0$$

for all $t \geq 0$. Then, the upper density of A is at most z_0.

In Section 2 we provide a proof where we make essential use of basic harmonic analysis, which we briefly recall. In the sections that follow we apply the main theorem in a variety of situations: sets avoiding one distance, sets avoiding two distances, and sets avoiding many distances. For the history of these Euclidean distance problems we refer to the surveys by Székely [18] and Székely and Wormald [8] and the references therein.

Sets avoiding one distance have been studied by combinatorialists because of their relation to the measurable chromatic number of the Euclidean space. This is the minimum number of colors one needs to color all points in $\mathbb{R}^n$ so that any two points at distance 1 receive different colors and so that points receiving the same color form Lebesgue measurable sets. Since every color class of a coloring provides a measurable set which avoids the distance 1, we have the inequality

$$m_1(\mathbb{R}^n) \cdot \chi_{\mathrm{m}}(\mathbb{R}^n) \geq 1. \tag{2}$$

For the plane it is only known that $5 \leq \chi_{\mathrm{m}}(\mathbb{R}^2) \leq 7$, where the lower bound is due to Falconer [10] and the upper bound comes e.g. from a coloring one constructs using a tiling of regular hexagons with circumradius slightly less than 1. Erdős conjectured that $m_1(\mathbb{R}^2) < 1/4$ so that (2) would yield an alternative proof of Falconer's result. So far the best known results on $m_1(\mathbb{R}^2)$ are the lower bound $m_1(\mathbb{R}^2) \geq 0.2293$ by Croft [7] and the upper bound $m_1(\mathbb{R}^2) \leq 12/43 \approx 0.2790$ by Székely [17]. In Section 3 we compute new upper bounds for $m_1(\mathbb{R}^n)$ for dimensions

$n = 2, \ldots, 24$ based on a strengthening of our main theorem by extra inequalities. These new upper bounds for $m_1(\mathbb{R}^n)$ imply by (2) new lower bounds for $\chi_{\mathrm{m}}(\mathbb{R}^n)$ in dimensions $3, \ldots, 24$.

If one considers sets which avoid more than one distance one can ask how N distances can be chosen so that the extreme density becomes as small as possible: What is the value of $\inf\{\, m_{d_1,\ldots,d_N}(\mathbb{R}^n) : d_1, \ldots, d_N > 0 \,\}$ for fixed N? For planar sets avoiding two distances Székely [17] showed that $\inf\{\, m_{d_1,d_2}(\mathbb{R}^2) : d_1, d_2 > 0 \,\} \leq m_{1,\sqrt{3}}(\mathbb{R}^2) \leq 2/11 \approx 0.181818$. In Section 4 we improve his result and show that $\inf\{\, m_{d_1,d_2}(\mathbb{R}^2) : d_1, d_2 > 0 \,\} \leq 0.141577$.

Recently, Bukh [5], using harmonic analysis and ideas resembling Szémeredi's regularity lemma, showed that $\inf\{\, m_{d_1,\ldots,d_N}(\mathbb{R}^n) : d_1, \ldots, d_N > 0 \,\}$ drops to zero exponentially in N. This implies a theorem of Furstenberg, Katznelson, and Weiss [13] that for every subset A in the plane which has positive upper density there is a constant d so that A does not avoid distances larger than d. Their original proof used tools from ergodic theory and measure theory. Alternative proofs have been proposed by Bourgain [4] using elementary harmonic analysis and by Falconer and Mastrand [12] using geometric measure theory. Bukh's result also implies that $m_{d_1,\ldots,d_N}(\mathbb{R}^n)$ becomes arbitrarily small if the distances $d_1, d_2, \ldots, d_N$ become arbitrarily small. This is originally due to Bourgain [4] and Falconer [11]. In Section 5 we give a short proof of Bukh's result using our main theorem. This proof has the additional advantage that it easily provides quantitative estimates about the spacing between the distances.

The idea of linear programming bounds for packing problems of discrete point sets in metric spaces goes back to Delsarte [9] and it has been successfully applied to a variety of situations. Cohn and Elkies [6] were the first who were able to set up a linear programming bound for packing problems in non-compact spaces; by then no less than 30 years since Delsarte's fundamental contribution had gone by. Our main theorem can be viewed as a continuous analogue to their linear programming bound.

2. Proof of the main theorem

For the proof of our main theorem elementary notions from harmonic analysis will be important. We recall these here. For details we refer to, e.g., the book by Katznelson [14].

A complex valued function $f\colon \mathbb{R}^n \to \mathbb{C}$ is called *periodic* if it is invariant under an n-dimensional discrete subgroup of $\mathbb{R}^n$ or, in other words, if there is a basis $b_1, \ldots, b_n$ of $\mathbb{R}^n$ so that for all $\alpha_1, \ldots, \alpha_n \in \mathbb{Z}$ we have $f(x + \sum_{i=1}^n \alpha_i b_i) = f(x)$. The set $L = \{\, \sum_{i=1}^n \alpha_i b_i : \alpha_i \in \mathbb{Z} \,\}$ is called the *period lattice* of f and $L^* = \{\, u \in \mathbb{R}^n : x \cdot u \in \mathbb{Z} \text{ for all } x \in L \,\}$ is called the *dual lattice* of L.

The *mean value* of a periodic function f is given by

$$M(f) = \lim_{T\to\infty} \frac{1}{\operatorname{vol}[-T,T]^n} \int_{[-T,T]^n} f(x)\,dx.$$

For two periodic functions f and g we write $\langle f, g\rangle = M(f\overline{g})$. We say that f is *square-integrable* if $\langle f, f\rangle < \infty$. By $\|f\| = \sqrt{\langle f, f\rangle}$ we denote its *norm*. If f and g are both square-integrable, then $\langle f, g\rangle$ exists. For $u \in \mathbb{R}^n$ we define the *Fourier coefficient* $\widehat{f}(u) = \langle f, e^{iu\cdot x}\rangle$. Notice that the support of $\widehat{f}$ is a discrete set, namely it lies in the dual lattice of the period lattice of f, scaled by 2π. Denote

$f_y(x) = f(y+x)$ for a vector $y \in \mathbb{R}^n$, then $\widehat{f_y}(u) = \widehat{f}(u)e^{iu\cdot y}$. For square-integrable, periodic functions f and g *Parseval's formula*

$$\langle f, g\rangle = \sum_{u\in\mathbb{R}^n} \widehat{f}(u)\overline{\widehat{g}(u)}$$

holds. By writing the latter sum we mean that we sum over the union of the supports of $\widehat{f}$ and $\widehat{g}$.

Proof of Theorem 1.1. Let A be a measurable subset of $\mathbb{R}^n$ that avoids distances $d_1, \ldots, d_N$. By 1_A we denote its characteristic function $1_A\colon \mathbb{R}^n \to \{0,1\}$ whose support is precisely A. Without loss of generality we can assume that 1_A is a periodic function; in this case we say that A is *periodic*.

Indeed, from any measurable set A which avoids distances $d_1, \ldots, d_N$ we can construct a periodic set which avoids distances $d_1, \ldots, d_N$ and with upper density arbitrarily close to the one of A. To do this we intersect A with a regular cube of side $2T$ so that $\operatorname{vol}(A \cap [-T,T]^n)/\operatorname{vol}[-T,T]^n$ is close to the upper density $\overline{\delta}(A)$ and so that $\operatorname{vol}([-T+d, T-d]^n)/\operatorname{vol}[-T,T]^n$, with $d = \max\{d_1,\ldots,d_N\}$, differs from 1 only negligibly. Then we construct the new periodic set by tiling $\mathbb{R}^n$ with copies of $A \cap [-T+d, T-d]^n$ centered at the points of the lattice $T\mathbb{Z}^n$.

By $A - y$ we denote the translation of the set A by the vector $-y \in \mathbb{R}^n$ so that its characteristic function satisfies $1_{A-y}(x) = 1_A(x+y) = (1_A)_y(x)$. The following two properties are crucial:

$$\langle 1_A, e^0\rangle = \overline{\delta}(A), \tag{3}$$

$$\langle 1_{A-y}, 1_A\rangle = \overline{\delta}(A \cap (A-y)), \text{ for all } y \in \mathbb{R}^n. \tag{4}$$

In particular, we have $\langle 1_A, 1_A\rangle = \overline{\delta}(A)$ and $\langle 1_{A-y}, 1_A\rangle = 0$ for all vectors y of Euclidean norm $d_1, \ldots, d_N$. Using Parseval's formula we rewrite (3) and (4) in terms of the Fourier coefficients of 1_A:

$$\begin{aligned} \widehat{1_A}(0) &= \overline{\delta}(A), \\ \sum_{u\in\mathbb{R}^n} |\widehat{1_A}(u)|^2 e^{iu\cdot y} &= \overline{\delta}(A \cap (A-y)) \text{ for all } y \in \mathbb{R}^n. \end{aligned}$$

Now we consider the function

$$\varphi(y) = \sum_{u\in\mathbb{R}^n} |\widehat{1_A}(u)|^2 e^{iu\cdot y} = \overline{\delta}(A \cap (A-y)), \tag{5}$$

which is called the *autocorrelation function* of 1_A. By taking spherical averages we construct from it a radial function f whose values only depend on the norm of the vectors. In other words, we set

$$f(y) = \frac{1}{\omega_n}\int_{S^{n-1}} \varphi(\|y\|\xi)\, d\omega(\xi).$$

Here ω denotes the standard surface measure on the unit sphere $S^{n-1} = \{\, \xi \in \mathbb{R}^n : \xi\cdot\xi = 1 \,\}$ and $\omega_n = \omega(S^{n-1}) = (2\pi^{n/2})/\Gamma(n/2)$. Clearly, $f(0) = \overline{\delta}(A)$, and $f(y) = 0$ whenever $\|y\| \in \{d_1, \ldots, d_N\}$. Because of the formula (cf. Schoenberg [16, (1.6)], see (1) for an explicit expression for Ω_n)

$$\frac{1}{\omega_n}\int_{S^{n-1}} e^{iu\cdot\xi}\, d\omega(\xi) = \Omega_n(\|u\|)$$

we can represent f in the form

$$f(y) = \sum_{t \geq 0} \alpha(t) \Omega_n(t\|x\|),$$

where $\alpha(t)$ is the sum of $|\widehat{1_A}(u)|^2$ for vectors u having norm t so that the $\alpha(t)$'s are real and nonnegative. Furthermore, $\alpha(0) = |\widehat{1_A}(0)|^2 = \overline{\delta}(A)^2$ and $\sum_{t \geq 0} \alpha(t) = f(0) = \overline{\delta}(A)$.

So the following linear program in the variables $\alpha(t)$ gives an upper bound for the upper density of any measurable set which avoids the distances $d_1, \ldots, d_N$

$$\begin{aligned} \sup\big\{ \alpha(0) \ : \ & \alpha(t) \geq 0 \text{ for all } t \geq 0, \\ & \sum_{t \geq 0} \alpha(t) = 1, \\ & \sum_{t \geq 0} \alpha(t) \Omega_n(td_k) = 0 \text{ for } k = 1, \ldots, N \big\}. \end{aligned} \tag{6}$$

Above, all but a countable subset of the $\alpha(t)$'s are zero. Note moreover that we used the normalization $\sum_{t \geq 0} \alpha(t) = 1$. This linear program has infinitely many variables $\alpha(t)$ but only $N + 1$ equality constraints. A dual of it is

$$\begin{aligned} \inf\big\{ z_0 \ : \ & z_0 + z_1 + z_2 + \cdots + z_N \geq 1, \\ & z_0 + z_1 \Omega_n(td_1) + z_2 \Omega_n(td_2) + \cdots + z_N \Omega_n(td_N) \geq 0 \\ & \quad \text{for all } t > 0 \big\}, \end{aligned} \tag{7}$$

which has $N + 1$ variables $z_0, z_1, z_2, \ldots, z_N$ and infinitely many constraints. As usual, weak duality holds between the pair of linear programs (6) and (7): If $\alpha(t)$ satisfies the conditions in (6) and if $(z_0, z_1, \ldots, z_N)$ satisfies the conditions in (7), then

$$\alpha(0) \leq \sum_{t \geq 0} \alpha(t)(z_0 + z_1 \Omega_n(td_1) + z_2 \Omega_n(td_2) + \cdots + z_N \Omega_n(td_N)) = z_0,$$

which finishes the proof of our main theorem. □

3. Sets avoiding one distance

It is notable that the linear programming bounds for the extreme density of sets avoiding exactly one distance allow for an analytic optimal solution. Since this problem is scaling invariant we can assume that we consider sets avoiding the unit distance $d_1 = 1$. Let $j_{\alpha,k}$ be the k-th positive zero of the Bessel function J_α. It is known that the absolute minimum of the function Ω_n is attained at $j_{n/2,1}$ (see Askey, Andrews, Roy [1, (4.6.2)], and Watson [20, Chapter 15, §31]). So, the point (z_0, z_1) which is determined by the equations

$$\begin{aligned} z_0 + z_1 &= 1 \\ z_0 + z_1 \Omega_n(j_{n/2,1}) &= 0 \end{aligned}$$

provides the optimal solution for the linear program in Theorem 1.1. Hence,

$$z_0 = \Omega_n(j_{n/2,1}) / (\Omega_n(j_{n/2,1}) - 1) \geq m_1(\mathbb{R}^n), \tag{8}$$

and this gives by (2) a lower bound for the measurable chromatic number, namely $\chi_{\mathrm{m}}(\mathbb{R}^n) \leq 1 - 1/\Omega_n(j_{n/2,1})$. It is interesting to notice that this lower bound coincides with the one provided by Bachoc, Nebe, Oliveira, and Vallentin [2, Corollary

8.2], albeit by a shift of one dimension. This shift of one dimension is due to the fact that Bachoc, Nebe, Oliveira, and Vallentin [2] study the problem of sets avoiding one distance on the $(n-1)$-dimensional unit sphere $S^{n-1} \subseteq \mathbb{R}^n$ and the lower bound for the measurable chromatic number $\chi_{\mathrm{m}}(\mathbb{R}^n)$ was obtained by upper bounding the density of sets in the unit sphere which avoid the distance d where d goes to zero. So, we see now that this limit process gives a lower bound for the measurable chromatic number of $\mathbb{R}^{n-1}$ and not only for the one of $\mathbb{R}^n$.

3.1. **Adding extra inequalities.** It is possible to strengthen the main theorem and the resulting bound (8) by introducing extra inequalities. Consider a regular simplex in $\mathbb{R}^n$ with edge length 1 having vertices $v_1, \ldots, v_{n+1}$. A set $A \subseteq \mathbb{R}^n$ which avoids the unit distance can only contain one vertex of this regular simplex. So we have for the autocorrelation function φ of the characteristic function 1_A defined in (5) that

$$\begin{aligned} \varphi(v_1) + \cdots + \varphi(v_{n+1}) &= \overline{\delta}(A \cap (A - v_1)) + \cdots + \overline{\delta}(A \cap (A - v_{n+1})) \\ &\leq \overline{\delta}(A) \;=\; \varphi(0), \end{aligned}$$

which means that we can strengthen the primal formulation of our linear programming bound (6) by the inequality

$$f(\|v_1\|) + \cdots + f(\|v_{n+1}\|) \leq 1, \tag{9}$$

If we center a regular simplex at the origin, then the above inequality specializes to

$$(n+1) f(\sqrt{1/2 - 1/(2n+2)}) \leq 1.$$

which gives the following strengthening of the dual formulation (7)

$$\begin{aligned} \inf \Big\{ z_0 + z_c \;:\; & z_c \geq 0, \\ & z_0 + z_1 + z_c(n+1) \geq 1, \\ & z_0 + z_1 \Omega_n(t) + z_c (n+1) \Omega_n(t\sqrt{1/2 - 1/(2n+2)}) \geq 0 \\ & \quad \text{for all } t \geq 0 \Big\}. \end{aligned}$$

In Table 3.1 we give the new upper bounds on $m_1(\mathbb{R}^n)$ we get for $n = 4, \ldots, 24$ by solving the linear program on a computer (we discuss numerical issues at the end of this section) which are improvements over the values which Székely and Wormald give in [19]. This in turn gives new lower bounds for the measurable chromatic number for $n = 4, \ldots, 24$.

However, in dimension 2 we only get an upper bound of 0.287119. To improve Székely's bound of $11/43 \approx 0.279069$ in the plane, we replace the regular triangle centered at the origin by more triangles. We use the following three triples of squared norms $(\|v_1\|^2, \|v_2\|^2, \|v_3\|^2)$ for (9): $(2.4, 2.4, 0.360314)$, $(3.1, 3.1, 6.524038)$ $(3.7, 3.7, 7.417141)$, where the last coordinate of (a, b, c) is a root of $3(a^2 + b^2 + c^2 + 1) - (a + b + c + 1)^2$. This condition assures that the determinant of the positive semidefinite Gram matrix

$$\begin{pmatrix} a & \frac{1}{2}(a+b-1) & \frac{1}{2}(a+c-1) \\ \frac{1}{2}(a+b-1) & b & \frac{1}{2}(b+c-1) \\ \frac{1}{2}(a+c-1) & \frac{1}{2}(b+c-1) & c \end{pmatrix}$$

n	best upper bound for $m_1(\mathbb{R}^n)$ previously known	new upper bound for $m_1(\mathbb{R}^n)$	best lower bound for $\chi_m(\mathbb{R}^n)$ previously known	new lower bound for $\chi_m(\mathbb{R}^n)$
2	0.279069 [17]	0.268412	5 [10]	
3	0.187500 [19]	0.165609	6 [10]	7
4	0.128000 [19]	0.112937	8 [19]	9
5	0.0953947 [19]	0.0752845	11 [19]	14
6	0.0708129 [19]	0.0515709	15 [19]	20
7	0.0531136 [19]	0.0361271	19 [19]	28
8	0.0346096 [19]	0.0257971	30 [19]	39
9	0.0288215 [19]	0.0187324	35 [19]	54
10	0.0223483 [19]	0.0138079	48 [2]	73
11	0.0178932 [19]	0.0103166	64 [2]	97
12	0.0143759 [19]	0.00780322	85 [2]	129
13	0.0120332 [19]	0.00596811	113 [2]	168
14	0.00981770 [19]	0.00461051	147 [2]	217
15	0.00841374 [19]	0.00359372	191 [2]	279
16	0.00677838 [19]	0.00282332	248 [2]	355
17	0.00577854 [19]	0.00223324	319 [2]	448
18	0.00518111 [19]	0.00177663	408 [2]	563
19	0.00380311 [19]	0.00141992	521 [2]	705
20	0.00318213 [19]	0.00113876	662 [2]	879
21	0.00267706 [19]	0.00091531	839 [2]	1093
22	0.00190205 [19]	0.00073636	1060 [2]	1359
23	0.00132755 [19]	0.00059204	1336 [2]	1690
24	0.00107286 [19]	0.00047489	1679 [2]	2106

TABLE 3.1. Upper bounds for $m_1(\mathbb{R}^n)$ and lower bounds for $\chi_{\mathrm{m}}(\mathbb{R}^n)$.

of the points v_1, v_2, v_3 of a corresponding regular simplex vanishes. Solving the corresponding linear program yields the new upper bound of 0.268412. We found the three triples by considering all triples (a, b, c) with $a, b = 0.1j$ with $j = 0, \ldots, 40$.

In dimension 3 we use three quadruples $(\|v_1\|^2, \|v_2\|^2, \|v_3\|^2, \|v_4\|^2)$ of squared norms for (9): $(0.3, 0.4, 0.4, 0.417157)$, $(1.9, 1.9, 1.9, 0.189372)$, $(2, 2, 2, 0.225148)$, where the last coordinate of (a, b, c, d) is a root of $3(a^2 + b^2 + c^2 + d^2 + 1) - 2(ab + ac + ad + bc + bd + cd) - 2(a + b + c + d)$. Solving the corresponding linear programming yields the new upper bound of 0.165609. We found the three quadruples by considering all triples (a, b, c, d) with $a, b, c = 0.1j$ with $j = 0, \ldots, 40$.

3.2. **Numerical calculations.** A few technical remarks concerning the numerical calculations are in order. For solving the linear programs we use the software `lpsolve` [3] and we generate the input using the program `GP/PARI` [15]. We discretize the conditions of the form

$$z_0 + z_1\Omega_n(t) + z_c(n+1)\Omega_n(t\sqrt{1/2 - 1/(2n+2)}) \geq 0 \quad \text{for all } t \geq 0$$

by discretizing the interval $[0, 20]$ into steps of size 0.0005.

Now we demonstrate in the case $n = 4$ how we turn the numerical calculations into a rigorous mathematical proof: The linear program has the optimal numerical solution $z_0 = 0.0826818$, $z_1 = 0.7660402$, $z_c = 0.0302556$. A lower bound of the minimum of the function

$$z(t) = z_0 + z_1\Omega_4(t) + 5z_c\Omega_4(\sqrt{2/5}t)$$

in $t \in [0, 20]$ is -0.00000006. The function $z(t)$ is positive for $t \geq 20$ because there $\Omega_4(t) \geq -0.02$ and $\Omega_4(\sqrt{2/5}t) \geq -0.04$ holds. Thus by adding 0.00000006 to z_0 we make sure that the new function $z(t)$ is nonnegative. This only slightly effects the value of the bound.

4. Planar sets avoiding two distances

In this section we quickly report on the problem to find the smallest extreme density a measurable set in the plane can have which avoids exactly two distances, i.e., $\inf\{\, m_{d_1,d_2}(\mathbb{R}^2) : d_1, d_2 > 0\,\}$. Székely [17] showed that this number is at most 2/11 by giving an upper bound for $m_{1,\sqrt{3}}(\mathbb{R}^2)$. By solving the corresponding linear program on the computer we improve his bound to $m_{1,\sqrt{3}}(\mathbb{R}^2) \leq 0.170213$. By adjusting the distances we can improve this further: $m_{1,j_{1,2}/j_{1,1}}(\mathbb{R}^2) \leq 0.141577$ where $j_{1,1}$ and $j_{1,2}$ are the first two positive zeros of the Bessel function J_1.

5. Sets avoiding many distances

In this section we give an alternative proof of the main result of Bukh [5, Theorem 1] about densities of sets avoiding many distances. He shows that the number $\inf\{\, m_{d_1,\ldots,d_N}(\mathbb{R}^n) : d_1, \ldots, d_N > 0\,\}$ drops to zero exponentially in N if the ratios $d_2/d_1, \ldots, d_N/d_{N-1}$ go to infinity. Bukh's proof is based on a so-called zooming out lemma which resembles Szemerédi's regularity lemma for dense graphs, whereas our proof is an easy consequence of Theorem 1.1 and simple properties of the function Ω_n.

Theorem 5.1. *For every positive integer N there is a number $r = r(N)$ strictly larger than 1 such that for distances $d_1, \ldots, d_N$ with*

$$d_2/d_1 > r,\ d_3/d_2 > r,\ \ldots,\ d_N/d_{N-1} > r \tag{10}$$

we have $m_{d_1,\ldots,d_N}(\mathbb{R}^n) \leq 2^{-N}$.

In the proof of Theorem 5.1 some facts about the function Ω_n will be useful. First, we have

$$|J_0(t)| \leq 1, \quad \text{and} \quad |J_\alpha(t)| \leq 1/\sqrt{2} \quad \text{for all } \alpha > 0 \tag{11}$$

(cf. (4.9.13) in Andrews, Askey, and Roy [1]). From this, it follows at once that $\lim_{t\to\infty} \Omega_n(t) = 0$ for $n > 2$. For $n = 2$ the same follows, e.g., from the asymptotic expansion for J_α (cf. (4.8.5) in Andrews, Askey, and Roy [1]).

Moreover,

$$\Omega_n(t) \geq -1/2, \quad \text{for all } n \geq 2,\ t \geq 0. \tag{12}$$

To see this, set $\alpha = (n-2)/2$. We use once again the fact that the global minimum of Ω_n is attained at $j_{\alpha+1,1}$, the first positive zero of $J_{\alpha+1,1}$ (cf. Section 3). We combine this fact with the inequality $j_{\alpha+1,1} > j_{\alpha,1} > \alpha$ (cf. Section 15.3 in Watson [20]) and with (11) to conclude that the global minimum of Ω_n has absolute value at most $\Gamma(\alpha+1)(2/\alpha)^\alpha$. Now we can use the inequality $\Gamma(\alpha+1) \leq \alpha e(\alpha/e)^\alpha$, valid for $\alpha > 2$, to obtain (12) for $n \geq 31$. The remaining cases one checks manually.

Proof of Theorem 5.1. Given $N > 0$, set $\varepsilon = 1/(N2^{N+1})$. Since $\Omega_n(0) = 1$ and since Ω_n is continuous, there is a number $t_0 > 0$ such that $\Omega_n(t) > 1 - \varepsilon$ for $t \le t_0$. Likewise, since $\lim_{t\to\infty} \Omega_n(t) = 0$, there is a number $t_1 > t_0$ such that $|\Omega_n(t)| < \varepsilon$ for $t \ge t_1$.

Set $r = r(N) = t_1/t_0$ and let distances $d_1, \ldots, d_N$ be given such that (10) is satisfied. With this we claim that, for $1 \le j \le N$,

$$\sum_{i=j}^{N} \frac{1}{2^{N-i+1}} \cdot \Omega_n(td_i) \ge -\frac{1}{2^{N-j+2}} - (N-j)\varepsilon.$$

Before we prove the claim, we show how to apply it. By taking $j = 1$ in the claim, and since by our choice of ε we have $-(N-1)\varepsilon \ge -1/2^{N+1}$, it follows that

$$\sum_{i=1}^{N} \frac{1}{2^{N-i+1}} \cdot \Omega_n(td_i) \ge -\frac{1}{2^{N}}.$$

Now we may set $z_0 = 1/2^N$ and $z_i = 1/2^{N-i+1}$ for $i = 1, \ldots, N$ and apply Theorem 1.1, proving our result.

To finish, we prove the claim by induction. For $j = N$, the statement follows immediately from (12). So suppose $1 < j \le N$. We distinguish two cases.

First, for $t \le t_0/d_{j-1}$, we have from the choice of t_0 that $\Omega_n(td_{j-1}) > 1 - \varepsilon$. Using this and the induction hypothesis, we then have that

$$\begin{aligned}\sum_{i=j-1}^{N} \frac{1}{2^{N-i+1}} \cdot \Omega_n(td_i) &= \frac{1}{2^{N-j+2}} \cdot \Omega_n(td_{j-1}) + \sum_{i=j}^{N} \frac{1}{2^{N-i+1}} \cdot \Omega_n(td_i)\\ &\ge \frac{1-\varepsilon}{2^{N-j+2}} - \frac{1}{2^{N-j+2}} - (N-j)\varepsilon\\ &\ge -\frac{1}{2^{N-j+3}} - (N-j+1)\varepsilon.\end{aligned}$$

Now suppose $t \ge t_0/d_{j-1}$. Observe that, for $j \le i \le N$, we have $td_i \ge t_0 d_i/d_{j-1} \ge t_0 r = t_1$, hence $|\Omega_n(td_i)| < \varepsilon$. So, by using (12), we have

$$\begin{aligned}\sum_{i=j-1}^{N} \frac{1}{2^{N-i+1}} \cdot \Omega_n(td_i) &= \frac{1}{2^{N-j+2}} \cdot \Omega_n(td_{j-1}) + \sum_{i=j}^{N} \frac{1}{2^{N-i+1}} \cdot \Omega_n(td_i)\\ &\ge -\frac{1}{2^{N-j+3}} - (N-j+1)\varepsilon,\end{aligned}$$

finishing the proof of the claim. □

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